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RESEARCH ARTICLE

BI-CUBIC B-SPLINE COLLOCATION SOLUTION TO TWO-DIMENSIONALCONDUCTION HEAT TRANSFER PROBLEMS

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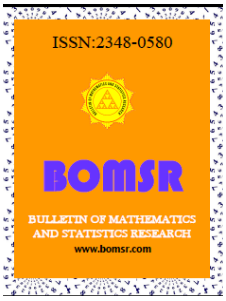

ABSTRACT

Tensor product of third degree B-spline basis functions is used as basis functions in collocation method for approximate solution of two-dimensional conduction heat transfer problems. Recursive form of B-spline function is used as basis in this present method. This method is applied to find the approximate solution of heat transfer governing partial differential equations with Boundary conditions for different kinds of domains. The results show efficiency and consistency of the present method. The easiness of the method reduces the complexity and time consuming when compared with other existed methods.

KeywordsB-Spline, Collocation, Perturbed Problems, Heat Transfer Problems,Laplace equation

1. INTRODUCTION

Heat transfer problems play the important role in boilers, condensers, air pre-heaters, economizers, electric motors, generators and transformers and in chemical reactions. The study of heat transfer problems gives the solutions to optimize the usage of materials and their performance in different systems [13] .Many numerical methods are developed to solve such type of Heat transfer problems. Widely used such numerical methods are finite volume method and finite element method [5, 6, 7, 8]. These methods depend on discretization of domains, conversion of strong form into weak form of the governing equation and evaluation of integrals to obtain linear system of equations. This process leads to errors like geometrical error, reduction in the continuity requirement of the approximating function which demands the fine mesh for an acceptable solution. In addition to these errors, mesh generation is more time consuming and costly.

Owing to the above mentioned difficulties in the mesh based methods, various point wise approximation techniques are developed like smoothed particle hydrodynamics (SPH) method, The element free Galerkin (EFG), Collocation methods etc. In collocation method, an approximating function is defined based on the nodal distribution only. In this method, many inter mediatory evaluations such as conversion of strong form differential equation to weak form, evaluation of integrals can be avoided.The use of B-Spline basis functions in collocation method improves the smoothness and accuracy of the solution. In this manuscript, a methodology is developed for the solution of Heat transfer problems using Bi-cubic B-spline Collocation method. Applicability and convergence of the present method is tested by considering numerical heat transfer problems

2. B-Spline Surfaces

A B-spline surface is defined [1,2,3,4] as

$$U(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} B_{i,j}\ M_{i,p}(x)\ N_{j,q}(y) \tag{1}$$

where $B_{i,j}$ are the vertices of the polygon net called control points, $M_{i,p}(x)$ is the pth degree B-spline basis function which is defined at the knot 'x' over the knot vector space KVx in X-direction and $N_{j,q}(y)$ is the qth degree B-spline basis function which is defined at the knot 'y' over the knot vector space KVy in Y-direction. The B-spline surface is also defined by rectangular array of control points. This form permits local control of curve shape. The degrees of its basis functions are defined independent of the control points.

3. B-Spline Collocation Method for 2D

Tensor product of B-spline basis function is used as basis function in collocation method to find the numerical solution for second order partial differential equations. Recursive form of B-spline function is employed as basis in normal collocation method. This method is developed based on the assumptions that the knot vectors are associated with the computational nodal points and constants in approximatesolution are treated as control points.

Considering the second order partial differential equation of the temperature distribution $U(x,y)$ which is the governing equation to the heat transfer problems over the region $a \le x \le b$, $c \le y \le d$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(x,y) \quad (2)$$

With the boundary conditions

$$\left\langle \begin{array}{ll} U(x,c) = g_1(x), & a \le x \le b \\ U(x,d) = g_2(x), & a \le x \le b \\ U(a,y) = h_1(y), & c \le y \le d \\ \quad and & \\ U(b,y) = h_2(y), & c \le y \le d \end{array} \right\rangle \tag{2a}$$

where a,b,c are constants, $g_1(x)$, $g_2(x)$ are functions in x, and $h_1(y), h_2(y)$ are functions in y.

