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RESEARCH ARTICLE

NEW REPRESENTATION OF SUPPORT OF A DISCRETE FUZZY NUMBER

THANGARAJ BEAULA[1], JOHNSON.B[2]

[1] PG and Research Department of Mathematics , TBML College , Porayar 609 307.
E-mail: edwinbeaula@yahoo.co.in.
[2.] DVC College of Education ,Srimushnam 608703.
E-mail: johnson.maths@gmail.com.

ABSTRACT

In this paper the support of a discrete fuzzy number is defined in a different way. The arithmetic operations viz : the addition and multiplication of discrete fuzzy numbers are defined using the newly coined representation.
KeyWords:Support,Discrete fuzzy number.

1 Introduction

Chang and Zadeh[2] introduced the concept of fuzzy number with the consideration of the properties of probability functions.Since then a lot of mathematicians have been studying on fuzzy number, and have obtained many results [1,3,4,5,7,9] .

In 2001 Voxman [6] introduced the concept of discrete fuzzy numbers, thediscrete fuzzy number can be used to represent the pixel value in thecentre point of a window [8]Also he gave out the canonical representation of discrete fuzzy numbers.

Guxiangwang and etal discussed the representationof cut-set form of the discrete fuzzy numbers,and using the representation it is shown that the usual addition of two discrete fuzzy numbers does not keep the closeness of the operation(at this point, discrete fuzzy numbers are not like non- discrete fuzzy numbers since non-discrete fuzzy numbers keep the closeness of usual addition[9] and defined a new addition for two discrete fuzzy numbers, which keeps the closeness of the operation.

In this paper the support of a discrete fuzzy number is coined differently. Further characterization properties regarding such representation are established. Also, the addition and multiplication of discrete fuzzy numbers are defined using the newly coined representation.

Definition 2.1 Let X be a universe of discourse, a fuzzy set is defined as $A = \{(x, \tilde{A}(x): x \in X\}$ characterizedbyamembershipfunction$\mu_A(x): X \to [0,1]$,where $\mu_A(x)$denotesthe degreeofmembershipofthe element x to the set A.

Definition2.2 Theα-cut, ${}^{\alpha}A$ ofa set$A$isthe crispsubsetof$A$withmembership gradesofatleastα.

Thatis,${}^{\alpha}A=\{x|\mu_A(x)\geq\alpha\}$.

Definition 2.3A fuzzy set on the real line R is a fuzzy number if it atleast possess the following properties.

1. Amustbe anormalfuzzyset,
2. Thealphalevels ${}^{\alpha}A$ mustbe closed forevery$\alpha\in[0,1]$.
3. Thesupportof$A$, ${}^{0+}A$mustbe bounded.

Definition 2.4 A fuzzy subset $u$of R with membership mapping$u: R \to [0,1]$ is called discrete fuzzy number (*DFN*) if its support is finite (ie) there are $x_1, x_2, \ldots\ldots\ldots.., x_n \in R$ with $x_1 < x_2 <\ldots\ldots < x_n$ such that $supp(u) = \{x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$, and there are natural numbers s, t with$1 \leq s \leq t \leq n$ such that

(i)$u(x_i) = 1$ foranynaturalnumberiwith$s \leq i \leq t$

(ii)$u(x_i) \leq u(x_j)$foreachnaturalnumbersi, jwith $1 \leq i \leq j \leq s$

(iii)$u(x_i) \geq u(x_j)$foreachnaturalnumberi, jwith $t \leq i \leq j \leq n$

3 New Representation of Support of a Discrete Fuzzy Number

Definition3.1 For $A\subset R$ denote $maxA = max\{x : x \in A\}$, and $min A = min\{x : x \in A\}$,Let u be a fuzzy setof R, and$[u]^0$ be finite

For any $r\in[0,1]$ and $x_0\in[u]^0$ denote$\bar{u}(r) = max[u]^r$, $\underline{u}(r) = min[u]^r$ for $r \leq 1$

where$[u]^r = \{x/\ u(x) \geq r\}$

Define the sets as

$[\bar{u}]_{r\leq1} = \{x \in [u]^0 / x \geq \underline{u}(r)\}$

$[\underline{u}]_{r\leq1} = \{x \in [u]^0 / x \leq \bar{u}(r)\}$

Also $[\underline{u}(r), \bar{u}(r)]_{r\leq1} = \{x \in [u]^0 / \quad \underline{u}(r) \leq x \leq \bar{u}(r)\}$

Define$\underline{[u]_{x_0}}_{r\leq1} = \{x\in[\underline{u}]_{r\leq1} : x\leq x_0\}$

$\overline{[u]}^{x_0}_{r\leq1} = \{x\in[\bar{u}]_{r\leq1} : x\geq x_0\}$

So,$[\underline{u}(r), \bar{u}(r)]^{x_0}_{r\leq1} = \{[\underline{u}(r), \bar{u}(r)]_{r\leq1} : \underline{u}(r) \leq x_0 \leq \bar{u}(r)\}$

