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RESEARCH ARTICLE

AN APPROACH TO RELATE THE TRIANGULAR AND HESSENBERG 'TFNM'

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ABSTRACT

In this paper, we recall the basic definitions are presented, some operations on triangular fuzzy numbers (TFNs) which is represented in L-R type are defined. We also have been defined the operations on triangular fuzzy matrices in L-R type. We yield the relation of the notion of triangular and Hessenberg of Triangular fuzzy number matrices(TFNM) are defined. We proposed some relevant properties are justified with counter examples.

Keywords : Fuzzy Arithmetic, Fuzzy number, Triangular fuzzy number (TFN), Triangular Triangular fuzzy matrix(TTFM), Hessenberg of Triangular fuzzy matrix(HTFM).

1. Introduction

Fuzzy sets have been introduced by Lofti.A.Zadeh[12] Fuzzy set theory permits the gradual assessments of the membership of elements in a set which is described in the interval [0,1]. It can be used in a wide range of domains where information is incomplete and imprecise. Interval arithmetic was first suggested by Dwyer [2] in 1951, by means of Zadeh's extension principle [13,14], the usual Arithmetic operations on real numbers can be extended to the ones defined on Fuzzy numbers. Dubosis and Prade [1] has defined any of the fuzzy numbers as a fuzzy subset of the real line [4]. A fuzzy number is a quantity whose values are imprecise, rather than exact as is the case with single – valued numbers.

Triangular fuzzy numbers (TFNs) are frequently used in application. It is well known that the matrix formulation of a mathematical formula gives extra facility to study the problem. Due to the presence of uncertainty in many mathematical formulations in different branches of science and technology.

We introduce triangular fuzzy matrices (TFMs), to the best of our knowledge. Fuzzy matrices were introduced for the first time by Thomason [11] who discussed the convergence of power of fuzzy matrix. Fuzzy matrices play an important role in scientific development. Two new operations and some applications of fuzzy matrices are given in [7,8,9,10].

Dinagar and Latha [12] proposed some types of type-2 triangular fuzzy matrices are investigated. Hessenberg matrices play an important role in many application and have been the object of several studies [3,5,6]. In recently Jaisankar et.al [5] On Hessenberg of triangular fuzzy number matrices is proposed.

The paper organized as follows, Firstly In section 2, we recall the definition of Triangular fuzzy number and some operations on triangular fuzzy numbers (TFNs) which is represented in L-R type are defined . In section 3, we have reviewed the definition of triangular fuzzy matrix (TFM) and some operations on Triangular fuzzy matrices (TFMs) is defined for the proposed representation. In section 4, we defined the notion of Triangular and Hessenberg TFM. In section 5, we have presented some properties of Triangular and Hessenberg TFM. Finally in section 6, conclusion is included.

2. Preliminaries

In this section, We recapitulate some underlying definitions and basic results of fuzzy numbers. In particular parametric representation of triangular fuzzy numbers and L-R representation of triangular fuzzy numbers.

Definition 2.1 Fuzzy Set

A fuzzy set is characterized by a membership function mapping the element of a domain, space or universe of discourse X to the unit interval [0,1]. A fuzzy set A in a universe of discourse X is defined as the following set of pairs

$$A = \{(x, \mu_A(x))\ ;\ x \in X\}$$

Here $\mu_A : X \to [0,1]$ is a mapping called the degree of membership function of the fuzzy set A and μ_A (x) is called the membership value of $x \in X$ on the fuzzy set A. These membership grades are often represented by real numbers ranging from [0,1].

Definition 2.2 Normal Fuzzy Set

A fuzzy set A of the universe of discourse X is called a normal fuzzy set implying that there exists at least one $x \in X$ such that μ_A (x) =1.

Definition 2.3 Convex Fuzzy Set

A fuzzy set A={(x, μ_A(x))}⊆ X is called Convex fuzzy set if all A_α are Convex set (i.e.,) for every element $x_1 \in A_\alpha$ and $x_2 \in A_\alpha$ for every $\alpha \in [0,1]$. $\lambda x_1 + (1-\lambda)x_2 \in A_\alpha$ for all $\lambda \in [0,1]$ otherwise the fuzzy set is called non-convex fuzzy set.

Definition 2.4 Fuzzy Number

A fuzzy set $\tilde{A}$defined on the set of real number R is said to be fuzzy number if its membership function has the following characteristics

- $i.$ $\tilde{A}$ is normal
- $ii.$ $\tilde{A}$ is convex
- $iii.$ The support of $\tilde{A}$ is closed and bounded then $\tilde{A}$ is called fuzzy number.

