**Vol.5.Issue.3.2017 (July-Sept)**
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**RESEARCH ARTICLE**

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

*A Peer Reviewed International Research Journal*

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

# PARTS OF A SET AND SUB-EXCEEDING FUNCTION : CODING AND DECODING

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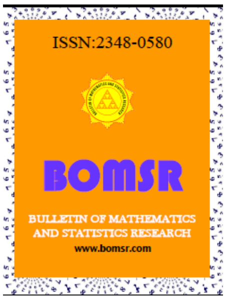


**ABSTRACT**

Let $n$ be an integer such that $n > 0$ and $\Omega$ set with cardinality $n-1$. The goal of this paper is to present a new way to encoding all subset of the set $\Omega$ by sub-exceeding function on $n$.

After giving some proprieties of the set $\mathcal{F}_n$ of sub-exceeding functions on $[n]$, we present a particular subset which we denote by $\mathcal{H}_n$ such that his cardinality is $2^{n-1}$.

Since this subset $\mathcal{H}_n$ has the same number of elements as the set of parts of $\Omega$ denoted $\mathcal{P}(\Omega)$, so we can construct a bijective map denoted $\psi$ from the set $\mathcal{H}_n$ to $\mathcal{P}(\Omega)$.

As results, given an arbitrary subset of $\Omega$, we can encode it by using a sub-exceeding function in $\mathcal{H}_n$. Reciprocally, if we have a sub-exceeding function in $\mathcal{H}_n$, we can decode it and find the subset of $\Omega$ that it represents. Moreover, we can also define in $\mathcal{H}_n$ the equivalent of the usual laws $\cup$, $\cap$ and $\overline{A}(A \in \Omega)$which operate in $\mathcal{P}(\Omega)$.

***Keywords***: Sub-exceeding function, Statistic permutation, Partition set

***2010 Mathematics Subject Classification***: 05A05; 05A18; 05A19; 18C35

# 1. Introduction

Let $n$ be an integer such that $n > 0$. For any map $f : [n] \rightarrow [n]$ such that $0 < f(i) \leq i$ for $i \in [n]$, D. Dumont and Viennot define this map by "sub-exceeding function on $[n]$". In this article, we denote by $\mathcal{F}_n$ the set of this sub-exceeding function. So

$$\mathcal{F}_n = \{ f : [n] \rightarrow [n] \mid 0 < f(i) \leq i \quad \forall i \in [n] \} \tag{1.1}$$

In 2001, Roberto Mantaci and Fanja Rakotondrajao [6] established a bijection between this set of sub-exceeding function on $[n]$ and the set of permutation $S_n$ . Following this results, we present a new way to encode all subset of any set $\Omega$ which is a set of $n-1$ objects.

This code can be done by processing a peculiar subset of $\mathcal{F}_n$ denoted by $\mathcal{H}_n$ which regroups all sub-exceeding functions $f$ such as the images $f(i)$ form an quasi-increasing sequence, i.e:

$$f(1) \leq f(2) \leq \cdots \leq f(n).$$

Here the set $\mathcal{H}_n$ can be written as

$$\mathcal{H}_n = \{ f \in \mathcal{F}_n \mid f(1) \leq f(2) \leq \cdots \leq f(n) \} \tag{1.2}$$

Given that an integer $n$ such that $n > 0$ and any set $\Omega$ with cardinality $n-1$, we can construct a bijection between $\mathcal{P}(\Omega)$ and the subset $\mathcal{H}_n$.

So, if we have a subset of $\Omega$, it is possible to encode this set using one and only one sub-exceeding function in $\mathcal{H}_n$ $(coding)$ and vice versa, given a sub-exceeding function in $\mathcal{H}_n$, we can find the subset of $\Omega$ that it represents *(Decoding)*.

Additionally, we can also define the corresponding in $\mathcal{H}_n$ of the operation $\cup$, $\cap$ and $\overline{A}(A \in \Omega)$ denoted by $\cup_{Rab}$, $\cap_{Rab}$ and $\overline{f}$ (the complement of $f$).

# 2 . Notation and preliminary

For any integer $n$ such that $n > 0$. We denote by

- $[n]$ : the set $\{1;\ 2; \ldots;\ n\}$;
- $\mathcal{P}(\Omega)$ : the set of all subset of $\Omega$ where $\Omega = \{\omega_1; \omega_2; \ldots; \omega_n\}$;
- $\mathcal{P}^k(\Omega)$ : the set of all parts of $\Omega$ which has $k$ elements;
- $S_n$ : the set of permutations on $[n]$.

Recall that a function $f$ said sub-exceeding function on $[n]$ if and only if for all $i \in [n]$ we have $f(i) \leq i$. So, $f$ can be represented by the word of $n$ letters $f(1)\, f(2) \ldots f\,(n)$. Thus we describe $f$ by his images i.e. $f = f(1)\, f(2) \ldots f\,(n)$ and we adopt in the whole continuation this annotation.

**Example 2.1.** $For\ n = 1;\ 2;\ 3$ we have:

$$\mathcal{F}_1 = \{1\}$$
$$\mathcal{F}_2 = \{11 ; 12\}$$
$$\mathcal{F}_3 = \{111 ; 112 ; 121 ; 122 ; 113 ; 123\}$$

## 2.1. Bijection between $\mathcal{F}_n$ And $S_n$

**Theorem 2.2.** Let $n$ be an integer such that $n > 0$, then the set $\mathcal{F}_n$ has cardinality $n!$ i.e.

$$Card\ \mathcal{F}_n = n!$$

***Proof .***

Let $f$ a sub-exceeding function in $\mathcal{F}_n$ where $n$ is an integer such that $n > 0$. So, we can write $f = f(1)\,f(2)\,\dots f(n)$ where $f(i) \leq i$ for all $i \in [n]$. Them, take $k$ an integer in $[n+1]$ and denote by $f'$ the function such that

$$f' = f(1)\,f(2)\dots f(n)\,k$$

Now, write by $\mathcal{F}'_n$ the set of this function $f'$ i.e.

$$\mathcal{F}'_n = \{\, f' = f(1)f(2)\dots f(n)\,k \mid f = f(1)f(2)\dots f(n) \in \mathcal{F}_n \; et \; k \in [n+1] \,\} \qquad (2.1)$$

Immediately, we see that $f' \in \mathcal{F}_{n+1}$ so,

$$\mathcal{F}'_n \subseteq \mathcal{F}_{n+1} \qquad (2.2)$$

and

$$Card\, \mathcal{F}'_n = (n+1) Card\; \mathcal{F}_n \qquad (2.3)$$

Let $f$ be a sub-exceeding function in $\mathcal{F}_{n+1}$ i.e. $f = f(1)f(2)\dots f(n)f(n+1)$ . From *(2.1)*, because $f(n+1) \in [n+1]$, we have $f \in \mathcal{F}'_n$ . Then

$$\mathcal{F}'_n \supseteq \mathcal{F}_{n+1} \qquad (2.4)$$

Consequently, from (2.2) and (2.4), we have

$$\mathcal{F}_{n+1} = \mathcal{F}'_n \qquad (2.5)$$

Finally, as $Card\; \mathcal{F}_1 = 1$, the equation (2.3) gives

$$Card\; \mathcal{F}_n = n! \qquad (2.6)$$

⊡

**Theorem 2.3** (Roberto Mantaci and Fanja Rakotondrajao)**.** Let $\phi$ be the map from $\mathcal{F}_n$ to $S_n$ defined by

$$\begin{array}{llll} \phi : & \mathcal{F}_n & \longrightarrow & S_n \\ & f & \longmapsto & \phi(f) = \big(n\,f(n)\big)\big(n-1\,f(n-1)\big)\dots\big(2\,f(2)\big)\big(1\,f(1)\big) = \sigma_f \end{array} \qquad (2.7)$$

Then $\phi$ is bijective.