Property 3.2 Let u be a fuzzy set of R, and$[u]^0$be finite,

then$[\bar{u}]_{r\leq1} \cup [\underline{u}]_{r\leq1} \cup [\underline{u}(r), \bar{u}(r)]_{r\leq1} = [u]^0$

Proof

Let $x \in [\bar{u}]_{r\leq1} \cup [\underline{u}]_{r\leq1} \cup [\underline{u}(r), \bar{u}(r)]_{r\leq1}$

Then $x \in [\bar{u}]_{r\leq1}$(or) $x \in [\underline{u}]_{r\leq1}$ (or) $x \in [\underline{u}(r), \bar{u}(r)]_{r\leq1}$

Suppose$x\in[\bar{u}]_{r\leq1} = \{x \in [u]^0 / x \geq \underline{u}(r)\}$ then$x \in [u]^0$ and$x \geq \underline{u}(r) = max[u]^r$

If $x \in \underline{[u]}_{r\le 1} = \{x \in [u]^0 \,/\, x \le \bar{u}(r)\}$ then $x \in [u]^0$ and $x \le \underline{u}(r) = min[u]^r$

Again if $x \in [\underline{u}(r), \bar{u}(r)]_{r \le 1} = \{x \in [u]^0 \,/\, \underline{u}(r) \le x \le \bar{u}(r)\}$ then $x \in [u]^0$ and $\underline{u}(r) \le x \le \bar{u}(r)$

It follows that in the above cases $x \in [u]^0$ and either $x \ge \underline{u}(r)$ or $x \le \underline{u}(r)$, $\bar{u}(r) \le x \le \underline{u}(r)$

Therefore $x \in [u]^0$ and $x \ge min[u]^r$ (or) $x \le max[u]^r$ (or) $min[u]^r \le x \le max[u]^r$

Conversely, suppose $x \in [u]^0 = \{x: u(x) \ge 0\}$ then by the definition of $[u]^0$

$[u]^0 = \{x/x \le max[u]^r$ (or) $x \ge min[u]^r$ (or) $min[u]^r \le x \le max[u]^r\}$

$= \{x/ x \le \bar{u}(r)$ (or) $x \ge \underline{u}(r)$ (or) $\bar{u}(r) \le x \le \underline{u}(r)\}$

$= \{x/ x \in \overline{[u]}_{r\le 1}$ (or) $x \in \underline{[u]}_{r \le 1}$ (or) $x \in [\underline{u}(r), \bar{u}(r)]_{r\le 1}\}$

Therefore $[u]^0 = \overline{[u]}_{r\le 1} \cup \underline{[u]}_{r\le 1} \cup [\underline{u}(r), \bar{u}(r)]_{r\le 1}$

Property 3.3 Let u be a fuzzy set of R, then $[u]^0$ be finite and $x_0 \in [u]^0$ then

(i) $x_0 \in \underline{[u]}_{r\le 1} \Leftrightarrow \underline{[u]}_{x_0\, r\le 1} \ne \varphi$

(ii) $x_0 \in \overline{[u]}_{r\le 1} \Leftrightarrow \overline{[u]}^{x_0}_{r\le 1} \ne \varphi$

(iii) $x_0 \in [\underline{u}(r), \bar{u}(r)]_{r\le 1} \Leftrightarrow [\underline{u}(r), \bar{u}(r)]^{x_0}_{r\le 1} \ne \varphi$

Proof:

(i) $x_0 \in \underline{[u]}_{r\le 1} \Leftrightarrow \underline{[u]}_{x_0\, r\le 1} \ne \varphi$

By definition,

$\underline{[u]}_{x_0\, r\le 1} = \{x \in \underline{[u]}_{r\le 1} : x \le x_0\}$

$\underline{[u]}_{r\le 1} = \{x \in [u]^0 / x \ge \bar{u}(r)\}$

Suppose $x_0 \in \underline{[u]}_{r\le 1}$

$\Rightarrow x_0 \in [u]^0$ and $x_0 \ge \bar{u}(r)$

$\Rightarrow u(x_0) > 0$ and $x_0 \ge max[u]^r$

$\Rightarrow u(x_0) \le 1$ and $x_0 \ge max[u]^r$

$\Rightarrow x_0 \in \underline{[u]}_{x_0\, r\le 1}$

Conversely, if $\underline{[u]}_{x_0\, r\le 1} \ne \varphi$ then $x \in \underline{[u]}_{x_0\, r\le 1}$

By definition $x \in \underline{[u]}_{r\le 1}$ and $x \le x_0$

$\Rightarrow x \in [u]^0$ and $x \ge \bar{u}(r)$ and $x \le x_0$

$\Rightarrow x \in [u]^0$ and $\bar{u}(r) \le x \le x_0$

$\Rightarrow x_0 \in \underline{[u]}_{r\le 1}$

(ii) $x_0 \in [\bar{u}]_{r\leq 1} \Leftrightarrow \overline{[u]}^{x_0}_{r\leq 1} \neq \varphi$

By definition $\overline{[u]}^{x_0}_{r\leq 1} = \{ x \in [\bar{u}]_{r\leq 1} : x \geq x_0 \}$