Let

$$U^h(x,y) = \sum_{i=-3}^{m-1} \sum_{j=-3}^{n-1} B_{i,j}\ M_{i,p}(x)\ N_{j,q}(y) \tag{3}$$

Where $B_{i,j}$'s are control points, $M_{i,p}$ is the pth degree B- spline basis junction in the X- direction and $N_{j,q}(y)$ qth degree B- spline basis junction in the Y- direction be the approximate solution of heat distribution function $U(x, y)$ for which the governing partial differential equation is equation (3).The function $U(x, y)$ is the temperature distribution over the considered computational domain. Particularly, the approximate solution is assumed based on the third degree B-spline basis function that is employed in the collocation method. In order to have the B-spline property which is partition of unity, three additional knots should be taken both sides of knot vector space

Let the computational domain be $a \le x \le b, c \le y \le d$, the nodes in X-direction are $\{a = x_1, x_2, x_3, . . . , x_{m-1}, x_m = b\}$ and the nodes in Y-direction $\{c = y_1, y_2, y_3, . . . , y_{n-1}, y_n = d\}$.

Assuming that the knot vectors in each direction are nodes in each direction respectively and control points are treated as constants or unknowns in equation (3)

The first and second order partial derivatives of approximate function with respect to x are derived by differentiating the component function $M_{i,p}(x)$ with respect to x. i.e.

$$\frac{\partial U^h(x,y)}{\partial x} = \sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j} \frac{\partial M_{i,p}(x)}{\partial x} N_{j,q}(y) \tag{4}$$

$$\frac{\partial^2 U^h(x,y)}{\partial x^2} = \sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j} \frac{\partial^2 M_{i,p}(x)}{\partial x^2} N_{j,q}(y) \tag{5}$$

Similarly, the first and second order partial derivatives of approximate solution with respect to y is given as

$$\frac{\partial U^h(x,y)}{\partial y} = \sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j} M_{i,p}(x) \frac{\partial N_{j,q}(y)}{\partial y}, \quad \frac{\partial^2 U^h(x,y)}{\partial y^2} = \sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j} M_{i,p}(x) \frac{\partial^2 N_{j,q}(y)}{\partial y^2} \tag{6}$$

Similarly,

$$\frac{\partial^2 U^h(x,y)}{\partial y \partial x} = \sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j} \frac{\partial M_{i,p}(x)}{\partial x} \frac{\partial N_{j,q}(y)}{\partial y} \tag{7}$$

4. Collocation Method

Collocation method is a numerical technique. It is used to establish the relations among control points which are used to express the linear combination of the base functions. This method converts assumed approximate solution in the form of system of linear equations and become a powerful tool in developing various approximate methods because of its point based and discrete nature. Residue is the difference between the exact solution and the approximate solution. The residue is made to zero at some discrete nodal values in order to get the constraints among the control points. General working procedure of collocation method.

If B-spline functions are used as basises in approximate solution [10, 11, 12] and the collocation procedure is followed to obtain the system of linear equations in control points of approximation solution then this method is known as B-spline collocation method.

Substituting the equations (3)-(7) in governing differential equation (2). Then we have

$$\sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j} \frac{\partial^2 M_{i,p}(x)}{\partial x^2} N_{j,q}(y) + \sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j} M_{i,p}(x) \frac{\partial^2 N_{j,q}(y)}{\partial y^2} = f(x,y) \tag{8}$$

Let (x_r, y_s) be the any nodal point in the computational domain and expanding the equation (9) then we have

$$\sum_{i=-3}^{m-1}\frac{\partial^2 M_{i,p}(x_r)}{\partial x^2}\left[B_{i,-3}N_{-3,q}(y_s)+B_{i,-2}N_{-2,q}(y_s)+B_{i,-1}N_{-1,q}(y_s)+\cdots+B_{i,n-1}N_{n-1,q}(y_s)\right]$$

$$+\sum_{i=-3}^{m-1}M_{i,p}(x_r)\left[B_{i,-3}\frac{\partial^2 N_{-3,q}(y_s)}{\partial y^2}+B_{i,-2}\frac{\partial^2 N_{-2,q}(y_s)}{\partial y^2}+B_{i,-1}\frac{\partial^2 N_{-1,q}(y_s)}{\partial y^2}+\cdots+B_{i,n-1}\frac{\partial^2 N_{n-1,q}(y_s)}{\partial y^2}\right]=f(x_r,y_s) \quad (9)$$

