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RESEARCH ARTICLE

The Behaviour of Maximum Planarity of $\Gamma(Z_{3p^2})$ and Extending it to the Consistency of Planarity

M.MALATHI[1], N.SELVI[2]
[1]Department of Mathematics, Saradha Gangadharan College, Puducherry
malathisundar.maths@gmail.com
[2] P.G and Research Department of Mathematics, A.D.M. College for Women (Autonomous), Nagappattinam
drnselvi@gmail.com

M.MALATHI

ABSTRACT

Let R be a commutative ring with unity and let Z(R) be its set of zero divisors. The zero divisor graph of R denoted by $\Gamma(\mathrm{Z_n})$ is a graph which is undirected with vertices $\mathrm{Z(R)^* = Z(R) - \{0\}}$, the set of non- zero divisors of R, and for distinct $\mathrm{x, y \in Z(R)^*}$, the vertices x and y are adjacent iff xy = 0. In this paper we analyze the crucial point of behaviour of maximum planarity in rectilinear crossing number of some zero divisor graphs, on adding or removing the edges by which one can able to predict using this behaviour that how far the graph retains its consistency.

Keywords: Rectilinear Crossing number, planar graph, Zero Divisor Graph.
Subject Classification: 05C25, 05C10, 52C35.

Introduction

A graph which can be drawn in the plane in such a way that edges meet only at points corresponding to their common ends is called a *Planar* graph, and such a drawing is called a *Planar embedding* of the graph. Let *G* be a graph drawn in the plane with the requirement that the edges are line segments, no three vertices are collinear, and no three edges may intersect in a point, unless the point is a vertex. Such a drawing is said to be a *Rectilinear drawing* of *G*. The rectilinear crossing number of *G*, denoted by $\overline{cr}(G)$, is the fewest number of edge crossings attainable over all rectilinear drawings of *G* [3]. Any such a drawing is called optimal. The idea of a zero divisor graph of a commutative ring was introduced by I. Beck in [2]. The zero divisor graph is very useful to find the algebraic structures and properties of rings. We mainly focus on D. F. Anderson and P. S. Livingston's zero divisor graphs.[1]

Definition - 1: If *a* and *b* are two non-zero elements of a ring Z_n such that *a.b* = 0, then '*a*' and '*b*' are the ***Zero divisors*** of commutative ring Z_n.

Definition - 2: If a graph $G' = (V, E')$ is a ***maximum planar subgraph*** of a graph $G = (V, E)$ such that there is no planar subgraph $G'' = (V, E'')$ of G with $|E''| > |E'|$, then G' is called a maximum planar subgraph of G.

An important consideration in the topological design of a network is fault tolerance. That is the ability of the network to provide service even when it contains a faulty component or components. The behaviour of a network in the presence of a fault can be analyzed by determining the effect that removing an edge or a vertex from its underlying graph G has on the fault tolerance criterion. It is often of interest to know how the value of a graph parameter is affected when a small change is made in a graph, for instance vertex or edge removal and edge addition. In this connection, we define the following six classes. We use acronyms to denote the following classes of graphs. [7,8]

[C- represents Changing, U- Unchanging, V-Vertex, E- Edge, R- Removal, A- Addition, **R**- Rectilinear, **C**- Crossing]

$$(CVR\boldsymbol{RC}) \qquad \overline{cr}(\Gamma(Z_n) - v) \neq \overline{cr}\big(\Gamma(Z_n)\big), \forall\, v \in V\big(\Gamma(Z_n)\big)$$
$$(CER\boldsymbol{RC}) \qquad \overline{cr}(\Gamma(Z_n) - e) \neq \overline{cr}\big(\Gamma(Z_n)\big), \forall\, e \in E\big(\Gamma(Z_n)\big)$$
$$(CEA\boldsymbol{RC}) \qquad \overline{cr}(\Gamma(Z_n) + e) \neq \overline{cr}\big(\Gamma(Z_n)\big), \forall\, e \in E\big(\overline{\Gamma(Z_n)}\big)$$
$$(UVR\boldsymbol{RC}) \qquad \overline{cr}(\Gamma(Z_n) - v) = \overline{cr}\big(\Gamma(Z_n)\big), \forall\, v \in V\big(\Gamma(Z_n)\big)$$
$$(UER\boldsymbol{RC}) \qquad \overline{cr}(\Gamma(Z_n) - e) \neq \overline{cr}\big(\Gamma(Z_n)\big), \forall\, e \in E\big(\Gamma(Z_n)\big)$$
$$(UEA\boldsymbol{RC}) \qquad \overline{cr}(\Gamma(Z_n) + e) = \overline{cr}\big(\Gamma(Z_n)\big), \forall\, e \in E\big(\overline{\Gamma(Z_n)}\big)$$

These six postulates have been approached individually in the literature with other terminology and several related problems. Using the above changing and unchanging Rectilinear crossing number terminology has been proposed in this chapter. It is useful to partition the vertices of $\Gamma(Z_n)$ into three sets, according to how their removal affects the Rectilinear crossing number.

Let $V = V^0 \cup V^+ \cup V^-$ for,

$$V^0 = \{v \in V\big(\Gamma(Z_n)\big) : \overline{cr}(\Gamma(Z_n) - v) = \overline{cr}\big(\Gamma(Z_n)\big)\}$$
$$V^+ = \{v \in V\big(\Gamma(Z_n)\big) : \overline{cr}(\Gamma(Z_n) - v) > \overline{cr}\big(\Gamma(Z_n)\big)\}$$
$$V^- = \{v \in V\big(\Gamma(Z_n)\big) : \overline{cr}(\Gamma(Z_n) - v) < \overline{cr}\big(\Gamma(Z_n)\big)\}$$

Similarly, the edge set can be partitioned into,

$$E^0 = \{e \in E\big(\Gamma(Z_n)\big) : \overline{cr}(\Gamma(Z_n) - e) = \overline{cr}\big(\Gamma(Z_n)\big)\}$$
$$E^- = \{e \in E\big(\Gamma(Z_n)\big) : \overline{cr}(\Gamma(Z_n) - e) < \overline{cr}\big(\Gamma(Z_n)\big)\}$$

Therefore any edge set in $\Gamma(Z_n)$ can be written as $E = E^0 \cup E^-$

Similarly we can define another new edge set E^+ such that

$$E^+ = \{e \in E\big(\overline{\Gamma(Z_n)}\big) : \overline{cr}(\Gamma(Z_n) + e) > \overline{cr}\big(\Gamma(Z_n)\big)\}$$

Detection of planarity in any graph:

Property 1: By Euler's Theorem, "A graph is planar iff 2e$\geq 3f$, where e is the edge and n is the number of vertices." We know that $\mathrm{e} - \mathrm{n} + 2 = \mathrm{f}$, and $e \leq 3n + 6$.