Here $(i\;f(i))$ is the permutation which transforms $i$ into $f(i)$ and $f(i)$ into $i$. Similarly $\big(i\;f(i)\big)\big(j\;f(j)\big)$ is the compose of two permutations $(i\;f(i))$ *and* $(j\;f(j))$. So

$$\sigma_f = \big(n\,f(n)\big)\big(n-1\,f(n-1)\big)\dots\big(1\,f(1)\big) = \big(n\,f(n)\big)\circ\big(n-1\,f(n-1)\big)\circ\dots\circ\big(1\,f(1)\big) \qquad (2.8)$$

**Example 2.4.** For $n = 3$ and $f = 122$, we have $\phi(f) = (3\,2)(2)(1) = 132$ where the permutation $(i, i)$ is simplified by $(i)$.

***Proof of theorem 2.3 .***

Since the cardinality of $\mathcal{F}_n$ and $S_n$ are the same and equal to $n!$, we have to prove that $\phi$ is injective.

Let $f$ and $g$ *be* two sub-exceeding functions such that $\phi(f) = \phi(g)$ i.e. $\sigma_f = \sigma_g$. these give us

$$\big(n\,f(n)\big)\dots\big(2\,f(2)\big)\big(1\,f(1)\big) = \big(n\,g(n)\big)\dots\big(2\,g(2)\big)\big(1\,g(1)\big) \qquad (2.9)$$

By calculating $\sigma_f(n)$ and $\sigma_g(n)$, we have $\sigma_f(n) = f(n)$ and $\sigma_g(n) = g(n)$. Therefore from the equality in (2.9) we see that $f(n) = g(n)$.

Multiply both sides of the equation (2.9) by $(n\,f(n))$ at left and by $(n\,g(n))$ at right, thus we have a new equality of permutation in $S_{n-1}$:

$$\big(n\,f(n)\big)\big(n\,f(n)\big)\dots\big(2\,f(2)\big)\big(1\;f(1)\big) = \big(n\,g(n)\big)\big(n\,g(n)\big)\dots\big(2\,g(2)\big)\big(1\;g(1)\big) \qquad (2.10)$$

$$\Downarrow$$

$$S_{n-1} \ni (n-1\ \ f\,(n-1))\ldots(2\,f\,(2))(1\ \ f\,(1)) = (n-1\ \ g(n-1))\ldots(2\,g(2))(1\ \ g(1)) \in S_{n-1} \quad (2.11)$$

Write

$$\sigma'_f = (n-1\ \ f\,(n-1))\ldots(2\,f\,(2))(1\,f\,(1))$$

And

$$\sigma'_g = (n-1\ \ g\,(n-1))\ldots(2\ \ g(2))(1\ \ g\,(1)).$$

We have $\sigma'_f(n-1) = f(n-1)$ and $\sigma'_g(n-1) = g(n-1)$. Thus the equality in (2.11) gives

$$f(n-1) = g(n-1).$$

Continuing this method, we finally have that

$$f\,(n) = g(n),\ f(n-1) = g(n-1), \ldots, f\,(2) = g(2), f\,(1) = g(1).$$

We have shown that the two sub-exceeding functions *f* and *g* are equal, Therefore the map $\phi$ is Injective from $\mathcal{F}_n$ to $S_n$.

In sum, the map $\phi$ is bijective from $\mathcal{F}_n$ to $S_n$.

⊡

## 2.2. Algorithm for the construction of $f \in \mathcal{F}_n$ from $\sigma \in S_n$

Let now $\sigma$ be a permutation in $S_n$. We will construct the corresponding sub-exceeding function $f$ i.e. $f = \phi^{-1}(\sigma)$ as described by the algorithm below:

Let be $\sigma = x_1\, x_2 \ldots x_n$ i.e. $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ x_1 & x_2 & \ldots & x_n \end{pmatrix}$. By the construction of $\phi$, we have

$$f(n) = x_n.$$

Now find $n$ in $x_1\, x_2 \ldots x_n$ and permute the place of $n$ and $x_n$. So we have a new permutation

$$\sigma' = \begin{pmatrix} 1 & 2 & \ldots & n \\ x_1 & x_2 & \ldots & x_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & \ldots & n \\ x'_1 & x'_2 & \ldots & x'_n \end{pmatrix} \in S_{n-1}.$$

Therefore, we have $f(n-1) = x'_{n-1}$. Now find $n-1$ in $x'_1\, x'_2 \ldots x'_{n-1}$ and permute the place of $n-1$ and $x'_{n-1}$. We continue to adopt this method until all of $f(i)$ are found.

**Example 2.5.** For $n = 9$, let be $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 4 & 3 & 7 & 6 & 9 & 8 & 5 \end{pmatrix}$. We immediately see that $f(9) = 5$. Now permute 9 and 5 in $\sigma$, so

$$\sigma' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 4 & 3 & 7 & 6 & 5 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 7 & 6 & 5 & 8 \end{pmatrix} \in S_8 \Rightarrow \quad f(8) = 8.$$

*Using the same method, we find*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 4 & 3 & 7 & 6 & 5 \end{pmatrix} \Rightarrow f(7) = 5$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 5 & 6 \end{pmatrix} \Rightarrow f(6) = 6\ et\ f(5) = 5$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \Rightarrow f(4) = 3$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \Rightarrow f(3) = 3$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow f(2) = 1\ \ et\ f(1) = 1$$

Finally,

$$h = 113356585 \in \mathcal{F}_9$$

# 3. The set $\mathcal{H}_n$ And his proprieties

**Definition 3.1.** For positive integers $n$ and $k$ such that $1 \leq k \leq n$, we denote by $\mathcal{H}_n^k$ the sub-set of $\mathcal{F}_n$ such that

$$\mathcal{H}_n^k = \{f \in \mathcal{F}_n \mid f(i) \leq f(i+1) \text{ for all } i \in [n-1] \text{ and } \{f(i)\}_{i\in[n]} = [k]\} \quad (3.1)$$

Here, $\mathcal{H}_n^k$ is the set of all sub-exceeding function of $\mathcal{F}_n$ which have a quasi-increasing image formed by all elements of $[k]$.