2.5 Parametric Representation of Triangular Fuzzy Number

Definition 2.5.1

A fuzzy number $\tilde{A}$ = (a, b, c) is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & ;\ -\infty < x \le a \\ \dfrac{x-a}{b-a} & ;\ a \le x \le b \\ \dfrac{x-c}{b-c} & ;\ b \le x \le c \\ 0 & ;\ c \le x < \infty \end{cases}$$

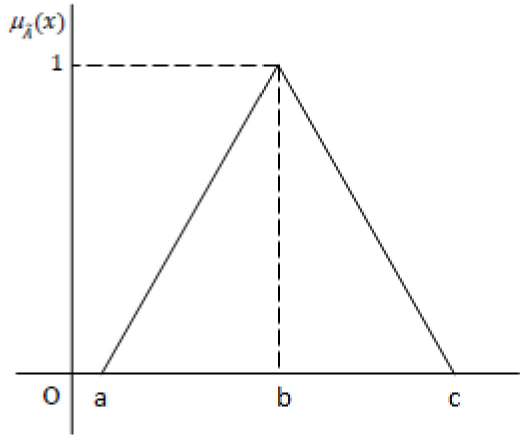

Fig:1

Definition 2.5.2 Zero Triangular Fuzzy Number

If $\tilde{A} = (0,0,0)$ then $\tilde{A}$ is said to be zero triangular fuzzy number. It is denoted by 0.

Definition 2.5.3 Zero Equivalent Triangular Fuzzy Number

A triangular fuzzy number $\tilde{A}$ is said to be a zero equivalent triangular fuzzy number, If $\Re(\tilde{A})$ =0. It is defined by $\tilde{0}$.

Definition 2.5.4 Unit Triangular Fuzzy Number

If $\tilde{A} = (1,1,1)$ then $\tilde{A}$ is said to be unit triangular fuzzy number. It is denoted by 1.

Definition 2.5.5 Unit Equivalent Triangular Fuzzy Number

A triangular fuzzy number $\tilde{A}$ is said to be unit equivalent triangular fuzzy number. If $\Re(\tilde{A}) = 1$. It is denoted by $\tilde{1}$.

2.6 L-R Parametric Representation of Triangular Fuzzy Number.

Definition 2.6.1

A triangular fuzzy number $\tilde{A}$ = (a,b,c), described in the definition 2.5.1. may also be represented as $\tilde{A}_{LR} = (m, \alpha, \beta)$ where $m = b, \alpha = b - a \geq 0, \beta = c - b \geq 0$.

A Fuzzy number $\tilde{A}_{LR} = (m, \alpha, \beta)$ is said to be a triangular fuzzy number if is membership function is given by

$$\mu_{\tilde{A}_{LR}}(\mathrm{x}) = \begin{cases} 0 & ; -\infty < x \leq \mathrm{m}\alpha \\ 1 - \dfrac{\mathrm{m} - \mathrm{x}}{\alpha} & ; \mathrm{m} - \mathrm{d} \leq \mathrm{x} < m \\ 1 - \dfrac{\mathrm{x} - \mathrm{m}}{\beta} & ; \mathrm{m} \leq \mathrm{x} < x + \beta \\ 0 & ; \mathrm{m} + \beta \leq \mathrm{x} < \infty \end{cases}$$

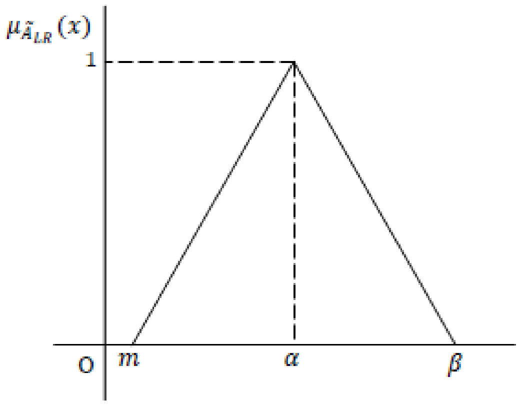

Fig:2

Definition 2.6.2 Zero Triangular Fuzzy Number

If $\tilde{A}_{LR} = (0,0,0)_{LR}$ then $\tilde{A}_{LR}$ is said to be zero triangular fuzzy number. It is denoted by 0.

Definition 2.6.3 Zero Equivalent Triangular Fuzzy Number

A triangular fuzzy number $\tilde{A}_{LR}$ is said to be a zero equivalent triangular fuzzy number. If $\Re(\tilde{A}_{LR})$ =0 it is defined by $\tilde{0}_{LR}$.

Definition 2.6.4 Unit Triangular Fuzzy Number

If $\tilde{A}_{LR} = (1,1,1)_{LR}$ then $\tilde{A}_{LR}$ is said to be unit triangular fuzzy number. It is denoted by $\tilde{1}_{LR}$.

Definition 2.6.5 Unit Equivalent Triangular Fuzzy Number

A triangular fuzzy number $\tilde{A}_{LR}$ is said to be unit equivalent triangular fuzzy number. If $\Re(\tilde{A}_{LR})$=1. It is denoted by $\tilde{1}$.

Definition 2.7 Ranking Function

An efficient approach for computing the fuzzy number is by the use of a ranking function $\Re: f(R) \to R$, where $F(R)$ is a set of fuzzy numbers defined on set of real numbers. which maps each fuzzy number into the real line, where a natural order exists.

$$\Re(\tilde{A}_{LR}) = m + \frac{\beta - \alpha}{4}$$

Definition 2.8 Arithmetic Operations on Triangular Fuzzy Numbers

Let $\tilde{A}_{LR} = (m_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{B}_{LR} = (m_2, \alpha_2, \beta_2)_{LR}$ be triangular fuzzy numbers then we defined.