Property 2: By Kuratowski's theorem, a graph is planar iff it doesnot have any subgraph of K_5 or $K_{3,3}$.

Observations of some general factors involving the change of graph into a planar graph:

The following are some techniques which is suggested for the possibility of the planarity in any graph.

(i) The graph may have a vertex with highest degree, that is maximum clique which is involved in maximum crossings, that may affect the planarity. So ***removal of any vertex with maximum clique*** in any graph or else, removal of the edges incident with that vertex, can take us to make a minimum planarity.

(ii) Any connected graph may contain one or many cycles. ***Removal of the vertex or the edges incident with those vertices forming the cycle*** of maximum length and involving in maximum number of crossings may help us to take the graph into planarity.

(iii) ***Removal of all the edges*** involving in crossings can finally lead us a way for the maximum planarity.

Consistency of Rectilinear Crossing Number of $\Gamma(Z_{3p^2})$:

In this section we evaluate the consistency of bipartite zero- divisor graph especially for $\Gamma(Z_{3p^2})$. From [4,5,6] we know that the complete graph $\Gamma(Z_{3p^2})$is planar when p = 2, 3. So we proceed by finding the consistency for $p \geq 5$.

Theorem -1: If $p \geq 5$ are prime numbers in $\Gamma(Z_{3p^2})$, then,

(i)There exist $v \in V(\Gamma(Z_{3p^2}))$ s.t, $\overline{cr}(\Gamma(Z_{3p^2}) - v) = \overline{cr}(\Gamma(Z_{3p^2})), then\ \Gamma(Z_{3p^2}) \in UVR\boldsymbol{RC}.$

(ii)There exist $v \in V(\Gamma(Z_{3p^2}))$ s.t, $\overline{cr}(\Gamma(Z_{3p^2}) - v) < \overline{cr}(\Gamma(Z_{3p^2}))$, $then\ \Gamma(Z_{3p^2}) \in CVR\boldsymbol{RC}.$

(iii)There exist $e \in E(\Gamma(Z_{3p^2}))$ s.t, $\overline{cr}(\Gamma(Z_{3p^2}) - e) = \overline{cr}(\Gamma(Z_{3p^2}))$,$then\ \Gamma(Z_{3p^2}) \in UER\boldsymbol{RC}.$

(iv)There exist $e \in E(\Gamma(Z_{3p^2}))$ s.t, $\overline{cr}(\Gamma(Z_{3p^2}) - e) < \overline{cr}(\Gamma(Z_{3p^2}))$,$then\ \Gamma(Z_{3p^2}) \in CER\boldsymbol{RC}.$

(v)There exist $e \in E(\overline{\Gamma(Z_{3p^2})})$ s.t, $\overline{cr}(\Gamma(Z_{3p^2}) + e) \geq \overline{cr}(\Gamma(Z_{3p^2}))$,$then\ \Gamma(Z_{3p^2}) \in CEA\boldsymbol{RC}.$

(vi) For any graph $\Gamma(Z_{3p^2})$, $V\left(\Gamma(Z_{3p^2})\right) = V^0 \cup V^-$.

(vii) For any graph $\Gamma(Z_{3p^2})$, $E\left(\Gamma(Z_{3p^2})\right) = E^0 \cup E^-$.

Proof: The vertex set of $\Gamma(Z_{3p^2})$ is

$$V\left(\Gamma(Z_{3p^2})\right) = \{3,3,3,..3(p^2-1), p, 2p, .., p(3p-1)\}$$

$$\left|V\left(\Gamma(Z_{3p^2})\right)\right| = p^2 + 2p - 1$$

The vertex set of $\Gamma(Z_{3p^2})$ can be partitioned into four vertex subsets namely, V_1, V_2, V_3, V_4

Let $V_1 = \{p^2, 2p^2\}$, $V_2 = \{3p, 6p, 9p, \dots, 3p(p-1)\}$, $V_3 = \{p, 2p, 4p, \dots, p(3p-1)\}$

And $V_4 = \{3,6,9, \dots, 3(p^2-1)\}$.

The drawing of Rectilinear crossing of $\Gamma(Z_{3p^2})$ is same as in theorem [5], by imbedding complete graph K'_{p+1} inside the complete bipartite graph $K_{p-1,2(p-1)}$, where $E(K'_{p+1})$ is,

$$E(K'_{p+1}) = \{(p^2, 2p^2), \ (p^2, 3p(p-1)), (p^2, 3p), \dots (2p^2, 3p(p-1))\} - (p^2, 2p^2)$$

As all the vertices of V_2 are adjacent to each and every vertices of V_3, which forms a complete bipartite graph $K_{2,p(p-1)}$.

Case (i): Let v be any vertex in V_4. From the drawing of $\overline{cr}\left(\Gamma(Z_{3p^2})\right)$, it is clear that the two vertices p^2 and $2p^2$ forms a complete bipartite subgraph $H = K_{2,p(p-1)}$ of $\Gamma(Z_{3p^2})$.

Suppose p=5, then we get $H = K_{2,20}$. Therefore, by Euler's Theorem the condition $e \leq 3n + 6$ should be satisfied. Since, $(V(H)) = 22, n(E(H)) = 40\ \ implies\ 40 \leq 3(22) + 6 = 72$. Therefore the condition $e \leq 3n + 6$ is satisfied, which implies that $H = K_{2,p(p-1)}$ is a planar subgraph of $\Gamma(Z_{3p^2})$. Therefore the removal of p(p-1) vertices of V_4 from $\overline{cr}\left(\Gamma(Z_{3p^2})\right)$

Doesnot affect the rectilinear crossing number of $\Gamma(Z_{3p^2})$. Therefore $\overline{cr}(\Gamma(Z_{3p^2}) - v) = \overline{cr}(\Gamma(Z_{3p^2}))$,. which implies $\Gamma(Z_{3p^2}) \in UVR\boldsymbol{RC}$.