**Example 3.1.** Take $n = 4$ and $k = 3$. We have here $f = 1123 \in \mathcal{H}_4^3$ because $\{f(i)\}_{i\in[4]}$ is an quasi-inceasing sequence and all of the elements of [3] are there. Now take $f = 1133$, although the sequence $\{f(i)\}_{i\in[4]}$ is quasi-inceasing , $f = 1133 \notin \mathcal{H}_4^3$ because $\{f(i)\}_{i\in[n]} \neq$ [3] (*Without* 2 *among the* $f(i)$)

From definition 3.1, we can write the set $\mathcal{H}_n$ as

$$\mathcal{H}_n = \bigcup_{k=1}^{n} \mathcal{H}_n^k \quad (3.2)$$

## 3.1. Iterative construction of the set $\mathcal{H}_n^k$

**Proposition 3.1.** Let $n$ and $k$ be two integers such that $1 \leq k \leq n$.

1. The set $\mathcal{H}_n^1 = \{f \mid f = 11 \dots .11_{n-fois}\}$ for all $n$.
2. For $n > 1$ and $k > 1$, we can construct all sub-exceeding functions in $\mathcal{H}_n^k$ by adding the integer $k$ at the end of the elements of $\mathcal{H}_{n-1}^{k-1}$ and $\mathcal{H}_{n-1}^k$.

**Example 3.2.** Take $n = 1; 2; 3$.

$$\mathcal{H}_1^1 = \{1\}$$

$$\mathcal{H}_2^1 = \{11\} \text{ and } \mathcal{H}_2^2 = \{12\}$$

$$\mathcal{H}_3^1 = \{111\}, \quad \mathcal{H}_3^2 = \{112; 122\} \text{ and } \mathcal{H}_3^3 = \{123\}$$

***Proof of proposition 3.1.***
Let $n$ and $k$ be two integers such that $1 \leq k \leq n$.

1. Take $f \in \mathcal{H}_n^1$ . Since $\mathcal{H}_n^1$ is the set of all sub-exceeding function in $\mathcal{F}_n$ which have a quasi-incresing sequence of image formed by the element of [1], necessary $f(i) = 1$ for all $i \in [n]$. Then
$$\mathcal{H}_n^1 = \{f \mid f = 11 \dots .11_{n-fois}\}.$$
2. For $n > 0$ and $k > 0$, take $f$a sub-exceeding function in $\mathcal{H}_{n-1}^{k-1} \cup \mathcal{H}_{n-1}^k$ and denote by $f'$ the sub-exceeding function such that $f' = f(1)f(2) \dots f(n-1)\, k$ i.e. adding the integer $k$ at the end of $f$.
   - If $f \in \mathcal{H}_{n-1}^{k-1}$, the sequance $(f(i))_{i\in[n-1]}$ is formed by all elements of $[k-1]$, then $(f'(i))_{i\in[n]}$ is formed by all the elements of $[k]$. So $f' \in \mathcal{H}_n^k$.

- If $f \in \mathcal{H}_{n-1}^{k}$, the sequence $(f(i))_{i\in[n-1]}$ is formed by all elements of $[k]$, then $(f'(i))_{i\in[n]}$ is also formed by all the elements of $[k]$. So $f' \in \mathcal{H}_n^k$.

Denote by $(\mathcal{H}_n^k)'$ the set of this fuction $f'$ i.e.

$$(\mathcal{H}_n^k)' = \{f' = f(1)f(2)\dots f(n-1)\,k \mid f = f(1)f(2)\dots f(n-1) \in \mathcal{H}_{n-1}^{k-1} \cup \mathcal{H}_{n-1}^{k}\}. \qquad (3.3)$$

Immediatemy, we see that

$$(\mathcal{H}_n^k)' \subseteq \mathcal{H}_n^k. \qquad (3.4)$$

Now let f be a sub-exceeding function in $\mathcal{H}_n^k$ i.e. $f = f(1)f(2)\dots f(n-1)\,f(n)$ where $f(i) \le f(i+1)$ for all $i \in [n]$ and the sequence $(f(i))_{i\in[n]}$ is formed by all elements of $[k]$. Necessary, $f(n) = k$, so the sequence $(f(i))_{i\in[n-1]}$ is formed by all elements of $[k-1]$ or $[k]$. Therefore

$$f = f(1)f(2)\dots f(n-1)\,k \in (\mathcal{H}_n^k)'.$$

Then

$$(\mathcal{H}_n^k)' \supseteq \mathcal{H}_n^k. \qquad (3.5)$$

Consequently, from (3.4) and (3.5), we have $(\mathcal{H}_n^k)' = \mathcal{H}_n^k$.

Finally, by the construction of the set $(\mathcal{H}_n^k)'$ in the equation (3.3), we have all sub-exceeding functions of $\mathcal{H}_n^k$ by adding the integer $k$ at the end of the elements of $\mathcal{H}_{n-1}^{k-1} \cup \mathcal{H}_{n-1}^{k}$.

⊡

Now, we have the following iteration table of $\mathcal{H}_n^k$

| $n$ \ $k$ | 1 | 2 | 3 | 4 | 5 | … |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | |
| 2 | 11 | 12 | | | | |
| 3 | 111 | 112<br>122 | 123 | | | |
| 4 | 1111 | 1112<br>1122<br>1222 | 1123<br>1223<br>1233 | 1234 | | |
| 5 | 11111 | 11112<br>11122<br>11222<br>12222 | 11123<br>11223<br>12223<br>11233<br>12233<br>12333 | 11234<br>12234<br>12334<br>12344 | 12345 | |
| … | … | … | … | … | … | … |

**Proposition 3.2**. Let $n$ and $k$ be two integers such that $1 \le k \le n$, so we have the following relations:

1. $\text{Card}\,\mathcal{H}_n^k = 1$ for all $n$ ,
2. For $n > 1$ and $k > 1$, we have $\text{Card}\,\mathcal{H}_n^k = \text{Card}\,\mathcal{H}_{n-1}^{k-1} + \text{Card}\,\mathcal{H}_{n-1}^{k}$,
3. $\text{Card}\,\mathcal{H}_n^k = \binom{n-1}{k-1}$ and $\text{Card}\,\mathcal{H}_n = 2^{n-1}$.

***Proof.***

1. From proposition 3.1. (1), we have immediately $\text{Card}\,\mathcal{H}_n^1 = 1$ for all $n \ge 1$.

2. By the construcion in the proposition 3.1. (2), we have all elements of $\mathcal{H}_n^k$ by adding the integer k at the end of the elements of $\text{Card}\,\mathcal{H}_{n-1}^{k-1}$ and $\text{Card}\,\mathcal{H}_{n-1}^{k}$. Which give us $\text{Card}\,\mathcal{H}_n^k = \text{Card}\,\mathcal{H}_{n-1}^{k-1} + \text{Card}\,\mathcal{H}_{n-1}^{k}$.
3. Morever, by the construction in proposition 3.1 too,we have the cardinal table of $\mathcal{H}_n^k$:

| $n$ \ $k$ | 1 | 2 | 3 | 4 | 5 | … |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | |
| 2 | 1 | 1 | | | | |
| 3 | 1 | 2 | 1 | | | |
| 4 | 1 | 3 | 3 | 1 | | |
| 5 | 1 | 4 | 6 | 4 | 1 | |
| … | | | | | | |

This table is a modified Pascal triangle. More precisely, it is shifted i.e. instead of $k \in \mathbb{N}$ We have $k \in \mathbb{N}^*$. Moreover, the values of the $i^{th}$ line in this table is the value of $(i-1)^{th}$ line of the Pascal triangle. Thus,

$$\begin{aligned}\text{Card}\,\mathcal{H}_n &= \sum_{k=1}^{n} \text{Card}\,\mathcal{H}_n^k \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} \\ &= 2^{n-1}.\end{aligned}$$

⊡

### 3.2. Some properties of the set $\phi(\mathcal{H}_n^k)$

The set $\phi(\mathcal{H}_n)$ is none other than the image of $\mathcal{H}_n = \cup_{k=1}^{n} \mathcal{H}_n^k$ by the bijective mapping $\phi$. Since $\mathcal{H}_n^i \cap \mathcal{H}_n^j = \emptyset$, this set can be written by

$$\phi(\mathcal{H}_n) = \phi\left(\bigcup_{k=1}^{n} \mathcal{H}_n^k\right) = \bigcup_{k=1}^{n} \phi(\mathcal{H}_n^k). \tag{3.6}$$

In this section we give some properties of the set $\phi(\mathcal{H}_n)$.