Addition

$\tilde{A}_{LR} + \tilde{B}_{LR}$ = $(m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$

Subtraction

$\tilde{A}_{LR} - \tilde{B}_{LR}$ = $(m_1 - m_2, \alpha_1 - \alpha_2, \beta_1 - \beta_2)_{LR}$

Multiplication

$\tilde{A}_{LR} \times \tilde{B}_{LR}$ = $(m\Re(\tilde{B}_{LR}), \alpha\Re(\tilde{B}_{LR}), \beta\Re(\tilde{B}_{LR}))_{LR}$

Where $\Re(\tilde{B}_{LR}) = m + \frac{\beta-\alpha}{4}$ or $\Re(\tilde{B}_{LR}) = m + \frac{\beta-\alpha}{4}$

where the parametric triangular fuzzy number is a symmetric number.

Division

$\tilde{A}_{LR} / \tilde{B}_{LR}$ = $\left(\frac{m}{\Re(\tilde{B}_{LR})}, \frac{\alpha}{\Re(\tilde{B}_{LR})}, \frac{\beta}{\Re(\tilde{B}_{LR})}\right)$

where the parametric triangular fuzzy number is a symmetric number.

3. Triangular Fuzzy Matrices (TFMS)

In this section, we introduced the triangular fuzzy matrix and the operations of the fuzzy matrices.

Definition 3.1 Triangular Fuzzy Matrix (TFM).

A triangular fuzzy matrix of order $m \times n$ is defined as $A_{LR} = (\tilde{a}_{ij})_{LR\, m\times n}$ where $\tilde{a}_{ij} = (m_{ij}, \alpha_{ij}, \beta_{ij})$ is the ijth element of A_{LR}.

Definition 3.2Operations on Triangular Fuzzy Matrices(TFMS)

As for classical matrices. We define the following operations on triangular fuzzy matrices.

Let $A_{LR} = (\tilde{a}_{ij})_{LR}$ and $B_{LR} = (\tilde{b}_{ij})_{LR}$ be two triangular fuzzy matrices (TFMs) of same order. Then we have the following,

i. $A_{LR} + B_{LR}$ = $(\tilde{a}_{ij} + \tilde{b}_{ij})_{LR}$
ii. $A_{LR} - B_{LR}$ = $(\tilde{a}_{ij} - \tilde{b}_{ij})_{LR}$
iii. For $A_{LR} = (\tilde{a}_{ij})_{LR} m \times n$ and $B_{LR} = (\tilde{b}_{ij})_{LR} n \times k$ then
iv. $AB = (c_{ij})_{LR} m \times k$ where $\tilde{c}_{ij} = \sum_{p=1}^{n} \tilde{a}_{ij}.\tilde{b}_{ij}$, $i = 1,2, \ldots m$ and $j = 1,2, \ldots., k$.

v. A_{LR}^{T} or $A'_{LR} = \left(\tilde{a}_{ij}\right)_{LR}$

vi. $KA_{LR} = \left(k\tilde{a}_{ij}\right)_{LR}$ where k is scalar.

4. Relation between Hessenberg and Triangular TFM.

In this section, We introduced the basic definitions of Triangular and Hessenberg of Triangular fuzzy number matrices are followed to justify the properties in further section.

Definition 4.1 Lower Triangular TFM.

A Square triangular fuzzy matrix is $A = \left(\tilde{a}_{ij}\right)$ called Lower triangular TFM if all the entries above the diagonals are zero.

$$i.e., \tilde{a}_{ij} = 0; i < j, \forall\, i, j = 1,2, \dots., n$$

Example

$$A = \begin{pmatrix} (1,2,3) & (0,0,0) & (0,0,0) \\ (2,4,6) & (-1,1,3) & (0,0,0) \\ (1,3,5) & (3,4,5) & (-1,1,6) \end{pmatrix}$$

Definition 4.2 Upper Triangular TFM.

A Square triangular fuzzy matrix $A = \left(\tilde{a}_{ij}\right)$ is called an upper triangular TFM if all the entries below the diagonals are zero.

$$i.e., \tilde{a}_{ij} = 0; i > j\ , \forall\ i, j = 1,2, \dots., n$$

Example

$$A = \begin{pmatrix} (1,2,3) & (1,3,5) & (-1,1,6) \\ (0,0,0) & (-1,1,3) & (1,2,3) \\ (0,0,0) & (0,0,0) & (-1,1,6) \end{pmatrix}$$

Definition 4.3 Triangular TFM

A Square triangular fuzzy matrix $A = \left(\tilde{a}_{ij}\right)$ is called triangular TFM. If it is either upper triangular TFM and lower triangular fuzzy matrix.

Definition 4.7 Lower Hessenberg TFM.