Case (ii): As the centre point of $\Gamma(Z_{3p^2})$ are p^2 and $2p^2$ in , V_1 which involves maximum number of crossings in $\overline{cr}(\Gamma(Z_{3p^2}))$. So removal of any vertex in V_1 gives drastic changes in the rectilinear crossing number. Therefore $\overline{cr}(\Gamma(Z_{3p^2}) - v) < \overline{cr}(\Gamma(Z_{3p^2}))$, $\forall\, \boldsymbol{v} \in V_1$ which implies $\Gamma(Z_{3p^2}) \in CVR\boldsymbol{RC}$.

Case (iii): Since using case (i), the removal of the vertex in the subgraph of $\overline{cr}(K_{2,p(p-1)})$ induced within $\overline{cr}(\Gamma(Z_{3p^2}))$ doesnot affect the rectilinear crossing number of $\overline{cr}(\Gamma(Z_{3p^2}))$. Similarly the removal of any edge from the crossings $\overline{cr}(K_{2,p(p-1)})$ doesnot affect the $\overline{cr}(\Gamma(Z_{3p^2}))$. Similarly from the figure it is clear that the lower boundary of the complete graph K'_{p+1} has edges, which doesnot involve in crossing of $\overline{cr}(\Gamma(Z_{3p^2}))$. So the removal of any edge $e \in E(K'_{p+1})$ from the lower boundary doesnot change any crossings. Therefore, $\overline{cr}(\Gamma(Z_{3p^2}) - e) = \overline{cr}(\Gamma(Z_{3p^2}))$, where $e \in E(K'_{p+1}) \subseteq \overline{cr}(\Gamma(Z_{3p^2}))$ Then, $\Gamma(Z_{3p^2}) \in UER\boldsymbol{RC}$.

Case (iv): From case (ii), we know that the centre point p^2 and $2p^2$ of $\Gamma(Z_{3p^2})$ in V_1 contributes maximum number of crossings in $\overline{cr}(\Gamma(Z_{3p^2}))$.Also, any vertex in K'_{p+1} has a maximum clique, which is an induced subgraph of $\Gamma(Z_{3p^2})$. Therefore removal of the edge other than the edge in lower boundary from $\overline{cr}(K'_{p+1})$ in $\overline{cr}(\Gamma(Z_{3p^2}))$, decreases the rectilinear crossing number of $\Gamma(Z_{3p^2})$. Therefore, $\overline{cr}(\Gamma(Z_{3p^2}) - e) < \overline{cr}(\Gamma(Z_{3p^2}))$, where $e \in E(K'_{p+1}) \subseteq \overline{cr}(\Gamma(Z_{3p^2}))$ where 'e' is an edge other than the edge in lower boundary. Then, $\Gamma(Z_{3p^2}) \in CER\boldsymbol{RC}$

Case (v): Let 'e' be an edge in $\overline{\Gamma(Z_{3p^2})}$. Then addition of one edge between p^2 and $2p^2$ in V_1, doesnot affect the crossing. Also addition of an edge between two consecutive vertices in V_3, doesnot increase the rectilinear crossing of $\Gamma(Z_{3p^2})$. Therefore any edge $e \in E\overline{(\Gamma(Z_{3p^2}))}$. In particular, $e \in V_1$ or V_3, such that,

$\overline{cr}(\Gamma(Z_{3p^2}) + e) = \overline{cr}(\Gamma(Z_{3p^2}))$, $\forall e \in V_1$ or V_3............(1.1)

Then $\Gamma(Z_{3p^2}) \in UEA\boldsymbol{RC}$.

Otherwise addition of an edge between non-consecutive vertices of either V_3 or V_4 from $\overline{cr}(K_{p-1,2(p-1)})$ or $\overline{cr}(K_{2,p(p-1)})$ respectively or else crossing between V_3 and V_4 itself leads to drastic changes in the rectilinear crossing number of $\Gamma(Z_{3p^2})$. Therefore for any edge $e \in E\overline{(\Gamma(Z_{3p^2}))}$, in particular, $e \in V_3$ or V_4 or $V_3 \cup V_4$, such that,

$\overline{cr}(\Gamma(Z_{3p^2}) + e) > \overline{cr}(\Gamma(Z_{3p^2}))$, $\forall e \in V_1$ or V_3 or $V_1 \cup V_3$ ....(1.2)

Then $\Gamma(Z_{3p^2}) \in CEA\boldsymbol{RC}$.

Using (1.1) and (1.2), we get,

$\overline{cr}(\Gamma(Z_{3p^2}) + e) \geq \overline{cr}(\Gamma(Z_{3p^2}))$, $\forall e \in V_1$ or V_3 or $V_1 \cup V_3$

Case (vi): Using case (i) and case (ii), removal of any vertex in $\Gamma(Z_{3p^2})$ either decreases or increases the rectilinear crossing number of $\Gamma(Z_{3p^2})$. Hence every vertex in $\Gamma(Z_{3p^2})$ belongs to either V^0 if , $\overline{cr}(\Gamma(Z_{3p^2}) - v) = \overline{cr}(\Gamma(Z_{3p^2}))$, or in V^- if , $\overline{cr}(\Gamma(Z_{3p^2}) - v) < \overline{cr}(\Gamma(Z_{3p^2}))$. Therefore for any graph $\Gamma(Z_{3p^2})$, $V\left(\Gamma(Z_{3p^2})\right) = V^0 \cup V^-$.

Case (vii): Using case (iii) and case (iv), removal of any edge in $\Gamma(Z_{3p^2})$ will either decreases or increases the rectilinear crossing number of $\Gamma(Z_{3p^2})$. Clearly, an edge 'e' in $\Gamma(Z_{3p^2})$ is in either E^0, if $\overline{cr}(\Gamma(Z_{3p^2}) - e) = \overline{cr}(\Gamma(Z_{3p^2}))$, or in E^- if , $\overline{cr}(\Gamma(Z_{3p^2}) - e) < \overline{cr}(\Gamma(Z_{3p^2}))$. Therefore for any graph $\Gamma(Z_{3p^2})$, $E\left(\Gamma(Z_{3p^2})\right) = E^0 \cup E^-$.