**Proposition 3.3.** Let $n$ and $k$ be two integers such that $1 \leq k \leq n$ and f a sub-exceeding function i$\mathcal{H}_{n-1}^{k-1} \cup \mathcal{H}_{n-1}^{k}$n.

Denoting by $\sigma_f$ the image of $f$ by $\phi$ and by composing it with the permutation $(n+1\ \ k)$, we have:

1. $\sigma'_f = (n+1)\,\sigma_f$ is a permutation in $\phi(\mathcal{H}_{n+1}^k)$.
2. For any integer $i \in [n]$, we have $\sigma'_f(i) = \sigma_f(i)$ excepted for $\sigma'_f\left(\sigma_f^{-1}(k)\right) = n+1$.

This proposition describes the construction of the elements of $\phi(\mathcal{H}_{n+1}^k)$ as follows: the elements of $\phi(\mathcal{H}_{n+1}^k)$ are obtained by taking all elements of $\phi(\mathcal{H}_n^{k-1} \cup \mathcal{H}_n^k)$ and replacing the integer $k$ of these permutations by $n+1$ and add this integer $k$ at the end.

***Proof.***

1. Take $f \in \mathcal{H}_n^{k-1} \cup \mathcal{H}_n^k$ and $\sigma_f$ his associated permutation in $S_n$ such that

$$\sigma_f = \big(n\ \ f(n)\big)\big(n-1\ \ f(n-1)\big)\ldots\big(1\ \ f(1)\big). \tag{3.7}$$

Let us calculate the composition of $(n+1\ \ k)$ with $\sigma_f$. So we have a new permutation

$$\sigma'_f = (n+1\ k) \circ \sigma_f = (n+1\ k)\big(n\ \ f(n)\big)\big(n-1\ \ f(n-1)\big)\ldots\big(1\ \ f(1)\big) \tag{3.8}$$

and that $\sigma'_f(n+1) = k$. So the last word of the permutaion $\sigma'_f$ is $k$.

Now, let $f' = f(1)f(2) \dots f(n)k$ i.e. f is the sub-exceeding function $f \in \mathcal{H}_n^{k-1} \cup \mathcal{H}_n^k$ by adding the integer $k$ at the and. By the constuctionin the proposition 3.1, we see that this new function is in $\mathcal{H}_{n+1}^k$. Since the associated permutation of $f'$ is none other than $\sigma'_f$, we have

$$\sigma'_f = (n+1\ \ k)\sigma_f \in \phi\left(\mathcal{H}_{n+1}^k\right). \qquad (3.9)$$

2. Let $\sigma_f = \left(n\ f(n)\right)\left(n-1\ f(n-1)\right) \dots \left(1\ f(1)\right) = x_1\, x_2 \dots x_n$ and $j$ the integer such that $\sigma_f(j) = x_j = k$ . Then, for any integer $i$ such that $i \notin \{j; n+1\}$, we have

$$(n+1\ \ k) \circ \sigma_f(i) = \sigma_f(i) = x_i\ .$$

For the integers $j$ and $n+1$, we have

$$(n+1\ \ k) \circ \sigma_f(n+1) = k,$$
$$(n+1\ \ k) \circ \sigma_f(j) = (n+1\ k)(k) = n+1.$$

⊡

**Example 3.3.**

Let $f = 11122$ in $\mathcal{H}_5^2$. So $\sigma_f = (5\ 2)(4\ 2)(3\ 1)(2\ 1)(1) = 43152$. By composing $\sigma_f$ with the permutation $(6\ 2)$, we have

$$\sigma_f' = (6\ 2)(5\ 2)(4\ 2)(3\ 1)(2\ 1)(1) = 431562$$

a permutation in $\mathcal{H}_6^2$. It's like replacing 2 by 6 in $\sigma_f$ and add 2 at the end.

**Definition 3.2.** Let $n$ and $k$ be two integers such that $1 \le k \le n$.

- For any permutation $\sigma$ in $S_n$ and for any integer $i$ such that $1 \le i \le n$, we define by $\mathrm{pos}(i)$ the position of $i$ in $\sigma$.
- For any sub-exceedinf function $f$ in $\mathcal{H}_n^k$ and for any integer $i$ below $k$, we define by $dpos(i)$ the last position of $i$ in $f$.

**Example 3.4.** In $S_5$, let $\sigma = 13542$ which gives us

$$\mathrm{pos}(1) = 1, \mathrm{pos}(2) = 5, \mathrm{pos}(3) = 2, \mathrm{pos}(4) = 4 \text{ and } \mathrm{pos}(5) = 3.$$

In $\mathcal{H}_9^4$, let $f = 111223334$. Here

$$\mathrm{dpos}(1) = 3, \qquad d\mathrm{pos}(2) = 5, \qquad d\mathrm{pos}(3) = 8 \text{ and } \mathrm{pos}(4) = 9.$$

**Proposition 3.4.** Let $n$ and $k$ be two integers such that $1 \le k \le n$.

1. For any permutation $\sigma_f$ of $\phi(\mathcal{H}_n^k)$ and for any integer $j$ such that $1 \le j \le n$, we have $\mathrm{pos}(j)|_{\sigma_f} = \mathrm{dpos}(j)|_f$ where $f$is the associated sob-exceeding function of $\sigma_f$.
2. The integers $1;\ 2;\ 3;\ \dots;\ k$ are always placed in ascending order in $\sigma_f$ i.e.
$$\mathrm{pos}(1) \le \mathrm{pos}(2) \le \dots \le \mathrm{pos}(k).$$
3. Let $\sigma_f$ and $\sigma_g$ be two permutation in $\phi(\mathcal{H}_n^k)$ such that $\mathrm{pos}(i)|_{\sigma_f} = \mathrm{pos}(i)|_{\sigma_g}$ for all $i \in [k]$. Then
$$\sigma_f = \sigma_g.$$

**Proof.**

1. For any $\sigma_f$ in $\phi(\mathcal{H}_n^k)$, the corresponding sub-exceeding function has the forme
$$f = 11 \dots 1_{i_1-\text{fois}}\, 22 \dots 2_{i_2-\text{fois}} \dots (k-1)(k-1) \dots (k-1)_{i_{k-1}-\text{fois}}\, kk \dots k_{i_k-\text{fois}}$$

where for any integer $j \in [k]$,

$$\mathrm{dpos}(j) = \sum_{l=1}^{j} i_l \quad \text{and} \quad \sum_{l=1}^{k} i_l = n.$$