$[\bar{u}]_{r\leq 1} = \{ x \in [u]^0 / x \leq \underline{u}(r) \}$

Suppose $x_0 \in [\bar{u}]_{r\leq 1}$

$\Rightarrow x_0 \in [u]^0$ and $x_0 \leq \bar{u}(r)$

$\Rightarrow u(x_0) > 0$ and $x_0 \leq min[u]^r$

$\Rightarrow u(x_0) \leq 1$ and $x_0 \leq min[u]^r$

$\Rightarrow x_0 \in \overline{[u]}^{x_0}_{r\leq 1}$

Conversely, if $\overline{[u]}^{x_0}_{r\leq 1} \neq \varphi$ then $x \in \overline{[u]}^{x_0}_{r\leq 1}$

So $x \in [\bar{u}]_{r\leq 1}$ and $x \geq x_0$

$\Rightarrow x \in [u]^0$ and $x \leq \bar{u}(r)$ and $x \geq x_0$

$\Rightarrow x \in [u]^0$ and $\bar{u}(r) \geq x \geq x_0$

$\Rightarrow x_0 \in [\bar{u}]_{r\leq 1}$

(iii) $x_0 \in [\underline{u}(r), \bar{u}(r)]_{r\leq 1} \Leftrightarrow [\underline{u}(r), \bar{u}(r)]^{x_0}_{r\leq 1} \neq \varphi$

By definition

$[\underline{u}(r), \bar{u}(r)]^{x_0}_{r\leq 1} = \{ [\underline{u}(r), \bar{u}(r)]_{r\leq 1} : \underline{u}(r) \leq x_0 \leq \bar{u}(r) \}$

$[\underline{u}(r), \bar{u}(r)]_{r\leq 1} = \left\{ x \in [u]^0 / \underline{u}(r) \leq x \leq \bar{u}(r) \right\}$

Suppose $x_0 \in [\underline{u}(r), \bar{u}(r)]_{r\leq 1}$

$\Rightarrow x_0 \in [u]^0$ and $\underline{u}(r) \leq x \leq \bar{u}(r)$

$\Rightarrow u(x_0) > 0$ and $min[u]^r \leq x \leq max[u]^r$

$\Rightarrow u(x_0) \leq 1$ and $min[u]^r \leq x \leq max[u]^r$

$\Rightarrow x_0 \in [\underline{u}(r), \bar{u}(r)]^{x_0}_{r\leq 1}$

Conversely,if $[\underline{u}(r), \bar{u}(r)]^{x_0}_{r\leq 1} \neq \varphi$ then $x \in [\underline{u}(r), \bar{u}(r)]^{x_0}_{r\leq 1}$

So $x \in [\underline{u}(r), \bar{u}(r)]_{r\leq 1}$ and $\bar{u}(r) \leq x_0 \leq \underline{u}(r)$

$\Rightarrow x \in [u]^0$ and $\bar{u}(r) \leq x \leq \underline{u}(r)$ and $\bar{u}(r) \leq x_0 \leq \underline{u}(r)$

$\Rightarrow x \in [u]^0$ and $\bar{u}(r) \leq x_0 \leq \underline{u}(r)$

$\Rightarrow x_0 \in [\underline{u}(r), \bar{u}(r)]_{r\leq 1}$

Property3.4 Let u be a fuzzy set of R, and $[u]^0$ be finite then if $x^{/}, x^{//} \in [u]^0$, the following condition holds.

(i) $x^{/} \leq x^{//} \Rightarrow \underline{[u]}_{x^{//}\, r\leq 1} \subset \underline{[u]}_{x^{/}\, r\leq 1}$

(ii) $x^{/} \leq x^{//} \Rightarrow \overline{[u]}^{x^{/}}_{r\leq 1} \subset \overline{[u]}^{x^{//}}_{r\leq 1}$

(iii) $x^{/} \leq x^{//} \Rightarrow [\underline{u}(r), \bar{u}(r)]^{x^{/}}_{r\leq 1} \subset [\underline{u}(r), \bar{u}(r)]^{x^{//}}_{r\leq 1}$

Proof

(i) By definition $\underline{[u]}_{x^{//}}$ $_{r\leq 1}$ = { $x \in \underline{[u]}_{r\leq 1} : x \leq x^{//}$ }

$\underline{[u]}_{r\leq 1} = \left\{ x \in [u]^0 \,/\, x \geq \bar{u}(r) \right\}$

Given $x^{/} \leq x^{//}$

$\Rightarrow x_0 \in \underline{[u]}_{x^{//}}$ $_{r\leq 1}$ = { $x \in \underline{[u]}_{r\leq 1} : x \leq x^{//}$ }

$\Rightarrow x_0 \in \underline{[u]}_{r\leq 1}$ and $x_0 \leq x^{//}$

$\Rightarrow x_0 \in \underline{[u]}_{r\leq 1}$ and $x_0 \leq x^{/} \leq x^{//}$

$\Rightarrow x_0 \in \underline{[u]}_{r\leq 1}$ and $x_0 \leq x^{/}$

$\Rightarrow x_0 \in \underline{[u]}_{x^{/}}$ $_{r\leq 1}$

(ii) By definition $\overline{[u]}^{x^{/}}$ $_{r\leq 1}$ = { $x \in \overline{[u]}_{r\leq 1} : x \geq x^{/}$ }