Theorem -2: For any graph $\Gamma(Z_{3p^2})$, $p > 3$, the consistency of Rectilinear crossing of $\overline{cr}(\Gamma(Z_{3p^2}))$,
(i) when removing the edges, the edge planarity is,

$$P\left[E\left(\Gamma(Z_{3p^2})\right)\right] = \{2(p-1)^2 - [2(p-1)^2 - 5(p-1) + 2]\} - \left\{\frac{(p+1)d-2}{2} - \sum_{n=2}^{d-1} n\right\}$$

(ii) When removing the edges involved in crossings is, $P[\overline{cr}(\Gamma(Z_{3p^2}))]$

$$= \left\{\overline{cr}\left(\frac{K_{p-1,4(p-1)}}{2} -\right) - 2\sum_{n=1}^{2p-3} n\right\} + \frac{(p+1)d}{24}(3(p+1)d - 6d - 2d^2 + 2)$$

$$-\left[1\sum_{n=1}^{d-2} n + 2\sum_{n=1}^{d-3} n + .. + (d-2)\sum_{n=1}^{d(d-1)} n\right]$$

Proof: The vertex set of $\Gamma(Z_{3p^2})$, is $V\left(\Gamma(Z_{3p^2})\right) = \{3,6,9,..3(p^2-1), p, 2p, ..p(3p-1)\}$,

Then, $\left|V\left(\Gamma(Z_{3p^2})\right)\right| = p^2 + 2p - 1$. The vertex set, $V\left(\Gamma(Z_{3p^2})\right)$ can be partitioned into, $V_1 = \{p^2, 2p^2\}, V_2 = \{3p, 6p, 9p, ..3p(p-1)\}$,

$$V_3 = \{p, 2p, 4p, ..p(3p-1)\}, V_4 = \{3,6,9,..3(p^2-1)\}$$

The Edge set, $E\left(\Gamma(Z_{3p^2})\right)$ is given by,

$$E\left(\Gamma(Z_{3p^2})\right) = \left\{\begin{array}{c} (p^2,3),..\left(p^2,3(p^2-1)\right),(2p^2,3),..\left(2p^2,3(p^2-1)\right), \\ (p^2,3p),..\left(p^2,3p(p-1)\right),(2p^2,3p),..\left(2p^2,3p(p-1)\right), \\ (p,3p),..\left(p,3p(p-1)\right),(2p,3p),..\left(2p,3p(p-1)\right), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \\ (p(3p-1),3p),\dots\dots\dots\dots\dots\dots.(p(3p-1),3p(p-1)) \end{array}\right\}$$

The Rectilinear drawing of $\Gamma(Z_{3p^2})$ is followed from the theorem in [5], where we find the Rectilinear crossing of $\overline{cr}\ (E(K'_{p+1}))$. That is the complete graph, whose edge sets are,

$$\overline{cr}\ (E(K'_{p+1})) = \left\{(p^2,3p),..\left(p^2,3p(p-1)\right),(2p^2,3p),..\left(2p^2,3p(p-1)\right)\right\} - (p^2,2p^2).$$

Similarly placing $K_{2,p(p-1)}$, a complete bipartite graph, between the vertices p^2 and $2p^2$, becomes planar. But the vertices of V_3 are placed on or below the crescent. Because splitting the vertices and placing them on both on top and bottom of the crescent , as in finding the rectilinear crossing of $\overline{cr}(\Gamma(Z_{3p^2}))$, doesnot help in reducing the crossing for attaining the planarity. Therefore all the vertices of V_3 are placed below the crescent (for convenience), hence forming a $K_{p+1,2(p-1)}$ complete bipartite graph. Now we are left with finding the planarity of complete and complete bipartite graph, an induced subgraph of $\Gamma(Z_{3p^2})$.

I.To find the complete planarity of $E(K'_{p+1})$ by removal of edges:

The total number of edges in $E(K'_{p+1})$ are $\frac{(p+1)d}{2} - 1$, denoted by n[$E(K'_{p+1})$]. Now removing the edges one by one from each vertex , we obtain a planar graph. We observe that drawing of all the edges incident with 3p doesnot disturbs the planarity. But the edges which are incident from the remaining vertices, disturbs the planarity of $E(K'_{p+1})$. That is the removal of $(d-1)$ edges incident with 6p, $(d-2)$ edges incident with p^2, $(d-3)$ edges incident with $2p^2$, $(d-4)$ edges incident with 9p, and continuing this process until removal of 1 edge from $3p(p-2)$. Therefore summing up we get removal of ,$\sum_{n=1}^{d-1} n = 1 + 2 + .. + (d-2) + (d-1)$ edges makes the subgraph $E(K'_{p+1})$ planar. Since there is no edge between p^2 and $2p^2$, subtracting that one edge we get, $\sum_{n=2}^{d-1} n - 1$ edges to be removed, denoted by, $n[E_1(K'_{p+1})]$. Therefore planarity of $E(K'_{p+1})$ by removal of edge is given by,

$P\left[E\left(K'_{p+1}\right)\right] = n\left[E\left(K'_{p+1}\right) - n\left[E_1\left(K'_{p+1}\right)\right]\right]$

$$= \left[\frac{(p+1)d}{2} - 1\right] - \sum_{n=1}^{d-1} n - 1 \quad = \frac{(p+1)d - 2}{2} - \sum_{n=1}^{d-1} n - 1$$

Where $d = (p+1) - 1 = p$

II.To find the complete planarity of $\overline{cr}[E(K'_{p+1})]$ by removal of edges:

The total number of edges crossings, $n\left[\overline{cr}\left(E\left(K'_{p+1}\right)\right)\right]$ is given by,

$$n\left[\overline{cr}\left(E\left(K'_{p+1}\right)\right)\right] = \frac{(p+1)d}{24}\left(3(p+1)d - 6d - 2d^2 + 2\right)$$

Now removing all the edges involved in crossings, we can make the edge crossing zero. That is removing all the edges incident with the vertex 6p, $\sum_{n=1}^{d-2} n$ crossings is removed. Similarly, removal of the edges incident with vertex p^2 we get $2\sum_{n=1}^{d-3} n$ crossings removed. Continuing this process for $\{2p^2, \dots 3p(p-3)\}$, finally we reach at removing $(d-2)\sum_{n=1}^{d-(d-1)} n$ crossings removed from the n vertex $3p(p-3)$. Summing up we get,

$$1\sum_{n=1}^{d-2} n + 2\sum_{n=1}^{d-3} n + \ldots + (d-2)\sum_{n=1}^{d-(d-1)} n$$