A Square triangular fuzzy matrix $A = (\tilde{a}_{ij})$ is called Lower Hessenberg triangular fuzzy matrix if all the entries above the first super diagonal are zero.

i.e. $\tilde{a}_{ij} = 0; i + 1 < j\ \ \forall\ \ i, j = 1,2, \dots, n$

Example

$$A = \begin{pmatrix} (1,2,3) & (2,4,6) & (0,0,0) \\ (-1,1,3) & (1,3,5) & (3,4,5) \\ (7,8,9) & (-1,1,6) & (1,2,3) \end{pmatrix}$$

Definition 4.8 Upper Hessenberg TFM.

A Square triangular fuzzy matrix $A = (\tilde{a}_{ij})$ is called an upper Hessenberg triangular fuzzy matrix if all the entries below the first sub diagonal are zero.

i.e. $\tilde{a}_{ij} = 0; i > j + 1\ \ \forall\ \ i, j = 1,2, \dots, n$

Example

$$A = \begin{pmatrix} (1,2,3) & (2,4,6) & (7,8,9) \\ (-1,1,3) & (1,3,5) & (3,4,5) \\ (0,0,0) & (-1,1,6) & (1,2,3) \end{pmatrix}$$

Definition 4.9 Hessenberg Triangular Fuzzy Matrix(HTFM).

A Square triangular fuzzy matrix $A = (\tilde{a}_{ij})$ is called Hessenberg triangular fuzzy matrix (HFTM). If it is either upper Hessenberg triangular fuzzy matrix and lower Hessenberg triangular fuzzy matrix.

5.SOME PROPERTIES OF HESSENBERG AND TRIANGULAR FUZZY MATRICES

In this section, we introduced the properties of Triangular and Hessenberg of Triangular fuzzy Matrices(TFMs).

5.1. Properties of Hessenberg Triangular Fuzzy Matrices

Property 5.1.1

The Sum of the lower triangular TFM and lower Hessenberg TFM of order n is a lower Hessenberg TFM of order n.

Proof:

Let $A = (\tilde{a}_{ij})$ be the lower triangular TFM and

Let $B = (\tilde{b}_{ij})$ be the lower Hessenberg TFM.

Consider a matrix.

$A = \begin{pmatrix} (1,2,3) & (0,0,0) & (0,0,0) \\ (2,4,6) & (-1,1,3) & (0,0,0) \\ (1,3,5) & (3,4,5) & (-1,1,6) \end{pmatrix}$ be a lower triangular TFM.

$B = \begin{pmatrix} (1,2,3) & (2,4,6) & (0,0,0) \\ (-1,1,3) & (1,3,5) & (3,4,5) \\ (7,8,9) & (-1,1,6) & (1,2,3) \end{pmatrix}$ be a lower Hessenberg TFM.

Now we have to transform the matrices into L-R type representation by the definition (**2.6.1**) then we get matrices are

$$A_{LR} = \begin{pmatrix} (2,1,1)_{LR} & (0,0,0)_{LR} & (0,0,0)_{LR} \\ (4,2,2)_{LR} & (1,2,2)_{LR} & (0,0,0)_{LR} \\ (3,2,2)_{LR} & (4,1,1)_{LR} & (1,2,5)_{LR} \end{pmatrix}$$

$$B_{LR} = \begin{pmatrix} (2,1,1)_{LR} & (4,2,2)_{LR} & (0,0,0)_{LR} \\ (1,2,2)_{LR} & (3,2,2)_{LR} & (4,1,1)_{LR} \\ (8,1,1)_{LR} & (1,2,5)_{LR} & (2,1,1)_{LR} \end{pmatrix}$$

$A_{LR} + B_{LR} = C_{LR}$ then $(\tilde{a}_{ij})_{LR} + (\tilde{b}_{ij})_{LR} = (\tilde{c}_{ij})_{LR}$ we have to prove C_{LR} is the lower Hessenberg triangular fuzzy matrix.

$$C_{LR} = \begin{pmatrix} (2,1,1)_{LR} & (0,0,0)_{LR} & (0,0,0)_{LR} \\ (4,2,2)_{LR} & (1,2,2)_{LR} & (0,0,0)_{LR} \\ (3,2,2)_{LR} & (4,1,1)_{LR} & (1,2,5)_{LR} \end{pmatrix} + \begin{pmatrix} (2,1,1)_{LR} & (4,2,2)_{LR} & (0,0,0)_{LR} \\ (1,2,2)_{LR} & (3,2,2)_{LR} & (4,1,1)_{LR} \\ (8,1,1)_{LR} & (1,2,5)_{LR} & (2,1,1)_{LR} \end{pmatrix}$$

$$= \begin{pmatrix} (2,1,1)_{LR} + (2,1,1)_{LR} & (0,0,0)_{LR} + (4,2,2)_{LR} & (0,0,0)_{LR} + (0,0,0)_{LR} \\ (4,2,2)_{LR} + (1,2,2)_{LR} & (1,2,2)_{LR} + (3,2,2)_{LR} & (0,0,0)_{LR} + (4,1,1)_{LR} \\ (3,2,2)_{LR} + (8,1,1)_{LR} & (4,1,1)_{LR} + (1,2,5)_{LR} & (1,2,5)_{LR} + (2,1,1)_{LR} \end{pmatrix}$$

$$C_{LR} = \begin{pmatrix} (4,2,2)_{LR} & (4,2,2)_{LR} & (0,0,0)_{LR} \\ (5,4,4)_{LR} & (4,4,4)_{LR} & (4,1,1)_{LR} \\ (11,3,3)_{LR} & (5,3,6)_{LR} & (3,3,6)_{LR} \end{pmatrix}$$

Hence, C_{LR} is a lower Hessenberg TFM of order n.