$\overline{[u]}_{r\leq 1} = \left\{ x \in [u]^0 \,/\, x \leq \underline{u}(r) \right\}$

Given $x^{/} \leq x^{//}$

$\Rightarrow x_0 \in \overline{[u]}^{x^{/}}$ $_{r\leq 1}$ = { $x \in \overline{[u]}_{r\leq 1} : x \geq x^{/}$ }

$\Rightarrow x_0 \in \overline{[u]}_{r\leq 1}$ and $x_0 \geq x^{/}$

$\Rightarrow x_0 \in \overline{[u]}_{r\leq 1}$ and $x_0 \geq x^{/} \geq x^{//}$

$\Rightarrow x_0 \in \overline{[u]}_{r\leq 1}$ and $x_0 \geq x^{//}$

$\Rightarrow x_0 \in \overline{[u]}^{x^{//}}$ $_{r\leq 1}$

(iii) By definition $[\underline{u}(r), \bar{u}(r)]_{r\leq 1}^{x^{/}}$ = { $x \in [\underline{u}(r), \bar{u}(r)]_{r\leq 1} : \underline{u}(r) \leq x^{/} \leq \bar{u}(r)$ }

$[\underline{u}(r), \bar{u}(r)]_{r\leq 1} = \left\{ x \in [u]^0 \,/\, \underline{u}(r) \leq x \leq \bar{u}(r) \right\}$

Given $x^{/} \leq x^{//}$

$\Rightarrow x_0 \in [\underline{u}(r), \bar{u}(r)]_{r\leq 1}^{x^{/}}$ = { $x \in [\underline{u}(r), \bar{u}(r)]_{r\leq 1} : \underline{u}(r) \leq x^{/} \leq \bar{u}(r)$ }

$\Rightarrow x_0 \in [\underline{u}(r), \bar{u}(r)]_{r\leq 1}$ and $\bar{u}(r) \leq x_0 \leq \underline{u}(r)$

$\Rightarrow x_0 \in [\underline{u}(r), \bar{u}(r)]_{r\leq 1}$ and $\bar{u}(r) \leq x^{//} \leq \underline{u}(r)$

$\Rightarrow x_0 \in [\underline{u}(r), \bar{u}(r)]_{r\leq 1}^{x^{//}}$

Property3.5 Let $u,v \in$ DFN then

(i) { $x \in [u]^0 + [v]^0 : \varphi \neq \underline{[u+v]}_{x_0}$ $_{r\leq 1} \subset [u+v]^r$ }

$= \{ x \in [u]^r + [v]^r : \varphi \neq \underline{[u+v]}_{x_0}$ $_{r\leq 1} \subset [u+v]^r \}$

(ii) { $x \in [u]^0 + [v]^0 : \varphi \neq \overline{[u+v]}^{x_0}$ $_{r\leq 1} \subset [u+v]^r$ }

$= \{ x \in [u]^r + [v]^r : \varphi \neq \overline{[u+v]}^{x_0}$ $_{r\leq 1} \subset [u+v]^r \}$

(iii) { $x \in [u]^0 + [v]^0 : \varphi \neq [\underline{u}(r), \bar{u}(r)]_{r\leq 1}^{x_0} + [\underline{v}(r), \bar{v}(r)]_{r\leq 1}^{x_0} \subset [u+v]^r$ }

$=\{x\in [u]^r+[v]^r : \varphi\neq[\underline{u}(r),\bar{u}(r)]_{r\leq 1}^{x_0}+[\underline{v}(r),\bar{v}(r)]_{r\leq 1}^{x_0}\subset [u+v]^r\}$

Proof

(i) Let $x_0\in [u]^0+[v]^0$ where $x_0=y_0+z_0$ where $y_0\in [u]^0, z_0\in[v]^0$

$x_0\in\underline{[u+v]}_x$ $_{r\leq 1}$ $\Leftrightarrow x_0\in\underline{[u+v]}$ $_{r\leq 1}$ and $x_0\leq x$

$\Leftrightarrow x_0\in [u+v]^0, x_0\geq \overline{[u+v]}(r)$

$\Leftrightarrow x_0\in [u+v]^r, x_0\geq max[u+v]^r$

$\Leftrightarrow x_0\in [u]^r+[v]^r, x_0\geq\overline{[u+v]}(r)$ since $[u+v]^r=[u]^r+[v]^r$

(ii) Let $x_0\in [u]^0+[v]^0$ where $x_0=y_0+z_0$ where $y_0\in [u]^0, z_0\in[v]^0$

$x_0\in\overline{[u+v]}^x$ $_{r\leq 1}$ $\Leftrightarrow x_0\in\overline{[u+v]}$ $_{r\leq 1}$ and $x_0\geq x$