Then we have

$$\sigma_f = \left[\left(\sum_{j=1}^{k} i_j\right) \ k\right]\left[\left(\left(\sum_{j=1}^{k} i_j\right) - 1\right) \ k\right] \dots \left[1 + \left(\sum_{j=1}^{k} i_j\right) \ k\right]\left[\left(\sum_{j=1}^{k-1} i_j\right) \ k-1\right] \dots \left[1 + \left(\sum_{j=1}^{k-2} i_j\right) \ k-1\right] \dots \big((i_1 + i_2 \ ) \ 2\big) \dots \big((i_1+1) \ \ 2\big)(i_1 \ \ 1) \dots (1)$$

However, since $\mathrm{dpos}(j)$ is the las integer such that the image by $f$ gives $j$, so we have

$$\sigma_f\big(\mathrm{dpos}(j)\big) = \left[\left(\sum_{l=1}^{k} i_l\right) \ k\right]\left[\left(\left(\sum_{l=1}^{k} i_l\right) - 1\right) \ k\right] \dots \left[1 + \left(\sum_{l=1}^{k-1} i_l\right) \ k\right]\left[\left(\sum_{l=1}^{k-1} i_l\right) \ k-1\right] \dots \left[1 + \left(\sum_{l=1}^{k-2} i_l\right) \ k-1\right] \dots \left[1 + \left(\sum_{l=1}^{j} i_l\right) \ j+1\right]\left[\left(\sum_{l=1}^{j} i_l\right) \ j\right] \left[\left(\left(\sum_{l=1}^{j} i_l\right) - 1\right) \ j\right] \dots \big((i_1 + i_2 \ ) \ 2\big) \dots \big((i_1+1) \ \ 2\big)(i_1 \ \ 1) \dots (1)|_{\mathrm{dpos}(j)}.$$

Then,

$$\sigma_f\big(\mathrm{dpos}(j)\big) = \left[\left(\sum_{l=1}^{j} i_l\right) \ j\right]\left[\left(\left(\sum_{l=1}^{j} i_l\right) - 1\right) \ j\right] \dots \left[\left(1 + \left(\sum_{l=1}^{j-1} i_l\right)\right) \ j-1\right]\left[\left(\sum_{l=1}^{j-1} i_l\right) \ j-1\right] \dots \left[\left(\left(\sum_{l=1}^{j-2} i_l\right) + 1\right) \ j-1\right] \dots \big((i_1 + i_2 \ ) \ 2\big) \dots \big((i_1+1) \ \ 2\big)(i_1 \ \ 1) \dots (1)|_{\mathrm{dpos}(j)},$$

Which gives $\sigma_f\big(\mathrm{dpos}(j)\big) = j$, then $dpos(j)|_f = pos(j)|_{\sigma_f}$.

2. From the result in the item (1)

$$\sum_{l=1}^{j} i_l = pos(j)|_{\sigma_f}.$$

However in $f$, the sequence $\left(\sum_{l=1}^{j} i_l\right)_j$ is strictly increasing, so $\mathrm{pos}(1) \le \mathrm{pos}(2) \dots \le \mathrm{pos}(k)$ which causes that the integers $1;\ 2;\ 3;\ \dots;\ k$ are always placed in ascending order in $\sigma_f$.

$$\mathrm{pos}(1) \le \mathrm{pos}(2) \le \dots \le \mathrm{pos}(k).$$

3. Let $\sigma_f$ and $\sigma_g$ be two permutation in $\phi(\mathcal{H}_n^k)$ such that $\mathrm{pos}(i)|_{\sigma_f} = \mathrm{pos}(i)|_{\sigma_g}$ for all $i \in [k]$. Now let $p_i = \mathrm{pos}(i)|_{\sigma_f} = \mathrm{pos}(i)|_{\sigma_g}$, then from the relation in (1), we have $p_i = \mathrm{dpos}(i)|_f = \mathrm{dpos}(i)|_g$. Thus, the corresponding sub-exceeding function are

$$f = 11 \dots 1_{p_1-\mathrm{fois}}\, 22 \dots 2_{(p_2 - p_1)-\mathrm{fois}} \dots (k-1)(k-1) \dots (k-1)_{(p_{k-1}-p_{k-2})-\mathrm{fois}}\, kk \dots k_{(n-p_{k-1})-\mathrm{fois}}$$

And similarly

$$g = 11 \dots 1_{p_1-\mathrm{fois}}\, 22 \dots 2_{(p_2 - p_1)-\mathrm{fois}} \dots (k-1)(k-1) \dots (k-1)_{(p_{k-1}-p_{k-2})-\mathrm{fois}}\, kk \dots k_{(n-p_{k-1})-\mathrm{fois}}.$$

Since the mapping $\phi$ is bijective and that $f = g$, we have

$$\sigma_f = \sigma_g.$$

⊡

**Corollary 3.5**. From Proposition 3.3 and Proposition 3.4, we can iterate the elements of $\phi(\mathcal{H}_n^k)$ for each $1 \leq k \leq n$ as shown in the following table:

| $n$ \ $k$ | 1 | 2 | 3 | 4 | 5 | … |
|---|---|---|---|---|---|---|
| 1 | 1 | | | | | |
| 2 | 21 | 12 | | | | |
| 3 | 321 | 312<br>132 | 123 | | | |
| 4 | 2341 | 4312<br>3142<br>1342 | 4123<br>1423<br>1243 | 1234 | | |
| 5 | 23451 | 53412<br>43152<br>31452<br>13452 | 45123<br>51423<br>15423<br>41253<br>14253<br>12453 | 51234<br>15234<br>12534<br>12354 | 12345 | |
| … | … | … | … | … | … | … |

As we have seen that $\mathrm{Card}\, \mathcal{H}_n^k = \binom{n-1}{k-1}$ and similarly $\mathrm{Card}\, \phi(\mathcal{H}_n^k) = \binom{n-1}{k-1}$, this value indicates the possible positions of the increasing sequence $1;\ 2;\ 3;\ \dots; (k-1)$ among the $n-1$ places in $\sigma^*$ where $\sigma^*$is the permutation $\sigma$ by deleting the last integer $k$.

**Example 3.6.** *The elements of* $\phi(\mathcal{H}_5^3)$ *are* $(45123;\ 51423;\ 15423;\ 41253;\ 14253;\ 12453)$. *By deleting the last integer* 3 *from all these permutations, we have:*

$$\phi(\mathcal{H}_5^3)^* = \ (\,4512;\ 5142;\ 1542;\ 4125;\ 1425;\ 1245\,)$$

*This set presents all the possible positions of 1 and 2 in ascending order in the 4 existing places.*

## 4. The bijection between $\mathcal{H}_n$ and $\mathcal{P}(\Omega)$

The aim of this section is to construct a bijection between $\mathcal{H}_n$ and $\mathcal{P}(\Omega)$. Given a subexceeding function in $\mathcal{H}_n$, we will specify the partition of $\Omega$ that it presents. Conversely, given a subset of $\Omega$, we will give his corresponding sub-exceeding function.