Hence, the property 5.1.1 is proved by the counter example as above.

Property 5.1.2

The Sum of the upper triangular TFM and upper Hessenberg TFM of order n is a upper Hessenberg Triangular TFM of order n.

Proof:

Let $A = (\tilde{a}_{ij})$ be the upper triangular TFM and

Let $B = (\tilde{b}_{ij})$ be the upper Hessenberg TFM.

consider a matrix

$A = \begin{pmatrix} (1,2,3) & (1,3,5) & (-1,1,6) \\ (0,0,0) & (-1,1,3) & (1,2,3) \\ (0,0,0) & (0,0,0) & (-1,1,6) \end{pmatrix}$ be a upper triangular TFM.

$B = \begin{pmatrix} (1,2,3) & (2,4,6) & (7,8,9) \\ (-1,1,3) & (1,3,5) & (3,4,5) \\ (0,0,0) & (-1,1,6) & (1,2,3) \end{pmatrix}$ be a upper Hessenberg TFM.

Now, We have to transform the matrices into L-R type representation by the definition (**2.6.1**) then we get matrices are

$$A_{LR} = \begin{pmatrix} (2,1,1)_{LR} & (4,2,2)_{LR} & (3,2,2)_{LR} \\ (0,0,0)_{LR} & (1,2,2)_{LR} & (4,1,1)_{LR} \\ (0,0,0)_{LR} & (0,0,0)_{LR} & (1,2,5)_{LR} \end{pmatrix}$$

$$B_{LR} = \begin{pmatrix} (2,1,1)_{LR} & (4,2,2)_{LR} & (8,1,1)_{LR} \\ (1,2,2)_{LR} & (3,2,2)_{LR} & (4,1,1)_{LR} \\ (0,0,0)_{LR} & (1,2,5)_{LR} & (2,1,1)_{LR} \end{pmatrix}$$

$A_{LR} + B_{LR} = C_{LR}$ then $(\tilde{a}_{ij})_{LR} + (\tilde{b}_{ij})_{LR} = (\tilde{c}_{ij})_{LR}$ we have to prove C_{LR} is the Upper Hessenberg triangular fuzzy matrix.

$$C_{LR} = \begin{pmatrix} (2,1,1)_{LR} & (4,2,2)_{LR} & (3,2,2)_{LR} \\ (0,0,0)_{LR} & (1,2,2)_{LR} & (4,1,1)_{LR} \\ (0,0,0)_{LR} & (0,0,0)_{LR} & (1,2,5)_{LR} \end{pmatrix} + \begin{pmatrix} (2,1,1)_{LR} & (4,2,2)_{LR} & (8,1,1)_{LR} \\ (1,2,2)_{LR} & (3,2,2)_{LR} & (4,1,1)_{LR} \\ (0,0,0)_{LR} & (1,2,5)_{LR} & (2,1,1)_{LR} \end{pmatrix}$$

$$C_{LR} = \begin{pmatrix} (4,2,2)_{LR} & (8,4,4)_{LR} & (11,3,3)_{LR} \\ (1,2,2)_{LR} & (4,4,4)_{LR} & (8,2,2)_{LR} \\ (0,0,0)_{LR} & (1,2,5)_{LR} & (3,3,6)_{LR} \end{pmatrix}$$

Hence, C_{LR} is a Upper Hessenberg triangular fuzzy matrix of order n.

Hence, the property 5.1.2 is proved by the counter example as above.

Property 5.1.3

The difference of the lower triangular TFM and lower Hessenberg TFM of order n is a lower Hessenberg Triangular TFM of order n.

Proof: Similar procedure to follows, that justifies the proposed illustration as above

Property 5.1.4

The difference of the upper triangular TFM and upper Hessenberg TFM of order n is a upper Hessenberg TFM of order n.

Proof: Similar procedure to follows, that justifies the proposed illustration as above

Property 5.1.5:

The Product of the Lower triangular TFM and Lower Hessenberg TFM of order n is a Lower Hessenberg TFM of order n.

Proof:

Let $A = (\tilde{a}_{ij})$ be the lower triangular TFM and

Let $B = (\tilde{b}_{ij})$ be the lower Hessenberg TFM.

consider a matrix

$A = \begin{pmatrix} (1,2,3) & (0,0,0) & (0,0,0) \\ (1,3,5) & (3,5,7) & (0,0,0) \\ (2,3,4) & (3,5,7) & (1,2,3) \end{pmatrix}$ be a Lower triangular TFM.