$\Leftrightarrow x_0\in [u+v]^0, x_0\leq \overline{[u+v]}(r)$

$\Leftrightarrow x_0\in [u+v]^r, x_0\leq min\,[u+v]^r$

$\Leftrightarrow x_0\in [u]^r+[v]^r, x_0\leq\underline{[u+v]}(r)$ since $[u+v]^r=[u]^r+[v]^r$

(iii) Let $x_0\in [u]^0+[v]^0$ where $x_0=y_0+z_0$ where $y_0\in [u]^0, z_0\in[v]^0$

$x_0\in[\underline{u}(r),\bar{u}(r)]_{r\leq 1}^{x}+[\underline{v}(r),\bar{v}(r)]_{r\leq 1}^{x}$

$\Leftrightarrow x_0\in [\underline{u}(r),\bar{u}(r)]_{r\leq 1}+[\underline{v}(r),\bar{v}(r)]_{r\leq 1}$ and $\underline{u}(r)+\underline{v}(r)\leq x_0\leq\bar{u}(r)+\bar{v}(r)$

$\Leftrightarrow x_0\in [u+v]^0$ and $min[u+v]^r\leq x_0\leq max[u+v]^r$

$\Leftrightarrow x_0\in [u+v]^r$ and $\underline{[u+v]}^r\leq x_0\leq \overline{[u+v]}^r$

$\Leftrightarrow x_0\in [u]^r+[v]^r$, and $\underline{[u+v]}(r)\leq x_0\leq \overline{[u+v]}(r)$, since $[u+v]^r=[u]^r+[v]^r$

Property3.6 Let $u,v\in$ DFN **,**then

(i) $\{x\in [u]^0[v]^0$ suchthat $\varphi\neq\underline{[uv]}_{x_0}$ $_{r\leq 1}\subset [uv]^r\}$

$=\{x\in[u]^r[v]^r$ suchthat $\varphi\neq\underline{[uv]}_{x_0}$ $_{r\leq 1}\subset [uv]^r\}$

(ii) $\{x\in [u]^0[v]^0$ suchthat $\varphi\neq\overline{[uv]}^{x_0}$ $_{r\leq 1}\subset [uv]^r\}$

$=\{x\in[u]^r[v]^r$ suchthat $\varphi\neq\overline{[uv]}^{x_0}$ $_{r\leq 1}\subset [uv]^r\}$

(iii) $\{x\in [u]^0[v]^0$ suchthat $\varphi\neq[\underline{u}(r),\bar{u}(r)]_{r\leq 1}^{x_0}[\underline{v}(r),\bar{v}(r)]_{r\leq 1}^{x_0}\subset [uv]^r\}$

$=\{x\in[u]^r+[v]^r$ suchthat $\varphi\neq[\underline{u}(r),\bar{u}(r)]_{r\leq 1}^{x_0}[\underline{v}(r),\bar{v}(r)]_{r\leq 1}^{x_0}\subset [uv]^r\}$

Proof

(i) Let $x_0\in [u]^0+[v]^0$ where $x_0=y_0+z_0$ where $y_0\in [u]^0, z_0\in[v]^0$

$x_0\in\underline{[uv]}_x$ $_{r\leq 1}\Leftrightarrow x_0\in\underline{[uv]}$ $_{r\leq 1}$ and $x_0\leq x$

$\Leftrightarrow x_0 \in [uv]^0$, $x_0 \geq [\overline{uv}](r)$

$\Leftrightarrow x_0 \in [uv]^r$, $x_0 \geq max[uv]^r$

$\Leftrightarrow x_0 \in [u]^r [v]^r$, $x_0 \geq [\overline{uv}]^r$ since $[uv]^r = [u]^r [v]^r$

(ii) $x_0 \in \overline{[uv]}^x_{r \leq 1} \Leftrightarrow x_0 \in \overline{[uv]}_{r \leq 1}$ and $x_0 \geq x$

$\Leftrightarrow x_0 \in [uv]^0$, $x_0 \leq [\underline{uv}](r)$

$\Leftrightarrow x_0 \in [uv]^r$, $x_0 \leq min[uv]^r$

$\Leftrightarrow x_0 \in [u]^r [v]^r$, $x_0 \leq [\underline{uv}]^r$ since $[uv]^r = [u]^r [v]^r$

(iii) $x_0 \in [\underline{u}(r), \overline{u}(r)]^x_{r\leq 1} [\underline{v}(r), \overline{v}(r)]^x_{r\leq 1}$

$\Leftrightarrow x_0 \in [\underline{u}(r), \overline{u}(r)]_{r\leq 1} [\underline{v}(r), \overline{v}(r)]_{r\leq 1}$ and $\underline{u}(r)\underline{v}(r) \leq x_0 \leq \overline{u}(r)\overline{v}(r)$

$\Leftrightarrow x_0 \in [uv]^r$, $x_0 \leq min[uv]^r \leq x_0 \leq max[uv]^r$

$\Leftrightarrow x_0 \in [uv]^r$, $[\underline{uv}]^r \leq x_0 \leq [\overline{uv}]^r$