### 4.1. Construction of the bijection

Recall that that $\mathcal{P}^l(\Omega)$ represents the set of the parts of $\Omega$ to $l$ elements.

**Theorem 4.1.** let $\psi$ be a map from $\mathcal{H}_n^k$ to $\mathcal{P}^{k-1}(\Omega)$ such that

$$\begin{array}{llll} \psi : & \mathcal{H}_n^k & \longrightarrow & \mathcal{P}^{k-1}(\Omega) \\ & f & \longmapsto & \psi(f) = \{\omega_{\mathrm{dpos}(1)};\ \omega_{\mathrm{dpos}(2)};\dots;\ \omega_{\mathrm{dpos}(k-1)}\} \end{array} \quad (4.1)$$

where

$$\Omega = \{\omega_1;\ \omega_2;\dots;\ \omega_{n-1}\}. \quad (4.2)$$

Thenthe mapping $\psi$ is bijective.

From this theorem, $\mathcal{H}_n^k$ represent the set of the parts of $\Omega$ to $k-1$ elements.

***Proof.***
Since $\mathrm{Card}\,\mathcal{H}_n^k$ is the same as $\mathrm{Card}\,\mathcal{P}^{k-1}(\Omega)$ :

$$\mathrm{Card}\,\mathcal{H}_n^k = \mathrm{Card}\,\mathcal{P}^{k-1}(\Omega) = \binom{n-1}{k-1}, \quad (4.3)$$

we have to show that $\psi$ is injective. Let $f$ and $g$ be two sub-exceeding function in $\mathcal{H}_n^k$ such that $\psi(f) = \psi(g)$, i.e.

$$\{\omega_{\mathrm{dpos}(1)|f};\ \omega_{\mathrm{dpos}(2)|f};\ \dots;\ \omega_{\mathrm{dpos}(k-1)|f}\} = \{\omega_{\mathrm{dpos}(1)|g};\ \omega_{\mathrm{dpos}(2)|g};\ \dots;\ \omega_{\mathrm{dpos}(k-1)|g}\}. \quad (4.4)$$

We know that

$$\mathrm{dpos}(1)|f \le \mathrm{dpos}(2)|f \le \dots \le \mathrm{dpos}(k-1)|f$$

and similarly

$$\mathrm{dpos}(1)|g \le \mathrm{dpos}(2)|g \le \dots \le \mathrm{dpos}(k-1)|g.$$

However, this two subset $\psi(f)$ and $\psi(g)$ are the same, necessarily

$$\mathrm{dpos}(1)|f = \mathrm{dpos}(1)|g;\ \ \mathrm{dpos}(2)|f = \mathrm{dpos}(2)|g;\ \ \dots;\ \ \mathrm{dpos}(k-1)|f = \mathrm{dpos}(k-1)|g. \quad (4.5)$$

Note that the equality (4.5) also means that

$$\mathrm{pos}(1)\big|\sigma_f = \mathrm{pos}(1)\big|\sigma_g;\ \ \mathrm{pos}(2)\big|\sigma_f = \mathrm{pos}(2)\big|\sigma_g;\ \ \dots;\ \ \mathrm{pos}(k-1)\big|\sigma_f = \mathrm{pos}(k-1)\big|\sigma_g.$$

Then according to the “Proposition (3.5) also means that
$\sigma_f = \sigma_g$.
Thus, using the bijection of $\phi$ (see 2.1), we have $f = g$ and the map $\psi$ is injective from $\mathcal{H}_n^k$ to $\mathcal{P}^{k-1}(\Omega)$, so bijective.

⊡

**Corollary 4.2.** *The map $\psi$ is also bijective from $\mathcal{H}_n$* to $\mathcal{P}(\Omega)$**.**
***Proof.***
As $\mathcal{H}_n = \bigcup_{k=1}^{n}\mathcal{H}_n^k$ and $\mathcal{H}_n^i \cap \mathcal{H}_n^j = \emptyset$, them the bijection of the map $\psi$ from $\mathcal{H}_n^k$ to $\mathcal{P}^{k-1}(\Omega)$ can be extand in $\mathcal{H}_n$ to $\mathcal{P}(\Omega)$**.**

## 4.2. Algorithm for the construction of the sub-exceeding function $f$ corresponding of a sub-set of Ω

Let $n$ and $k$ be two integers such that $1 \le k \le n$ and $\Omega$ the set of $n-1$ objects such that $\Omega = \{\omega_1;\ \omega_2;\ \dots;\ \omega_{n-1}\}$.
Now, let $A$ be a subset of $\Omega$ such that $\mathrm{A} = \{a_1;\ a_2;\ \dots;\ a_k\}$. To construct the corresponding sub-exceeding function $f$ in $\mathcal{H}_n^k$ for this subset, we must order the element of $A$ according to their position in $\Omega$ . thus we have $\mathrm{A} = \{\omega_{j_1};\ \omega_{j_2};\ \dots;\ \omega_{j_k}\}$ where $j_i$ is the position of $a_i$ in $\Omega$ and that $j_1 < j_2 < \dots < j_k$. We can now write:

$$f = 11\dots1_{j_1-\mathrm{fois}}\,22\dots2_{(j_2-\,j_1)-\mathrm{fois}}\dots(k-1)(k-1)\dots(k-1)_{(j_{k-1}-j_{k-2})-\mathrm{fois}}\,kk\dots k_{(n-p_{k-1})-\mathrm{fois}}\,. \quad (4.6)$$

**Example 4.3.** Let $\Omega = \{a; b; c; d; e; f\}$. As $\mathrm{card}(\Omega) = 6$, so the set of parts of this set is isomorphic to $\mathcal{H}_7$.

Given $f = 1122344$ in $\mathcal{H}_7^4$, we have here

$$\mathrm{dpos}(1) = 2, \qquad \mathrm{dpos}(2) = 4, \qquad \mathrm{dpos}(3) = 5.$$

So the part of $\Omega$ which it represents is formed by the second element, the fourth element and the fifth element.

$$\Omega_f = \{b, d, e\}.$$

Now, let $\Omega_1 = \{f, a, e, d\}$ and $\Omega_2 = \{f, a, b\}$ be two subset of $\Omega$. After ordering $\Omega_1$ depending on their place in $\Omega$, we have

$$\Omega_1 = \{a, d, e, f\} = \{\omega_1, \omega_4, \omega_5, \omega_6\}.$$

The corresponding sub-exceeding function of $\Omega_1$ is in $\mathcal{H}_7^5$ such that

$$\mathrm{dpos}(1) = 1, \qquad \mathrm{dpos}(2) = 4, \qquad \mathrm{dpos}(3) = 5 \ \text{ and } \ \mathrm{dpos}(4) = 6.$$

So

$$f = 1\,2\,2\,2\,3\,4\,5\,.$$

After ordering $\Omega_2$ depending on their place in $\Omega$, we have

$$\Omega_1 = \{a, b, f\} = \{\omega_1, \omega_2, \omega_6\}.$$

Then, the corresponding sub-exceeding function of $\Omega_2$ is in $\mathcal{H}_7^4$ such that

$$\mathrm{dpos}(1) = 1, \qquad \mathrm{dpos}(2) = 2 \ \text{ and } \ \mathrm{dpos}(3) = 6.$$

So

$$f = 1\,2\,3\,3\,3\,3\,4.$$

**Corollary 4.4.** from the construction of $\psi$, we find that the sub-exceeding function $f = 11 \dots 11$ always represent the empty set $(\emptyset)$ and the function $f = 123 \dots n$ always represent the set $\Omega$ where $\mathrm{Card}\,\Omega = n - 1$.