Expressing the above formulation in matrix form,

$$[A_{cd}]*[B] = f(x_r,y_s) \quad (10)$$

$$[A_{cd}]=\begin{bmatrix} A'_{-3} & A'_{-2} & A'_{-1} & \cdots & A'_{(m-1)}\end{bmatrix}_{1\times(m+2)*(n+2)}$$

$$[B]=[B_{-3}\cdots B_{(m-1)}]^T_{(m+2)*(n+2)\times 1}$$

$$A_{(m-1)}=\begin{bmatrix}\frac{\partial^2 M_{(m-1),p}(x_r)}{\partial x^2}N_{-3,q}(y_s)+M_{(m-1),p}(x_r)\frac{\partial^2 N_{-3,q}(y_s)}{\partial y^2}\\ \vdots\\ \frac{\partial^2 M_{(m-1),p}(x_r)}{\partial x^2}N_{n-1,q}(y_s)+M_{(m-1),p}(x_r)\frac{\partial^2 N_{n-1,q}(y_s)}{\partial y^2}\end{bmatrix}^T_{1\times(n+2)}$$

$$B_{(m-1)}=\begin{bmatrix}B_{(m-1),-3}\\ \vdots\\ B_{(m-1),(n-1)}\end{bmatrix}_{(n+2)\times 1}$$

$$[A_{cd}]*[B] = \lfloor f(x_r,y_s)\rfloor \quad (11)$$

Equation (11) is the matrix form of equation (10) at single computational domain node point (x_r, y_s).It is the equation in $(m+2)\times(n+2)$ control points which is the result of assumption that the approximate solution (3) satisfies the governing partial differential equation (2) at the computational domain point (x_r, y_s).The computational domain consists (m)*(n) node points . The equation (10) should be evaluated at each these computational domain node and assumed that these node points are collocation points also. Then the equation (11) becomes system of (m)*(n) linear equations in (m+3)*(n+3) control points which are unknowns.

The Matrix form of above system of (m)*(n) linear equations is given below as

$$[A_{cd}]_{m*n\times(m+2)*(n+2)}*[B]_{(m+2)*(n+2)\times 1}=\lfloor f(x_r,y_s)\rfloor_{m*n\times 1} \quad (12)$$

The Matrix A_{cd} is not square matrix and yet Boundary conditions are not applied to approximate solution. Applying Boundary conditions to approximate solution (eq3)

The total number of nodal points on boundary is $(2m+2n-4)$ and the assumption is that the approximate solution satisfies the boundary conditions then applying equation (3) for given boundary conditions (2a),we have

Along the boundary points, we have

$$U(x,y)=U^h(x,y)=\sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1}B_{i,j}\,M_{i,p}(x)\,N_{j,q}(y)$$

Applying all the boundary conditions to approximate solution, we get

- Taking the boundary along y=c ,we have

$$y=c, U^h(x,y)=g_1(x)$$

$$\sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j}\ M_{i,p}(x)\ N_{j,q}(c)=g_1(x)$$

$$\sum_{i=-3}^{m-1} M_{i,p}(x)\begin{bmatrix} B_{i,-3}N_{-3,q}(c)+B_{i,-2}N_{-2,q}(c)+B_{i,-1}N_{-1,q}(c)+\ .\ .\ . \\ +B_{i,n-1}N_{n-1,q}(c) \end{bmatrix}=g_1(x)$$

Expressing the above equation into the matrix form , we have

$A_b * B = g_1(x)$ (13)

where $A_b = \begin{bmatrix} M_{-3,p}(x)N_{-3,q}(c) \\ . \quad . \quad . \\ . \quad . \quad . \\ . \quad . \quad . \\ M_{-3,p}(x)N_{(n-1),q}(c) \\ M_{-2,p}(x)N_{-3,q}(c) \\ \ldots\ldots \\ M_{-2,p}(x)N_{(n-1),q}(c) \\ M_{-1,p}(x)N_{-3,q}(c) \\ \ldots\ldots \\ M_{-1,p}(x)N_{(n-1),q}(c) \\ M_{(m-1),p}(x)N_{-3,q}(c) \\ M_{(m-1),p}(x)N_{-2,q}(c) \\ \ldots\ldots \\ M_{(m-1),p}(x)N_{(n-1),q}(c) \end{bmatrix}^T_{1\times(m+2)*(n+2)}$

The matrix form (13) is extended to all nodal points along the line y=c, then the system of m-linear equations are obtained in (m+3)*(n+3) control points and its matrix form is given as

$$(A_b)_{m\times(m+2)*(n+2)} * B = [g_1(x)]_{m\times 1} \tag{14}$$

where 'x' takes the m-nodes along the line y=c;

Similarly along the remaining boundary lines, we have along the above line i.e y=d

$$(A_a)_{m\times(m+2)*(n+2)} * B = [g_2(x)]_{m\times 1}. \tag{15}$$

where x takes above boundary line nodes.