$\Leftrightarrow x_0 \in [u]^r [v]^r$, $[\underline{uv}]^r \leq x_0 \leq [\overline{uv}]^r$, since $[uv]^r = [u]^r [v]^r$

Definition 3.7 Let $u,v \in$ DFN, $r \in [0,1]$,define addition " $\tilde{+}$ " of $[u]^r$ and $[v]^r$ as follows

$[u]^r \tilde{+} [v]^r = \{x \in [u]^r + [v]^r : \varphi \neq \underline{[u+v]}_{x_0}\ {}_{r \leq 1} \subset [u+v]^r\}$

$\cup \{x \in [u]^r + [v]^r : \varphi \neq \overline{[u+v]}^{x_0}\ {}_{r \leq 1} \subset [u+v]^r\}$

$\cup \{x \in [u]^r + [v]^r : \varphi \neq [\underline{u}(r), \overline{u}(r)]^{x_0}_{r\leq 1} + [\underline{v}(r), \overline{v}(r)]^{x_0}_{r\leq 1} \subset [u+v]^r\}$

Definition 3.8 Let $u,v \in$ DFN, $r \in [0,1]$,define multiplication " $\tilde{\times}$ " of $[u]^r$ and $[v]^r$ as follows

$[u]^r \tilde{\times} [v]^r = \{x \in [u]^r [v]^r : \varphi \neq \underline{[uv]}_{x_0}\ {}_{r \leq 1} \subset [uv]^r\}$

$\cup \{x \in [u]^r [v]^r : \varphi \neq \overline{[uv]}^{x_0}\ {}_{r \leq 1} \subset [uv]^r\}$

$\cup \{x \in [u]^r [v]^r : \varphi \neq [\underline{u}(r), \overline{u}(r)]^{x_0}_{r\leq 1} [\underline{v}(r), \overline{v}(r)]^{x_0}_{r\leq 1} \subset [uv]^r\}$

Theorem3.9 Let $u,v \in$DFN and $A_r = [u]^r \tilde{+} [v]^r$ for any $r \in [0,1]$,A_r is a finite set and satisfying the conditions.

(i) There exist $x_r, y_r, (x_r, y_r) \in A_0$ with $x_r \leq (x_r, y_r) \leq y_r$ such that $A_r = \{ z \in A_0 : x_r \leq z \leq y_r \}$

(ii) $A_{r_2} \subset A_{r_1}$ for any $r_1, r_2 \in [0,1]$ with $0 \leq r_1 \leq r_2 \leq 1$

(iii) For any $r_0 \in [0,1]$,There exists real number $r_0^/$ with $0 < r_0^/ < r_0$ such that $A_{r_2} = A_{r_1}$ (i.e. $A_r = A_{r_1}$ for any $r \in [r_0^/, r_0]$)

then there exists a unique $\omega \in$ DFN such that $[\omega]^r = [u]^r \tilde{+} [v]^r$ for any $r \in [0,1]$.

Proof

For any $r \in [0,1]$,it is clear that $[u]^r$, $[v]^r$ are finite . since $u,v \in$DFN and by the definition of $[u]^r \tilde{+} [v]^r$

It is obvious that A_r is a finite set. It asserts to show that $[u]^r \tilde{+} [v]^r$ satisfies condition (i)

Let $r \in [0,1]$, $x_r = min([u]^r \tilde{+} [v]^r)$, $y_r = max([u]^r \tilde{+} [v]^r)$

$(x_r, y_r) = [\underline{u}(r), \bar{u}(r)] \tilde{+} [\underline{v}(r), \bar{v}(r)]$ then it is obvious that

$x_r, y_r, (x_r, y_r) \in A_0$ and $x_r \leq (x_r, y_r) \leq y_r$ By the definition of $[u]^r \tilde{+} [v]^r$ and by property 3.2,

$$x_r \in \{x \in [u]^r + [v]^r : \varphi \neq \underline{[u+v]}_{x_0}\ {}_{r \leq 1} \subset [u+v]^r\}$$

$$\text{and } y_r \in \{x \in [u]^r + [v]^r : \varphi \neq \overline{[u+v]}^{x_0}\ {}_{r \leq 1} \subset [u+v]^r\}$$

$$\text{and } (x_r, y_r) \in \{x \in [u]^r + [v]^r : \varphi \neq [\underline{u}(r), \bar{u}(r)]_{r\leq 1}^{x_0} + [\underline{v}(r), \bar{v}(r)]_{r\leq 1}^{x_0} \subset [u+v]^r\}$$

so $\underline{[u+v]}_{x_r}\ {}_{r \leq 1} \subset [u+v]^r$, $\overline{[u+v]}^{y_r}\ {}_{r \leq 1} \subset [u+v]^r$ and $[u+v]^{(x_r, y_r)}\ {}_{r \leq 1} \subset [u+v]^r\}$

Now to show that $\{z \in [u]^0 \tilde{+} [v]^0$ such that $x_r \leq z \leq y_r\} = [u]^r \tilde{+} [v]^r$