Crossings removed and is denoted by $P\left[\overline{cr}\left(E\left(K'_{p+1}\right)\right)\right] = n\left[\overline{cr}\left(E\left(K'_{p+1}\right)\right)\right] - n\left[\overline{cr}\left(E_1\left(K'_{p+1}\right)\right)\right]$

$$= \frac{(p+1)d}{24}\left(3(p+1)d - 6d - 2d^2 + 2\right) - 1\sum_{n=1}^{d-2} n + 2\sum_{n=1}^{d-3} n + \ldots + (d-2)\sum_{n=1}^{d-(d-1)} n$$

III.To find the complete planarity of $K_{p+1,2(p-1)}$ by removal of edge crossings:

As all the $2(p-1)$ vertices in V_3 are placed at the bottom of the crescent vertically, the total number of edges incident with these $2(p-1)$ vertices are denoted by $n\left[E(K_{p+1,2(p-1)})\right]$ and is given by, $n\left[E(K_{p+1,2(p-1)})\right] = 2(p-1)^2$. The edges incident from the vertex ‘p’ to K'_{p+1} doesnot affect the planarity. Similarly the remaining vertices incident to the vertices $3p$ and $3p(p-1)$ that are placed on the two ends of the crescent also doesnot disturbs the planarity. Therefore removing the rest of the edges, a complete planar graph is obtained. The number of edges that are removed is denoted by $n\left[E_1(K_{p+1,2(p-1)})\right]$ and is given by, $n\left[E_1(K_{p+1,2(p-1)})\right] = 2(p-1)^2 - 5(p-1) + 2$

Therefore planarity obtained after removal of the edges from $E(K_{p+1,2(p-1)})$ is given by,

$P\left[E(K_{p+1,2(p-1)})\right] = n\left[E(K_{p+1,2(p-1)})\right]$- $n\left[E_1(K_{p+1,2(p-1)})\right]$

$$= 2(p-1)^2 - \left[2(p-1)^2 - 5(p-1) + 2\right] = 5(p-1) - 2 = 5p - 7$$

IV.To find the complete planarity of $\overline{cr}(K_{p+1,2(p-1)})$ by removal of edge crossings:

As the vertices in V_1 are non-adjacent to all the vertices of V_3, so modifying $\overline{cr}(K_{p+1,2(p-1)})$ as $\overline{cr}\left(\frac{K_{p-1,2(p-1)+2(p-1)}}{2}\right) = E\left(\overline{cr}\left(\frac{K_{p-1,4(p-1)}}{2}\right)\right)$. The total crossing is denoted by $n\left[E\left(\overline{cr}\left(\frac{K_{p-1,4(p-1)}}{2}\right)\right)\right]$ and is obtained by using Zarankewicz conjecture. That is,

$$n\left[E\left(\overline{cr}\left(\frac{K_{p-1,4(p-1)}}{2}\right)\right)\right] = \frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{4(p-1)}{2}\right\rfloor\left\lfloor\frac{4p-5}{2}\right\rfloor$$

Now removing the crossings from the vertex other than ‘p’, we obtain the planarity by removal of crossings, which is denoted by $n\left[E_1\left(\overline{cr}\left(\frac{K_{p-1,4(p-1)}}{2}\right)\right)\right]$. Removal of the edges incident to $6p, 9p, ..3p(p-2)$ from all the vertices in $V_3 - p$ we obtain the planarity which is given by,

$$n\left[E_1\left(\overline{cr}\left(\frac{K_{p-1,4(p-1)}}{2}\right)\right)\right] = 2\left(\sum_{n=1}^{\left\lfloor\frac{p}{3}\right\rfloor} n\right)\left(\sum_{n=1}^{2p-3} n\right)$$

Therefore planarity obtained after the removal of crossings is given by,

$$P\left[E\left(\overline{cr}\left(\frac{K_{p-1,4(p-1)}}{2}\right)\right)\right] = n\left[E\left(\overline{cr}\left(\frac{K_{p-1,4(p-1)}}{2}\right)\right)\right] - n\left[E_1\left(\overline{cr}\left(\frac{K_{p-1,4(p-1)}}{2}\right)\right)\right]$$

$$= \left\{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{4(p-1)}{2}\right\rfloor\left\lfloor\frac{4p-5}{2}\right\rfloor\right\} - \left\{2\sum_{n=1}^{\left\lfloor\frac{p}{3}\right\rfloor} n \sum_{n=1}^{2p-3} n\right\}$$

Case (i): Let $p = 5$

The vertex set of $\Gamma(Z_{75})$, is $V(\Gamma(Z_{75})) = \{3,6,9,..72,5,10,..60\}$,

Then, $|V(\Gamma(Z_{75}))| = 34$

The vertex set, $V(\Gamma(Z_{75}))$ can be partitioned into, $V_1 = \{25,50\}, V_2 = \{15,30,45,..,60\}$,

$$V_3 = \{5,10,20,35,40,55,65,70\}, V_4 = \{3,6,9,..72\}$$

The Edge set, $E(\Gamma(Z_{75}))$ is given by,

$$E(\Gamma(Z_{75})) = \left\{\begin{array}{c}(25,3),..(25,72),(50,3),..(50,72),\\(25,15),..(25,60),(50,15),..(50,60),\\(5,15),..(5,60),(10,15),..(10,60),\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.\\(70,15),\dots\dots\dots\dots\dots\dots.(70,60)\end{array}\right\}$$

Therefore $\overline{cr}\ (E(K_6')) = \{(25,15),..(25,60),(50,15),..(50,60)\} - (25,50)$.