Along the left boundary nodal point

$$(A_l)_{n\times(m+2)*(n+2)} * B = [h_1(y)]_{n\times 1} \tag{16}$$

Along the right boundary nodal points, we have the linear system of n- equations, so we have

$$(A_r)_{n\times(m+2)*(n+2)} * B = [h2(y)]_{n\times 1} \tag{17}$$

Assembling all the systems of linear equations which are generated over the computational domain boundary points i.e combining all the equations (14) (15), (16) and (17), then we have

$$\begin{bmatrix} A_b \\ A_a \\ A_l \\ A_r \end{bmatrix}_{si} * \quad [B]_{(m+3)*(n+3)\times 1} = \begin{bmatrix} g_1(x) \\ g_2(x) \\ h_1(y) \\ h_2(y) \end{bmatrix}_{(2m+2n-4)\times 1} \tag{18}$$

where $si = (2m+2n-4)\times(m+2)*(n+2)$

Assembling all the matrices in order to form the global matrix, they are (13) and (18)

$$[A_{cd}]_{m*n\times(m+2)*(n+2)} * [B]_{(m+3)*(n+3)\times 1} = \lfloor f(x_r, y_s) \rfloor_{m*n\times 1}$$

and

$$\begin{bmatrix} A_b \\ A_a \\ A_l \\ A_r \end{bmatrix}_{si} * \quad [B]_{(m+3)*(n+3)\times 1} = \begin{bmatrix} g_1(x) \\ g_2(x) \\ h_1(y) \\ h_2(y) \end{bmatrix}_{(2m+2n-4)\times 1}$$

where $si = (2m + 2n - 4) \times (m + 2) * (n + 2)$

i.e.The number of equations generated in the computational domain are m*n and the number of equations obtained by using boundary conditions are 2m+2n but corner nodes are used two times therefore deducting these repetitions then we have 2m+2n-4 boundary conditions are only included in constructing the global matrix. The repeated evaluations at corner nodes are neglected

$$\begin{bmatrix} A_{cd} \\ A_b \\ A_a \\ A_l \\ A_r \end{bmatrix}_{(m*n+2m+2n-4)\times(m+2)*(n+2)} * \quad [B]_{(m+2)*(n+2)\times 1} = \begin{bmatrix} f(x_r, y_s \\ g_1(x) \\ g_2(x) \\ h_1(y) \\ h_2(y) \end{bmatrix}_{(m*n+2m+2n-4)\times 1}$$

This can be written in the simplified matrix form, we have

$$[A]*[B]=[C] \qquad (19)$$

where

$$A = \begin{bmatrix} A_{cd} \\ A_b \\ A_a \\ A_l \\ A_r \end{bmatrix}_{(m*n+2m+2n-4)\times(m+3)*(n+3)}$$

$$C = \begin{bmatrix} f(x_r, y_s \\ g_1(x) \\ g_2(x) \\ h_1(y) \\ h_2(y) \end{bmatrix}_{(m*n+2m+2n-4)\times 1}$$

$$[B] = \begin{bmatrix} B_{-3,-3} & B_{-3,-2} & B_{-3,-1} & \cdot & \cdot & \cdot & B_{-3,n-1} \\ & \cdot & & \cdot & & \cdot & \\ & \cdot & & \cdot & & \cdot & \\ & \cdot & & \cdot & & \cdot & \\ B_{m-1,-3} & B_{m-1,-2} & B_{m-1,-1} & \cdot & \cdot & \cdot & B_{m-1,n-1} \end{bmatrix}_{(m+3)\times(n+3)}$$

Solve the equation 19 for unknown constants (control points). Replacing these values with unknowns in equation 3 then the approximation solution becomes known approximate solution to the equation 2.

The whole assembly of matrices and its solution is obtained by coding in Matlab as based on the implementation procedure given below.

5. Numerical Example

In this section two dimensional numerical experiments are considered as in the part of testing of applicability of bi-cubic B-spline collocation method to various kinds of boundary value problems.

Laplace equation with the non–zero boundary conditions is considered under the numerical experiment 1 and compared the obtain solution with the well established numerical technique Finite Element Method by 64 nodes. This problem is solved by the present method considering the 2 sets of control points 7 X 7 (49 control points)and7 X 9(63 control points) where the first digit represents the number of control points in X-direction and the second digit represents the number of control points in Y-direction. The results are presented in table 1.

Governing differential equation [13] for the temperature distribution $U(x,y)$ is

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0 \qquad\qquad 0\le x,y\le 1$$

With the boundary conditions

$$U(0,y)=100^{\circ}C\ ;U(x,0)=100^{\circ}C;\ U(1,y)=100^{\circ}C;\ U(x,1)=500^{\circ}C$$

Given computational domain [0 1]×[0,1]and boundary conditions are shown in below Figure 1.