$B = \begin{pmatrix} (2,3,4) & (3,5,7) & (0,0,0) \\ (2,4,6) & (1,3,5) & (1,2,3) \\ (4,6,8) & (4,5,6) & (2,3,4) \end{pmatrix}$ be a Lower Hessenberg TFM.

Now, We have to transform the matrices into LR type representation by the definition (2.6.1) then we get matrices are

$$A_{LR} = \begin{pmatrix} (2,1,1)_{LR} & (0,0,0)_{LR} & (0,0,0)_{LR} \\ (3,2,2)_{LR} & (5,2,2)_{LR} & (0,0,0)_{LR} \\ (3,1,1)_{LR} & (5,2,2)_{LR} & (2,1,1)_{LR} \end{pmatrix}$$

$$B_{LR} = \begin{pmatrix} (3,1,1)_{LR} & (5,2,2)_{LR} & (0,0,0)_{LR} \\ (4,2,2)_{LR} & (3,2,2)_{LR} & (2,1,1)_{LR} \\ (6,2,2)_{LR} & (5,1,1)_{LR} & (3,1,1)_{LR} \end{pmatrix}$$

$$C_{LR} = A_{LR}.B_{LR}$$

$$= \begin{pmatrix} (2,1,1)_{LR} & (0,0,0)_{LR} & (0,0,0)_{LR} \\ (3,2,2)_{LR} & (5,2,2)_{LR} & (0,0,0)_{LR} \\ (3,1,1)_{LR} & (5,2,2)_{LR} & (2,1,1)_{LR} \end{pmatrix} . \begin{pmatrix} (3,1,1)_{LR} & (5,2,2)_{LR} & (0,0,0)_{LR} \\ (4,2,2)_{LR} & (3,2,2)_{LR} & (2,1,1)_{LR} \\ (6,2,2)_{LR} & (5,1,1)_{LR} & (3,1,1)_{LR} \end{pmatrix}$$

$$= \begin{pmatrix} (6,3,3)_{LR} & (10,5,5)_{LR} & (0,0,0)_{LR} \\ (29,14,14)_{LR} & (30,16,16)_{LR} & (10,4,4)_{LR} \\ (41,17,17)_{LR} & (40,16,16)_{LR} & (16,7,7)_{LR} \end{pmatrix}$$

Hence, C_{LR} is a Lower Hessenberg triangular fuzzy matrix of order n.

Hence, the property 5.1.5 is proved by the counter example as above.

Property 5.1.6

The Product of an upper Triangular TFM and upper Hessenburg TFM of order n is a upper Hessenbarg TFM of order n.

Proof:

Let $A = (\tilde{a}_{ij})$ be the Upper triangular TFM and

Let $B = (\tilde{b}_{ij})$ be the Upper Hessenberg TFM.

consider a matrix

$$A = \begin{pmatrix} (1,2,3) & (1,3,5) & (2,3,4) \\ (0,0,0) & (3,5,7) & (3,5,7) \\ (0,0,0) & (0,0,0) & (1,2,3) \end{pmatrix} \text{ be a upper triangular TFM.}$$

$$B = \begin{pmatrix} (2,3,4) & (3,5,7) & (4,6,8) \\ (2,4,6) & (1,3,5) & (1,2,3) \\ (0,0,0) & (4,5,6) & (2,3,4) \end{pmatrix} \text{ be a upper Hessenberg TFM.}$$

Now, We have to transform the matrices into LR type representation by the definition (2.6.1) then we get matrices are

$$A_{LR} = \begin{pmatrix} (2,1,1)_{LR} & (3,2,2)_{LR} & (3,1,1)_{LR} \\ (0,0,0)_{LR} & (5,2,2)_{LR} & (5,5,2)_{LR} \\ (0,0,0)_{LR} & (0,0,0)_{LR} & (2,1,1)_{LR} \end{pmatrix}$$

$$B_{LR} = \begin{pmatrix} (3,1,1)_{LR} & (5,2,2)_{LR} & (6,2,2)_{LR} \\ (4,2,2)_{LR} & (3,2,2)_{LR} & (2,1,1)_{LR} \\ (0,0,0)_{LR} & (5,1,1)_{LR} & (3,1,1)_{LR} \end{pmatrix}$$

$$C_{LR} = A_{LR}.B_{LR}$$

$$= \begin{pmatrix} (2,1,1)_{LR} & (3,2,2)_{LR} & (3,1,1)_{LR} \\ (0,0,0)_{LR} & (5,2,2)_{LR} & (5,5,2)_{LR} \\ (0,0,0)_{LR} & (0,0,0)_{LR} & (2,1,1)_{LR} \end{pmatrix} . \begin{pmatrix} (3,1,1)_{LR} & (5,2,2)_{LR} & (6,2,2)_{LR} \\ (4,2,2)_{LR} & (3,2,2)_{LR} & (2,1,1)_{LR} \\ (0,0,0)_{LR} & (5,1,1)_{LR} & (3,1,1)_{LR} \end{pmatrix}$$

$$C_{LR} = \begin{pmatrix} (18,11,11)_{LR} & (34,16,16)_{LR} & (27,13,13)_{LR} \\ (20,8,8)_{LR} & (40,16,16)_{LR} & (25,10,10)_{LR} \\ (0,0,0)_{LR} & (10,5,5)_{LR} & (6,3,3)_{LR} \end{pmatrix}$$

Hence, C_{LR} is a Upper Hessenberg triangular fuzzy matrix of order n.