## 5. The operations $\cup_{rab}$ $\cap_{rab}$ and $\bar{f}$ in $\mathcal{H}_n$

As $\mathcal{P}(\Omega)$ is stable by the operations $\cup$, $\cap$ and by passing to the complement, then we present in this section the equivalent of these operations in $\mathcal{H}_n$.

**Definition 5.1.** Let $f$ be a sub exceeding function in $\mathcal{H}_n^k$, we define the indicator of $f$ the set $I_f$ such that

$$I_f = \{dpos(1); dpos(2); \dots; dpos(k-1)\}.$$

**Example 5.1.** In $\mathcal{H}_9$, take $f = 112223445 \in \mathcal{H}_9^5$, then the set $I_f$ is

$$I_f = \{2; 5; 6; 8\}.$$

### 5.1. The operation $\cup_{rab}$ in $\mathcal{H}_n$

**Definition 5.2.** Let $f$ and $g$ be two sub-exceeding functions who belong respectively in $\mathcal{H}_n^k$ and $\mathcal{H}_n^l$. We denote by $(f \cup_{rab} g)$ the sub-exceeding function $h$ such that

$$h = (f \cup_{rab} g) \in \mathcal{H}_n^{m+1} \quad \text{with} \quad I_h = I_f \cup I_g \text{ and } m = card\, I_h .$$

**Example 5.2.** Let $f$ and $g$ be two sub-exceeding functions such that $f = 1122344 \in \mathcal{H}_7^4$ and $g = 1222345 \in \mathcal{H}_7^5$. We have here $I_f = \{2; 4; 5\}$ and $I_g = \{1; 4; 5; 6\}$. Then, the sub-exceeding function $h = (f \cup_{rab} g)$ has his indicator the set $I_h = \{1; 2; 4; 5; 6\}$. From this indicator, we have $h \in \mathcal{H}_7^6$ such that

$$h = 1233456.$$

**Theorem 5.3.** Let $f$ and $g$ be two sub-exceeding functions in $\mathcal{H}_n$ which respectively represents $\Omega_f$ and $\Omega_g$ (*two parts of Ω*). Then, $\mathcal{H}_n$ is stable by $\cup_{rab}$ and $(f \cup_{rab} g)$ represent $\Omega_f \cup \Omega_g$.

***Proof.***
Set

$$I_f = \{\text{dpos}(1)|_f;\ \text{dpos}(2)|_f \dots ; \text{dpos}(k-1)|_f\}$$

and

$$I_f = \{\text{dpos}(1)|_g;\ \text{dpos}(2)|_g \dots ; \text{dpos}(l-1)|_g\}.$$

After ordering, we may write $I_f \cup I_g$ as follows:

$$I_f \cup I_g = \{\text{dpos}(1); \text{dpos}(2); \dots ; \text{dpos}(m)\}.$$

From this form, this set of $m$ elements is the indicator of the sub-exceeding function $(f \cup_{rab} g)$. Thus $\mathcal{H}_n$ is stable by $\cup_{rab}$.
Moreover, the set $\Omega_f \cup \Omega_g$ is such that

$$\Omega_f \cup \Omega_g = \{\omega_{\text{dpos}(1)};\ \omega_{\text{dpos}(2)}; \dots ;\ \omega_{\text{dpos}(m)}\}$$

By the construction of the bijective mapping $\psi$, the sub-exceeding function $h = (f \cup_{rab} g)$ with his indicator $I_f \cup I_g$ represent $\Omega_f \cup \Omega_g$.

**Example 5.4.** Let $f$ and $g$ be two sub-exceeding functions such that $f = 1122344 \in \mathcal{H}_7^4$ and $g = 1222345 \in \mathcal{H}_7^5$. We have here $I_f = \{2; 4; 5\}$ and $I_g = \{1; 4; 5; 6\}$. Then, the sub-exceeding function $h = (f \cup_{rab} g)$ has his indicator the set $I_h = \{1; 2; 4; 5; 6\}$. From this indicator, we have $h \in \mathcal{H}_7^6$ such that

$$h = 1233456.$$

Moreover, the two parts of $\Omega$ such that these two functions $f$ and $g$ represents are

$$\Omega_f = \{\omega_2;\ \omega_4;\ \omega_5\} \text{ and } \Omega_g = \{\omega_1;\ \omega_4;\ \omega_5;\ \omega_6\}$$

Thus

$$\Omega_f \cup \Omega_g = \{\omega_1;\ \omega_2;\ \omega_4;\ \omega_5;\ \omega_6\}$$

Therefore, $h = 1233456$ represent $\Omega_f \cup \Omega_g$.

## 5.1. The operation $\cap_{rab}$ in $\mathcal{H}_n$

**Definition 5.3.** Let $f$ and $g$ be two sub-exceeding functions who belong respectively in $\mathcal{H}_n^k$ and $\mathcal{H}_n^l$. We denote by $(f \cap_{rab} g)$ the sub-exceeding function $h$ such that

$$h = (f \cap_{rab} g) \in \mathcal{H}_n^{m+1} \quad \text{with} \quad I_h = I_f \cap I_g \text{ and } m = card\, I_h .$$

**Example 5.5.** Let $f$ and $g$ be two sub-exceeding functions such that $f = 1122344 \in \mathcal{H}_7^4$ and $g = 1222345 \in \mathcal{H}_7^5$. We have here $I_f = \{2; 4; 5\}$ and $I_g = \{1; 4; 5; 6\}$. Then, the sub-exceeding function $h = (f \cup_{rab} g)$ has his indicator the set $I_h = \{4; 5\}$. From this indicator, we have $h \in \mathcal{H}_7^3$ such that

$$h = 1111233.$$

**Theorem 5.6.** Let $f$ and $g$ be two sub-exceeding functions in $\mathcal{H}_n$ which respectively represents $\Omega_f$ and $\Omega_g$ (*two parts of* $\Omega$). Then, $\mathcal{H}_n$ is stable by $\cap_{rab}$ and $(f \cap_{rab} g)$ represent $\Omega_f \cap \Omega_g$.