Let $x' \in \{z \in [u]^0 \tilde{+} [v]^0$ such that $x_r \leq z \leq y_r\}$

then $x' \in [u]^0 \tilde{+} [v]^0 = [u+v]^0$ and such that $x_r \leq x' \leq y_r$

By property 3.2, $x' \in \underline{[u+v]}\ {}_{r \leq 1}$ (or) $x' \in \overline{[u+v]}\ {}_{r \leq 1}$(or) $x' \in [\underline{u}(r), \bar{u}(r)]_{r\leq 1} + [\underline{v}(r), \bar{v}(r)]_{r\leq 1}$

By property 3.3, $\underline{[u+v]}_{x'}\ {}_{r \leq 1} \neq \varphi$(or) $\overline{[u+v]}^{x'}\ {}_{r \leq 1} \neq \varphi$ (or) $[\underline{u}(r), \bar{u}(r)]_{r\leq 1}^{x'} + [\underline{v}(r), \bar{v}(r)]_{r\leq 1}^{x'} \neq \varphi$

In addition by property 3.4it is clear that

$\underline{[u+v]}_{x'}\ {}_{r \leq 1} \subset \underline{[u+v]}_{x_r}\ {}_{r \leq 1}$, $\overline{[u+v]}^{x'}\ {}_{r \leq 1} \subset \overline{[u+v]}^{y_r}\ {}_{r \leq 1}$

Also $[\underline{u}(r), \bar{u}(r)]_{r\leq 1}^{x'} + [\underline{v}(r), \bar{v}(r)]_{r\leq 1}^{x'} \subset [u+v]^{(x_r, y_r)}\ {}_{r \leq 1}$

$\underline{[u+v]}_{x'}\ {}_{r \leq 1} \subset [u+v]^r$, $\overline{[u+v]}^{x'}\ {}_{r \leq 1} \subset [u+v]^r$ and $[\underline{u}(r), \bar{u}(r)]_{r\leq 1}^{x'} + [\underline{v}(r), \bar{v}(r)]_{r\leq 1}^{x'} \subset [u+v]^r$

Therefore by the property3.5,

$x' \in \{x \in [u]^r + [v]^r : \varphi \neq \underline{[u+v]}_{x}\ {}_{r \leq 1} \subset [u+v]^r\}$

(or) $x' \in \{x \in [u]^r + [v]^r : \varphi \neq \overline{[u+v]}^{x}\ {}_{r \leq 1} \subset [u+v]^r\}$

(or) $x' \in \{x \in [u]^r + [v]^r : \varphi \neq [\underline{u}(r), \bar{u}(r)]_{r\leq 1}^{x} + [\underline{v}(r), \bar{v}(r)]_{r\leq 1}^{x} \subset [u+v]^r\}$

so $x' \in \{[u]^r + [v]^r\}$

On the other hand, if $\tilde{x} \in [u]^r \tilde{+} [v]^r$ it asserts that $x_r \leq \tilde{x} \leq y_r$ by the definition of x_r and y_r

it isclear that $\tilde{x} \in [u]^0 + [v]^0 = [u]^0 \tilde{+} [v]^0$ by the definition of $[u]^r \tilde{+} [v]^r$

So $\tilde{x} \in \{z \in [u]^0 \tilde{+} [v]^0$ such that $x_r \leq \tilde{x} \leq y_r\}$

$\Rightarrow \{z \in [u]^0 \tilde{+} [v]^0$ such that $x_r \leq \tilde{x} \leq y_r\} = [u]^r \tilde{+} [v]^r$

So $[u]^r \tilde{+} [v]^r$ satisfies the condition (i)

(ii) To prove $[u]^{r_2} \tilde{+} [v]^{r_2} \subset [u]^{r_1} \tilde{+} [v]^{r_1}$ holds for any $r_1, r_2 \in [0,1]$with $0 \leq r_1 \leq r_2 \leq 1$

Let $x' \in [u]^{r_2} \tilde{+} [v]^{r_2}$ by the definition of $[u]^{r_2} \tilde{+} [v]^{r_2}$

$x' \in \{x \in [u]^{r_2} + [v]^{r_2} : \varphi \neq \underline{[u+v]}_{x}\ {}_{r \leq 1} \subset [u+v]^{r_2}\}$

(or) $x^{/} \in \{x \in [u]^{r_2} + [v]^{r_2} : \varphi \neq \overline{[u+v]}^{x}{}_{r \le 1} \subset [u+v]^{r_2}\}$

(or) $x^{/} \in \{ x \in [u]^{r_2} + [v]^{r_2} : \varphi \neq [\underline{u}(r), \bar{u}(r)]^{x}_{r \le 1} + [\underline{v}(r), \bar{v}(r)]^{x}_{r \le 1} \subset [u+v]^{r_2} \}$

That is $x^{/} \in [u]^{r_2} + [v]^{r_2}$, $\underline{[u+v]}_{x^{/}}{}_{r \le 1} \neq \varphi, \underline{[u+v]}_{x^{/}}{}_{r \le 1} \subset [u+v]^{r_2}$