Hence, the property 5.1.6 is proved by the counter example as above.

Property 5.1.7

The transpose of Sum of the lower triangular TFM and lower Hessenberg TFM of order n is an upper Hessenberg TFM of order n.

Proof:

Let $A = (\tilde{a}_{ij})$ be the lower triangular TFM and

Let $B = (\tilde{b}_{ij})$ be the lower Hessenberg TFM.

Consider a matrix.

$$A = \begin{pmatrix} (1,2,3) & (0,0,0) & (0,0,0) \\ (2,4,6) & (-1,1,3) & (0,0,0) \\ (1,3,5) & (3,4,5) & (-1,1,6) \end{pmatrix}$$ be a lower triangular TFM.

$$B = \begin{pmatrix} (1,2,3) & (2,4,6) & (0,0,0) \\ (-1,1,3) & (1,3,5) & (3,4,5) \\ (7,8,9) & (-1,1,6) & (1,2,3) \end{pmatrix}$$ be a lower Hessenberg TFM.

Now we have to transform the matrices into LR type representation by the definition (2,6,1) then we get matrices are

$$A_{LR} = \begin{pmatrix} (2,1,1)_{LR} & (0,0,0)_{LR} & (0,0,0)_{LR} \\ (4,2,2)_{LR} & (1,2,2)_{LR} & (0,0,0)_{LR} \\ (3,2,2)_{LR} & (4,1,1)_{LR} & (1,2,5)_{LR} \end{pmatrix}$$

$$B_{LR} = \begin{pmatrix} (2,1,1)_{LR} & (4,2,2)_{LR} & (0,0,0)_{LR} \\ (1,2,2)_{LR} & (3,2,2)_{LR} & (4,1,1)_{LR} \\ (8,1,1)_{LR} & (1,2,5)_{LR} & (2,1,1)_{LR} \end{pmatrix}$$

Now, transpose of A_{LR} and B_{LR} is

$$A_{LR}^T = \begin{pmatrix} (2,1,1)_{LR} & (4,2,2)_{LR} & (3,2,2)_{LR} \\ (0,0,0)_{LR} & (1,2,2)_{LR} & (4,1,1)_{LR} \\ (0,0,0)_{LR} & (0,0,0)_{LR} & (1,2,5)_{LR} \end{pmatrix} B_{LR}^T = \begin{pmatrix} (2,1,1)_{LR} & (1,2,2)_{LR} & (8,1,1)_{LR} \\ (4,2,2)_{LR} & (3,2,2)_{LR} & (1,2,5)_{LR} \\ (0,0,0)_{LR} & (4,1,1)_{LR} & (2,1,1)_{LR} \end{pmatrix}$$

$A_{LR}^T + B_{LR}^T = C_{LR}^T$ then $(\tilde{a}_{ji})_{LR} + (\tilde{b}_{ji})_{LR} = (\tilde{c}_{ji})_{LR}$ we have to prove C_{LR}^T is the upper Hessenberg triangular fuzzy matrix.

$$C_{LR}^T = \begin{pmatrix} (2,1,1)_{LR} & (4,2,2)_{LR} & (3,2,2)_{LR} \\ (0,0,0)_{LR} & (1,2,2)_{LR} & (4,1,1)_{LR} \\ (0,0,0)_{LR} & (0,0,0)_{LR} & (1,2,5)_{LR} \end{pmatrix} + \begin{pmatrix} (2,1,1)_{LR} & (1,2,2)_{LR} & (8,1,1)_{LR} \\ (4,2,2)_{LR} & (3,2,2)_{LR} & (1,2,5)_{LR} \\ (0,0,0)_{LR} & (4,1,1)_{LR} & (2,1,1)_{LR} \end{pmatrix}$$

$$C_{LR}^T = \begin{pmatrix} (4,2,2)_{LR} & (5,4,4)_{LR} & (11,3,3)_{LR} \\ (4,2,2)_{LR} & (4,4,4)_{LR} & (5,3,6)_{LR} \\ (0,0,0)_{LR} & (4,1,1)_{LR} & (3,3,6)_{LR} \end{pmatrix}$$

Hence, C_{LR}^T is a upper Hessenberg triangular fuzzy matrix of order n.

Hence, the property 5.1.7 is proved by the counter example as above.

Property 5.1.8

The transpose of the Sum of an upper triangular TFM and upper Hessenberg TFM of order n is a lower Hessenberg TFM of order n.