I.To find the complete planarity of $E(K_6')$ by removal of edges:

Total number of edges in $E(K_6') = 14 = \frac{30}{2} - 1 = \frac{(p+1)d}{2} - 1$

The removal of edges in $E(K_6')$ are as follows:

Removal of $4 = (d-1)$ edges incident with $6p = 30$,

Removal of $3 = (d-2)$ edges incident with $p^2 = 25$,

Removal of $2 = (d-3)$ edges incident with $2p^2 = 50$,

Removal of $1 = (d-4)$ edges incident with $3p(p-2) = 45$,

Summing up and subtracting the edge (25,50) we get,

$$n[E(K_6')] = 9 = (1+2+3+4) - 1 = \sum_{n=1}^{4} n - 1 = \sum_{n=1}^{d-1} n - 1$$

Therefore planarity of $E(K_6')$ by removal of edges is,

$$P[E(K_6')] = n[E(K_6')] - n[E_1(K_6')] = 5 = 14 - 9 = \left(\frac{30}{2} - 1\right) - \left((1+2+3+4) - 1\right)$$

$$= \left[\frac{(p+1)d}{2} - 1\right] - \sum_{n=1}^{d-1} n - 1$$

II.To find the complete planarity of $\overline{cr}[E(K_6')]$ by removal of edges:

Total number of edge crossings is given by,

$$n[\overline{cr}[E(K_6')]] = 15 = \frac{6.5}{24}(3(6)(5) - 6(5) - 2(25) + 2)$$
$$= \frac{(p+1)d}{24}(3(p+1)(d) - 6(d) - 2(d^2) + 2)$$

The removal of crossings in $\overline{cr}[E(K_6')]$ are as follows:

Removal of $1(1+2+3) = 1\sum_{n=1}^{3} n = \sum_{n=1}^{d-2} n$ crossings incident with vertex $30 = 6p$

Removal of $2(1+2) = 2\sum_{n=1}^{2} n = 2\sum_{n=1}^{d-3} n$ crossings incident with vertex $25 = 6p^2$

Removal of $3(1) = 3\sum_{n=1}^{2} n = 2\sum_{n=1}^{d-3} n$ crossings incident with vertex $50 = 6p^2$

Summing up we get,

$$n[\overline{cr}[E_1(K_6')]] = 15 = 6 + 6 + 3 = 1(6) + 2(3) + 3 = 1(1+2+3) + 2(1+2) + 3(1)$$
$$1\sum_{n=1}^{d-2} n + 2\sum_{n=1}^{d-3} n + .. + (d-2)\sum_{n=1}^{d-(d-1)} n$$

Therefore planarity obtained after the removal of crossings is given by,

$$P[\overline{cr}(E(K_6'))] = n[\overline{cr}(E(K_6'))] - n[\overline{cr}(E_1(K_6'))] = 0 = 15 - 15$$

$$= \frac{6.5}{24}(3(6)(5) - 6(5) - 2(25) + 2) - [1(6) + 2(3) + 3]$$

$$= \frac{(p+1)d}{24}(3(p+1)d - 6d - 2d^2 + 2) - 1\sum_{n=1}^{d-2} n + 2\sum_{n=1}^{d-3} n + .. + (d-2)\sum_{n=1}^{d-(d-1)} n$$

III.To find the planarity of $K_{6,8}$ by removal of edge crossings:

$$n\big[E(K_{6,8})\big] = 32 = 2(p-1)^2$$
$$n\big[E_1(K_{6,8})\big] = 14 = 32 - 20 + 2 = 2(p-1)^2 - 5(p-1) + 2$$

Therefore planarity obtained after removal of the edges from $E(K_{6,8})$ is given by,

$$P\big[E\big(K_{6,8}\big)\big] = n\big[E\big(K_{6,8}\big)\big] - P\big[E_1\big(K_{6,8}\big)\big] = 18 = 32 - 14$$
$$= 2(5-1)^2 - \{2(5-1)^2 - 5(5-1) + 2\}$$
$$= 2(p-1)^2 - \{2(p-1)^2 - 5(p-1) + 2\} = 5p - 7$$

IV.To find the complete planarity of $\overline{cr}(K_{6,8})$ by removal of edge crossings:

$$n\left[E\left(\overline{cr}\left(\frac{K_{4,16)}}{2}\right)\right)\right] = \frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{4(p-1)}{2}\right\rfloor\left\lfloor\frac{4p-5}{2}\right\rfloor$$
$$= \frac{1}{2}\left\lfloor\frac{5-1}{2}\right\rfloor\left\lfloor\frac{5-2}{2}\right\rfloor\left\lfloor\frac{4.4}{2}\right\rfloor\left\lfloor\frac{15}{2}\right\rfloor$$
$$= \frac{1}{2}\left\lfloor\frac{4}{2}\right\rfloor\left\lfloor\frac{3}{2}\right\rfloor\left\lfloor\frac{16}{2}\right\rfloor\left\lfloor\frac{15}{2}\right\rfloor = \frac{1}{2} \times 2 \times 1 \times 8 \times 7 = 56$$
$$n\left[E_1\left(\overline{cr}\left(\frac{K_{4,16}}{2}\right)\right)\right] = 2\left(\sum_{n=1}^{\left\lfloor\frac{p}{3}\right\rfloor} n\right)\left(\sum_{n=1}^{2p-3} n\right) = 2\left(\sum_{n=1}^{1} n\right)\left(\sum_{n=1}^{7} n\right)$$
$$= 2(1)(1+2+3+4+5+6+7) = 2(28) = 56$$

Therefore Planarity obtained for $\Gamma(Z_{75})$ after the removal of crossings is given by,

$$P\left[E\left(\overline{cr}\left(\frac{K_{p-1,4(p-1)}}{2}\right)\right)\right] = P\left[E\left(\overline{cr}\left(\frac{K_{4,16)}}{2}\right)\right)\right]$$
$$= n\left[E\left(\overline{cr}\left(\frac{K_{4,16)}}{2}\right)\right)\right] - n\left[E_1\left(\overline{cr}\left(\frac{K_{4,16)}}{2}\right)\right)\right] = 0 = 56 - 56$$

$$=\left\{\frac{1}{2}\left\lfloor\frac{5-1}{2}\right\rfloor\left\lfloor\frac{5-2}{2}\right\rfloor\left\lfloor\frac{4(4)}{2}\right\rfloor\left\lfloor\frac{20-5}{2}\right\rfloor\right\}-2(1)(1+2+3+4+5+6+7)$$

$$=\left\{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{4(p-1)}{2}\right\rfloor\left\lfloor\frac{4p-5}{2}\right\rfloor\right\}-\left\{2\sum_{n=1}^{\left\lfloor\frac{p}{3}\right\rfloor}n\sum_{n=1}^{2p-3}n\right\}$$

The complete Planarity obtained for $\Gamma(Z_{75})$ after the removal of edges is given by,

$$P[E(\Gamma(Z_{75}))]=P[E(K_6')]+p\big[E\big(K_{6,8}\big)\big]=23=5+18=(14-9)+(32-14)$$

$$=\left(\frac{30}{2}-1\right)-\big((1+2+3+4)-1\big)+[2(5-1)^2-5(5-1)+2]$$

$$\left(\frac{(p+1)d}{2}-1\right)-\left(\sum_{n=1}^{d-1}n-1\right)+[2(p-1)^2-\{2(p-1)^2-5(p-1)+2\}]$$