***Proof.***
Set

$$I_f = \{\text{dpos}(1)|_f;\ \text{dpos}(2)|_f \dots; \text{dpos}(k-1)|_f\}$$

and

$$I_f = \{\text{dpos}(1)|_g;\ \text{dpos}(2)|_g \dots; \text{dpos}(l-1)|_g\}.$$

After ordering, we may write $I_f \cap I_g$ as follows:

$$I_f \cap I_g = \{\text{dpos}(1); \text{dpos}(2); \dots; \text{dpos}(m)\}.$$

From this form, this set of $m$ elements is the indicator of the sub-exceeding function $(f \cup_{rab} g)$. Thus $\mathcal{H}_n$ is stable by $\bigcap_{rab}$.
Moreover, the set $\Omega_f \cap \Omega_g$ is such that

$$\Omega_f \cap \Omega_g = \{\omega_{\text{dpos}(1)};\ \omega_{\text{dpos}(2)}; \dots;\ \omega_{\text{dpos}(m)}\}$$

By the construction of the bijective mapping $\psi$, the sub-exceeding function $h = (f \cap_{rab} g)$ with his indicator $I_f \cap I_g$ represent $\Omega_f \cap \Omega_g$.
**Example 4.4.** Let $f$ and $g$ be two sub-exceeding functions such that $f = 1122344 \in \mathcal{H}_7^4$ and $g = 1222345 \in \mathcal{H}_7^5$. We have here $I_f = \{2; 4; 5\}$ and $I_g = \{1; 4; 5; 6\}$. Then, the sub-exceeding function $h = (f \cap_{rab} g)$ has his indicator the set $I_h = \{4; 5\}$. From this indicator, we have $h \in \mathcal{H}_7^3$ such that

$$h = 1111233.$$

Moreover, the two parts of $\Omega$ such that these two functions $f$ and $g$ represents are

$$\Omega_f = \{\omega_2;\ \omega_4;\ \omega_5\} \text{ and } \Omega_g = \{\omega_1;\ \omega_4;\ \omega_5;\ \omega_6\}$$

Thus

$$\Omega_f \cap \Omega_g = \{\ \omega_4;\ \omega_5\}$$

Therefore, $h = 1111233.$ represent $\Omega_f \cap \Omega_g$.

## 5.3. The operation $\bar{f}$ complementary to $f$ in $\mathcal{H}_n$

**Definition 5.4.** Let $f$ be a sub exceeding function in $\mathcal{H}_n^k$. We define by $\bar{f}$ the complementary sub-exceeding of $f$ such that $\bar{f} \in \mathcal{H}_n^{n-k+1}$ with indicator the set $I_{\bar{f}} = \overline{I_f}$. Here, $\overline{I_f}$ is the complementary set of $I_f$ in $[n-1]$.

**Example 5.9.** Let $f$ be a sub-exceeding function in $\mathcal{H}_7^4$ such that $f = 1122344$. We have here $I_f = \{2; 4; 5\}$ and $\overline{I_f} = \{1; 3; 6\}$. The complementary sub-exceeding function of $f$ denoted by $\bar{f}$ is then

$$\bar{f} = 1223334 \in \mathcal{H}_7^4$$

**Theorem 5.10.** Let $f$ be a sub-exceeding function in $\mathcal{H}_n$ which represent $\Omega_f$. Then $\bar{f}$ represent the set $\overline{\Omega_f}$ the complementary of $\Omega_f$ in $\Omega$.

**Proof.**

Let $f \in \mathcal{H}_n^k$ and $\Omega = \{\omega_1; \omega_2; \dots; \omega_{n-1}\}$. So $I_f = \{dpos(1)|_f; dpos(2)|_f; \dots; dpos(k-1)|_f\}$ and $\overline{I_f} = [n-1] \setminus \{dpos(1)|_f; dpos(2)|_f; \dots; dpos(k-1)|_f\}$. After ordering the elements of $\overline{I_f}$, we can write

$$\overline{I_f} = \{\mathrm{dpos}(1)|_{\bar{f}}; \mathrm{dpos}(2)|_{\bar{f}}; \dots; \mathrm{dpos}(n-k)|_{\bar{f}}\}.$$

Then, the sub-exceeding function $\bar{f}$ with his indicator $\overline{I_f}$ represent

$$\Omega_{\bar{f}} = \left\{\omega_{\mathrm{dpos}(1)|_{\bar{f}}}; \omega_{\mathrm{dpos}(2)|_{\bar{f}}}; \dots; \omega_{\mathrm{dpos}(n-k)|_{\bar{f}}}\right\}$$

Since

$$\left\{\omega_{\mathrm{dpos}(1)|_{\bar{f}}}; \omega_{\mathrm{dpos}(2)|_{\bar{f}}}; \dots; \omega_{\mathrm{dpos}(n-k)|_{\bar{f}}}\right\} \cap \Omega_f = \emptyset$$

and

$$\left\{\omega_{\mathrm{dpos}(1)|_{\bar{f}}}; \omega_{\mathrm{dpos}(2)|_{\bar{f}}}; \dots; \omega_{\mathrm{dpos}(n-k)|_{\bar{f}}}\right\} \cup \Omega_f = \Omega,$$

then we find that

$$\overline{\Omega_f} = \Omega_{\bar{f}} = \left\{\omega_{\mathrm{dpos}(1)|_{\bar{f}}}; \omega_{\mathrm{dpos}(2)|_{\bar{f}}}; \dots; \omega_{\mathrm{dpos}(n-k)|_{\bar{f}}}\right\}.$$

So, $\bar{f}$ represent the complement of $\Omega_f$ in $\Omega$.

$\boxdot$

## 6. Conclusion

After recalling that the set $\mathcal{F}_n$ of sub-exceeding functions on $[n]$ and the permutation set $S_n$ are isomorphic by a map denoted $\phi$ (see section 2). So, we can construct an algorithm to transform any permutation $\sigma \in S_n$ in to a sub-exceeding function on $[n]$.
After that, we define a particular subset of $\mathcal{F}_n$ denoted $\mathcal{H}_n$ in the section (3) and construct there elements by iterations on $[n]$. Recall that

$$\mathcal{H}_n^k = \{f \in \mathcal{F}_n \mid f(i) \le f(i+1) \text{ for all } i \in [n-1] \text{ and } \{f(i)\}_{i \in [n]} = [k]\}$$

and

$$\mathcal{H}_n = \bigcup_{k=1}^{n} \mathcal{H}_n^k \quad .$$

In the section (4), we established a map $\psi$ from $\mathcal{H}_n^k$ to $\mathcal{P}^{k-1}(\Omega)$ such that $\mathcal{P}^l(\Omega)$ represent the set of the parts of $\Omega$ to $l$ elements where $\Omega$ is the set of $n-1$ objects. So we have found that the map $\psi$ is bijective. Then, $\mathcal{H}_n^k$ represents all parts of $\Omega$ with $k-1$ elements and as result too, the bijection of the map $\psi$ from $\mathcal{H}_n^k$ to $\mathcal{P}^{k-1}(\Omega)$ can be extand in $\mathcal{H}_n$ to $\mathcal{P}(\Omega)$. So $\mathcal{H}_n$ represents all subset of $\Omega$.

Therefore, if we have a part of the set $\Omega$, it is possible to encode this part using one and only one sub-exceeding fuction in $\mathcal{H}_n$ *(Coding)* and vice verca, given a sub-exceeding function in $\mathcal{H}_n$, we can find the part of $\Omega$ who it represents *(Decoding)*.

In section (5), since $\mathcal{P}(\Omega)$is stable by the operations $\cup$, $\cap$ and by passing to the complement, then we have found the operation $\cup_{rab}$ $\cap_{rab}$ and $\bar{f}$ which are the equivalent of these laws in $\mathcal{H}_n$.

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