Proof:

Let $A = (\tilde{a}_{ij})$ be the lower triangular TFM and

Let $B = (\tilde{b}_{ij})$ be the lower Hessenberg TFM.

consider a matrix

$$A = \begin{pmatrix} (1,2,3) & (1,3,5) & (-1,1,6) \\ (0,0,0) & (-1,1,3) & (1,2,3) \\ (0,0,0) & (0,0,0) & (-1,1,6) \end{pmatrix}$$ be a upper triangular TFM.

$B = \begin{pmatrix} (1,2,3) & (2,4,6) & (7,8,9) \\ (-1,1,3) & (1,3,5) & (3,4,5) \\ (0,0,0) & (-1,1,6) & (1,2,3) \end{pmatrix}$ be a upper Hessenberg TFM.

Now, We have to transform the matrices into LR type representation by the definition (2.6.1) then we get matrices are

$$A_{LR} = \begin{pmatrix} (2,1,1)_{LR} & (4,2,2)_{LR} & (3,2,2)_{LR} \\ (0,0,0)_{LR} & (1,2,2)_{LR} & (4,1,1)_{LR} \\ (0,0,0)_{LR} & (0,0,0)_{LR} & (1,2,5)_{LR} \end{pmatrix}$$

$$B_{LR} = \begin{pmatrix} (2,1,1)_{LR} & (4,2,2)_{LR} & (8,1,1)_{LR} \\ (1,2,2)_{LR} & (3,2,2)_{LR} & (4,1,1)_{LR} \\ (0,0,0)_{LR} & (1,2,5)_{LR} & (2,1,1)_{LR} \end{pmatrix}$$

Now Transpose of A_{LR} and B_{LR} is

$$A_{LR}^T = \begin{pmatrix} (2,1,1)_{LR} & (0,0,0)_{LR} & (0,0,0)_{LR} \\ (4,2,2)_{LR} & (1,2,2)_{LR} & (0,0,0)_{LR} \\ (3,2,2)_{LR} & (4,1,1)_{LR} & (1,2,5)_{LR} \end{pmatrix}$$

$$B_{LR}^T = \begin{pmatrix} (2,1,1)_{LR} & (1,2,2)_{LR} & (0,0,0)_{LR} \\ (4,2,2)_{LR} & (3,2,2)_{LR} & (1,2,5)_{LR} \\ (8,1,1)_{LR} & (4,1,1)_{LR} & (2,1,1)_{LR} \end{pmatrix}$$

$A_{LR}^T + B_{LR}^T = C_{LR}^T$ then $(\tilde{a}_{ji})_{LR} + (\tilde{b}_{ji})_{LR} = (\tilde{c}_{ji})_{LR}$ we have to prove C_{LR}^T is the lower Hessenberg triangular fuzzy matrix.

$$C_{LR}^T = \begin{pmatrix} (2,1,1)_{LR} & (0,0,0)_{LR} & (0,0,0)_{LR} \\ (4,2,2)_{LR} & (1,2,2)_{LR} & (0,0,0)_{LR} \\ (3,2,2)_{LR} & (4,1,1)_{LR} & (1,2,5)_{LR} \end{pmatrix} + \begin{pmatrix} (2,1,1)_{LR} & (1,2,2)_{LR} & (0,0,0)_{LR} \\ (4,2,2)_{LR} & (3,2,2)_{LR} & (1,2,5)_{LR} \\ (8,1,1)_{LR} & (4,1,1)_{LR} & (2,1,1)_{LR} \end{pmatrix}$$

$$C_{LR}^T = \begin{pmatrix} (4,2,2)_{LR} & (1,2,2)_{LR} & (0,0,0)_{LR} \\ (8,4,4)_{LR} & (4,4,4)_{LR} & (1,2,5)_{LR} \\ (11,3,3)_{LR} & (8,2,2)_{LR} & (3,3,6)_{LR} \end{pmatrix}$$

Hence, C_{LR}^T is a lower Hessenberg triangular fuzzy matrix of order n.

Hence, the property 8 is proved by the counter example as above.

Property 5.1.9

The transpose of difference of the lower triangular TFM and lower Hessenberg TFM of order n is a upper Hessenberg TFM of order n.

Proof: Similar procedure to follows, that justifies the proposed illustration as above

Property 5.1.10

The transpose of difference of the upper triangular TFM and upper Hessenberg TFM of order n is a lower Hessenberg TFM of order n.

Proof: Similar procedure to follows, that justifies the proposed illustration as above

Property 5.1.11:

The transpose of Product of the Lower triangular TFM and Lower Hessenberg TFM of order n is a upper Hessenberg TFM of order n.

Proof: Similar procedure to follows, that justifies the proposed illustration as above

Property 5.1.12

The transpose of Product of the upper triangular TFM and upper Hessenberg TFM of order n is a lower Hessenberg TFM of order n.

Proof: Similar procedure to follows, that justifies the proposed illustration as above

CONCLUSION

In this paper, we find a new matrix namely HTFM that is related to the TTFM. Those matrices are arises in the form of some new relevant properties are concluded. Using these proposed types of matrices are involving the notion like determinant of matrix, adjoint of matrix and inverse on matrix can be studied in future.

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