The complete Planarity obtained for $\Gamma(Z_{75})$ after the removal of crossings is given by,

$$P\big[\overline{cr}\big(E(\,\Gamma(Z_{75})\,)\big)\big]=P\big[\overline{cr}\big(E(\,\Gamma(K_6')\,)\big)\big]+P\left[\overline{cr}\left(E\left(\frac{K_{4,16)}}{2}\right)\right)\right]$$

$$=0+0=(15-15)+(56-56)$$

$$=\frac{6\times 5}{24}(3(6)(5)-6(5)-2(25)+2)-(1(6)+2(3)+3)$$

$$+\frac{1}{2}\left\lfloor\frac{5-1}{2}\right\rfloor\left\lfloor\frac{5-2}{2}\right\rfloor\left\lfloor\frac{4(5-1)}{2}\right\rfloor\left\lfloor\frac{4(5)-5}{2}\right\rfloor-2(1)[1+2+3+4+5+6+7]$$

$$=\frac{(p+1)d}{24}(3(p+1)d-6d-2d^2+2)-1\sum_{n=1}^{d-2}n+2\sum_{n=1}^{d-3}n+..+(d-2)\sum_{n=1}^{d-(d-1)}n$$

$$+\left\{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{4(p-1)}{2}\right\rfloor\left\lfloor\frac{4p-5}{2}\right\rfloor\right\}-\left\{2\sum_{n=1}^{\left\lfloor\frac{p}{3}\right\rfloor}n\sum_{n=1}^{2p-3}n\right\}$$

Case (ii): Let $p=7$

The vertex set of $\Gamma(Z_{147})$, is $V\big(\Gamma(Z_{147})\big)=\{3,6,9,..144,5,10,..140\}$, Then, $\big|V\big(\Gamma(Z_{147})\big)\big|=62$

The vertex set, $V\big(\Gamma(Z_{147})\big)$ can be partitioned into, $V_1=\{49,98\}, V_2=\{21,42,63,..,126\}$,

$$V_3=\{5,10,20,35,..,140\}, V_4=\{3,6,9,..144\}$$

The Edge set, $E\big(\Gamma(Z_{147})\big)$ is given by,

$$E\big(\Gamma(Z_{75})\big)=\begin{Bmatrix}(49,3),..(49,144),(98,3),..(98,144),\\(49,21),..(49,126),(98,21),..(98,126),\\(5,21),..(5,126),(10,21),..(10,126),\\\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.\\(140,21),\dots\dots\dots\dots\dots\dots.(140,126)\end{Bmatrix}$$

Therefore $\overline{cr}\ (E(K_8'))=\{(49,21),..(49,126),(98,21),..(98,126)\}-(49,98)$.

I.To find the complete planarity of $E(K_8')$ by removal of edges:

Total number of edges in $E(K_8')=27=\frac{56}{2}-1=\frac{(p+1)d}{2}-1$

The removal of edges in $E(K_8')$ are as follows:

Removal of $6=(d-1)$ edges incident with $6p=42$,

Removal of $5=(d-2)$ edges incident with $9p=63$,

Removal of $4=(d-3)$ edges incident with $p^2=49$,

Removal of $3=(d-4)$ edges incident with $2p^2=98$,

Removal of $2=(d-5)$ edges incident with $3p(p-3)=44$,

Removal of $1=(d-6)$ edges incident with $3p(p-2)=105$, Summing up and subtracting the edge (49,98) we get,

$$n[E_1(K_8')] = 20 = (1+2+3+4+5+6) - 1 = \sum_{n=1}^{6} n - 1 = \sum_{n=1}^{d-1} n - 1$$

Therefore planarity of $E(K_8')$ by removal of edges is,

$$P[E(K_8')] = n[E(K_8')] - n[E_1(K_8')] = 7 = 27 - 20 = \left(\frac{56}{2} - 1\right) - \left((1+2+3+4+5+6) - 1\right)$$

$$= \left[\frac{(p+1)d}{2} - 1\right] - \sum_{n=1}^{d-1} n - 1$$

II.To find the complete planarity of $\overline{cr}[E(K_8')]$ by removal of edges:

Total number of edge crossings is given by,

$$n[\overline{cr}[E(K_8')]] = 70 = \frac{8.7}{24}(3(8)(7) - 6(7) - 2(49) + 2)$$

$$= \frac{(p+1)d}{24}(3(p+1)(d) - 6(d) - 2(d^2) + 2)$$

The removal of crossings in $\overline{cr}[E(K_8')]$ are as follows:

Removal of $1(1+2+3+4+5) = 1\sum_{n=1}^{5} n = 1\sum_{n=1}^{d-2} n$ crossings incident with vertex $42 = 6p$

Removal of $2(1+2+3+4) = 2\sum_{n=1}^{4} n = 2\sum_{n=1}^{d-3} n$ crossings incident with vertex $63 = 9p$

Removal of $3(1+2+3) = 2\sum_{n=1}^{3} n = 2\sum_{n=1}^{d-4} n$ crossings incident with vertex $49 = p^2$

Removal of $4(1+2) = 2\sum_{n=1}^{2} n = 2\sum_{n=1}^{d-5} n$ crossings incident with vertex $98 = 2p^2$

Removal of $5(1) = 5\sum_{n=1}^{1} n = 3\sum_{n=1}^{d-(d-1)} n$ crossings incident with vertex 105= $3p(p-2)$

Summing up we get,

$$n[\overline{cr}[E(K_6')]] = 70 = 15 + 20 + 18 + 12 + 5 = 1(15) + 2(10) + 3(6) + 4(3) + 5$$

$$= 1(1+2+3+4+5) + 2(1+2+3+4) + 3(1+2+3) + 4(1+2) + 5(1)$$

$$= 1\sum_{n=1}^{d-2} n + 2\sum_{n=1}^{d-3} n + .. + 5\sum_{n=1}^{d-(d-1)} n$$

Therefore planarity obtained after the removal of crossings is given by,

$$P[\overline{cr}(E(K_8'))] = n[\overline{cr}(E(K_8'))] - n[\overline{cr}(E_1(K_8'))] = 0 = 70 - 70$$

$$= \frac{8.7}{24}(3(8)(7) - 6(7) - 2(49) + 2) - [1(15) + 2(10) + 3(6) + 4(3) + 5(1)]$$

$$= \frac{(p+1)d}{24}(3(p+1)d - 6d - 2d^2 + 2) - 1\sum_{n=1}^{d-2} n + 2\sum_{n=1}^{d-3} n + .. + 5\sum_{n=1}^{d-(d-1)} n$$