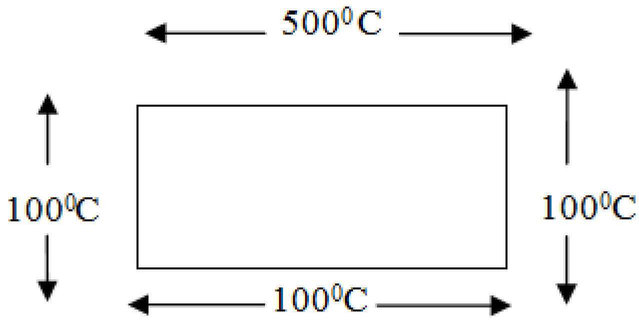

Figure1. Computational domain [0 1]×[0,1]

Case (i) : Partition of the domain for 7×7

Dividing each side 7 uniform parts of the interval [0, 1] gives the total 49 collocation points for the square computational domain. A third degree B-spline basis function is employed in collocation method to obtain the numerical solution for the numerical example1. Knot vectors associates the nodal points in each direction. So, the knot vectors in X-direction are KVx = {0 0.1667 0.3333 0.5000 0.6667 0.8333 1.} and the knot vectors in Y-direction KVy = {0 0.1667 0.3333 0.5000 0.6667 0.8333 1}.Three additional knot vectors are added both side of the knot vector space for both directions in order to maintain the partition of unity property of B-spline basis function. These additional knots are considered only to find the weights of knots at inside the knots in computational domain. These knots are not treated as collocation points because these knots are outside of the domain. The equation 4.4 is approximate solution with the assumption that the cubic degree B-spline is used as the basis function in collocation method. The computational domain for 7×7 partition has 49 nodes which are treated as knots to find B-spline base function.

The nodes in the computational domain are taken as the collocation points and the assumption that the approximate solution satisfies governing differential equation at these collocation points. This gives the system of 49-linear equations in control points (unknowns).24 nodes are boundary points. The boundary conditions are given for the given governing differential equation. The approximate solution is the solution to the governing differential equation as the assumption in collocation method.Therfore; the equation4.4 should satisfy the boundary conditions also.

Based on the implantation procedure which is given below for this numerical example1 is implemented in Matlab

- Implementation procedure of the method is given below
- Assumption of approximate solution as the Cartesian product of B-spline basis function ineach direction (3)
- Substuiting the approximation solution (eq3) in governing differential equation (eq2)

- System of linear equations is developed (eq.12)
- Imposing the boundary conditions (eq.14, eq. 15, eq. 16 & eq.17)
- Assembling all the equations (eq.13, eq. 14, eq. 15, eq.16&eq.17)
- Solve (eq.19) for control points [B]
- Substitute these control points (constants in eq.3) in eq.3

Table 1. Gives the different type of solution values at node (.5, .5)

Node	Present method		Finite Element Method (8X8 partition)	Analytical Solution
	7X7 partition	7X9 partition		
(.5,.5)	200.7047^0C	200.1005^0C	200^0C	200.11^0C

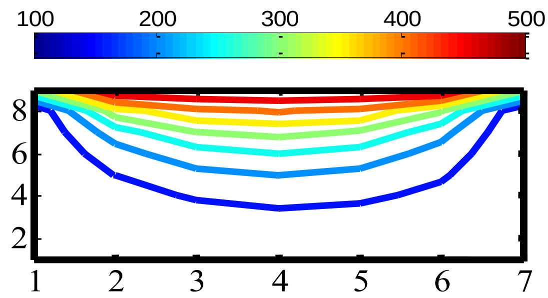

Figure 2.Temperature contours by the present method

6. Results and Discussion

Laplace 2-D heat conduction problem with the boundary conditions is illustrated to demonstrate the present method. Tested the method by changing the number of collocation points.Intially, computational domain is made into 7×7 partitions and estimated the temperature at the node (.5,.5) in the computational domain. These estimated values are shown in Table1.The temperature at mid-point (.5, .5) is 200.11^0C.When increased the number of partitions of computational domain 7×7 to 7×9 , it is observed that temperature at the mid (.5, .5) is improved from 200.7047^0Cto 200.1005^0C.This shows the convergence of the present method.

7. Conclusions

Bi-cubic B-spline collocation method is demonstrated in this paper to solve heat conduction transfer problems by taking the numerical example. The obtained results are good agreement with the exact values. Adaptability and convergence of the present is tested for the same problem by increasing the number of nodal points. This method can be applied more problems for different types computational domains and for different types of heat conduction problems.

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