III.To find the planarity of $K_{8,12}$ by removal of edge crossings:

$$n\left[E(K_{8,12})\right] = 72 = 36 \times 2 = 2(p-1)^2$$

$$n\left[E_1(K_{8,12})\right] = 44 = 2 \times 36 - 5 \times 6 + 2 = 2(p-1)^2 - 5(p-1) + 2$$

Therefore planarity obtained after removal of the edges from $E(K_{8,12})$ is given by,

$$P\left[E\left(K_{8,12}\right)\right] = n\left[E\left(K_{8,12}\right)\right] - P\left[E_1\left(K_{8,12}\right)\right] = 28 = 72 - 44$$

$$= 2(7-1)^2 - \{2(7-1)^2 - 5(7-1) + 2\}$$

$$= 2(p-1)^2 - \{2(p-1)^2 - 5(p-1) + 2\} = 5p - 7$$

IV.To find the complete planarity of $\overline{cr}\left(K_{8,12}\right)$ by removal of edge crossings:

$$n\left[E\left(\overline{cr}\left(\frac{\left(K_{6,24}\right)}{2}\right)\right)\right] = \frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{4(p-1)}{2}\right\rfloor\left\lfloor\frac{4p-5}{2}\right\rfloor$$

$$= \frac{1}{2}\left\lfloor\frac{7-1}{2}\right\rfloor\left\lfloor\frac{7-2}{2}\right\rfloor\left\lfloor\frac{4.6}{2}\right\rfloor\left\lfloor\frac{23}{2}\right\rfloor$$

$$=\frac{1}{2}\left\lfloor\frac{6}{2}\right\rfloor\left\lfloor\frac{5}{2}\right\rfloor\left\lfloor\frac{24}{2}\right\rfloor\left\lfloor\frac{23}{2}\right\rfloor=\frac{1}{2}\times 3\times 2\times 12\times 11=396$$

$$n\left[E_1\left(\overline{cr}\left(\frac{(K_{6,24})}{2}\right)\right)\right]=2\left(\sum_{n=1}^{\left\lfloor\frac{p}{3}\right\rfloor}n\right)\left(\sum_{n=1}^{2p-3}n\right)=2\left(\sum_{n=1}^{2}n\right)\left(\sum_{n=1}^{11}n\right)$$

$$=2(1+2)(1+2+3+4+..+11)=2(3)(66)=396$$

Therefore Planarity obtained for $\Gamma(Z_{75})$ after the removal of crossings is given by,

$$P\left[E\left(\overline{cr}\left(\frac{K_{p-1,4(p-1)}}{2}\right)\right)\right]=P\left[E\left(\overline{cr}\left(\frac{K_{6,24}}{2}\right)\right)\right]$$

$$=\ n\left[E\left(\overline{cr}\left(\frac{K_{6,24}}{2}\right)\right)\right]-n\left[E_1\left(\overline{cr}\left(\frac{K_{6,24}}{2}\right)\right)\right]=0=396-396$$

$$=\left\{\frac{1}{2}\left\lfloor\frac{7-1}{2}\right\rfloor\left\lfloor\frac{7-2}{2}\right\rfloor\left\lfloor\frac{4(6)}{2}\right\rfloor\left\lfloor\frac{23}{2}\right\rfloor\right\}-2(1+2)(1+2+3+4+..+11)$$

$$=\left\{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{4(p-1)}{2}\right\rfloor\left\lfloor\frac{4p-5}{2}\right\rfloor\right\}-\left\{2\sum_{n=1}^{\left\lfloor\frac{p}{3}\right\rfloor}n\sum_{n=1}^{2p-3}n\right\}$$

The complete Planarity obtained for $\Gamma(Z_{147})$ after the removal of edges is given by,

$$P[E(\Gamma(Z_{147}))]=P[E(K_8')]+p\big[E\big(K_{8,12}\big)\big]=35=7+28=(27-20)+(72-44)$$

$$=\left(\frac{56}{2}-1\right)-\big((1+2+3+4+5+6)-1\big)+[2(7-1)^2-5(7-1)+2]$$

$$=\left(\frac{(p+1)d}{2}-1\right)-\left(\sum_{n=1}^{d-1}n-1\right)+[2(p-1)^2-\{2(p-1)^2-5(p-1)+2\}]$$

The complete Planarity obtained for $\Gamma(Z_{147})$ after the removal of crossings is given by,

$$P\big[\overline{cr}\big(E(\,\Gamma(Z_{147})\,)\big)\big]=P\big[\overline{cr}\big(E(\,\Gamma(K_8')\,)\big)\big]+P\left[\overline{cr}\left(E\left(\frac{K_{6,24}}{2}\right)\right)\right]$$

$$=0+0=(70-70)+(396-396)$$

$$=\frac{8\times 7}{24}(3(8)(7)-6(7)-2(49)+2)-\big(1(15)+2(10)+3(6)+4(3)+5(2)\big)$$

$$+\frac{1}{2}\left\lfloor\frac{7-1}{2}\right\rfloor\left\lfloor\frac{7-2}{2}\right\rfloor\left\lfloor\frac{4(7-1)}{2}\right\rfloor\left\lfloor\frac{4(7)-5}{2}\right\rfloor-2(1)[1+2+3+..+11]$$

$$=\frac{(p+1)d}{24}(3(p+1)d-6d-2d^2+2)-1\sum_{n=1}^{d-2}n+2\sum_{n=1}^{d-3}n+..+(d-2)\sum_{n=1}^{d-(d-1)}n$$

$$+\left\{\frac{1}{2}\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{4(p-1)}{2}\right\rfloor\left\lfloor\frac{4p-5}{2}\right\rfloor\right\}-\left\{2\sum_{n=1}^{\left\lfloor\frac{p}{3}\right\rfloor}n\sum_{n=1}^{2p-3}n\right\}$$

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