(or) $x^{/} \in [u]^{r_2} + [v]^{r_2}$, $\overline{[u+v]}^{x^{/}}{}_{r \le 1} \neq \varphi, \overline{[u+v]}^{x^{/}}{}_{r \le 1} \subset [u+v]^{r_2}$

(or) $x^{/} \in [u]^{r_2} + [v]^{r_2}$, $[\underline{u}(r), \bar{u}(r)]^{x^{/}}_{r \le 1} + [\underline{v}(r), \bar{v}(r)]^{x^{/}}_{r \le 1} \neq \varphi,$

$[\underline{u}(r), \bar{u}(r)]^{x^{/}}_{r \le 1} + [\underline{v}(r), \bar{v}(r)]^{x^{/}}_{r \le 1} \subset [u+v]^{r_2}$

Since $[u]^{r_2} \subset [u]^{r_1}$ and $[v]^{r_2} \subset [v]^{r_1}$ it is clear that $[u]^{r_2} + [v]^{r_2} \subset [u]^{r_1} + [v]^{r_1}$

So $x^{/} \in [u]^{r_1} + [v]^{r_1}$, $\underline{[u+v]}_{x^{/}}{}_{r \le 1} \neq \varphi, \underline{[u+v]}_{x^{/}}{}_{r \le 1} \subset [u+v]^{r_1}$

(or) $x^{/} \in [u]^{r_1} + [v]^{r_1}$, $\overline{[u+v]}^{x^{/}}{}_{r \le 1} \neq \varphi, \overline{[u+v]}^{x^{/}}{}_{r \le 1} \subset [u+v]^{r_1}$

(or) $x^{/} \in [u]^{r_1} + [v]^{r_1}$, $[\underline{u}(r), \bar{u}(r)]^{x^{/}}_{r \le 1} + [\underline{v}(r), \bar{v}(r)]^{x^{/}}_{r \le 1} \neq \varphi,$

$[\underline{u}(r), \bar{u}(r)]^{x^{/}}_{r \le 1} + [\underline{v}(r), \bar{v}(r)]^{x^{/}}_{r \le 1} \subset [u+v]^{r_1}$

(ie) $x^{/} \in \{ x \in [u]^{r_1} + [v]^{r_1} : \varphi \neq \underline{[u+v]}_{x}{}_{r \le 1} \subset [u+v]^{r_1} \}$

(or) $x^{/} \in \{ x \in [u]^{r_1} + [v]^{r_1} : \varphi \neq \overline{[u+v]}^{x}{}_{r \le 1} \subset [u+v]^{r_1} \}$

(or) $x^{/} \in \{ x \in [u]^{r_1} + [v]^{r_1} : \varphi \neq [\underline{u}(r), \bar{u}(r)]^{x}_{r \le 1} + [\underline{v}(r), \bar{v}(r)]^{x}_{r \le 1} \subset [u+v]^{r_1} \}$

Therefore $x^{/} \in [u]^{r_1} + [v]^{r_1}$ so $[u]^{r_2} \tilde{+} [v]^{r_2} \subset [u]^{r_1} \tilde{+} [v]^{r_1}$ holds.

Now to show that $[u]^{r} \tilde{+} [v]^{r}$ also satisfies condition(iii)

Let $r_0 \in [0,1]$ from $u,v \in$ DFN then there exists $r_0^{/} \in \mathcal{R}$ with $0 < r_0^{/} < r_0$

such that $[u]^{r} = [u]^{r_0}$ and $[v]^{r} = [v]^{r_0}$ so $[u]^{r} + [v]^{r} = [u]^{r_0} + [v]^{r_0}$ holds.

Therefore $[u]^{r} \tilde{+} [v]^{r} = \{ x \in [u+v]^{0} : [u+v]^{r} \supset \underline{[u+v]}_{x}{}_{r \le 1} \neq \varphi\}$

$\bigcup \{ x \in [u+v]^{0} : [u+v]^{r} \supset \overline{[u+v]}^{x}{}_{r \le 1} \neq \varphi\}$

$\bigcup \{ x \in [u+v]^{0} : [u+v]^{r} \supset [\underline{u}(r), \bar{u}(r)]^{x}_{r \le 1} + [\underline{v}(r), \bar{v}(r)]^{x}_{r \le 1}{}_{r \le 1} \neq \varphi\}$

$= \{ x \in [u+v]^{0} : [u+v]^{r_0} \supset \underline{[u+v]}_{x}{}_{r \le 1} \neq \varphi\}$

$\bigcup \{ x \in [u+v]^{0} : [u+v]^{r_0} \supset \overline{[u+v]}^{x}{}_{r \le 1} \neq \varphi\}$

$\bigcup \{ x \in [u+v]^{0} : [u+v]^{r_0} \supset [\underline{u}(r), \bar{u}(r)]^{x}_{r \le 1} + [\underline{v}(r), \bar{v}(r)]^{x}_{r \le 1}{}_{r \le 1} \neq \varphi\}$

$= [u]^{r_0} \tilde{+} [v]^{r_0}$

(ie) $[u]^{r} \tilde{+} [v]^{r}$ also satisfies the condition (iii) there fore there exists a unique $\omega \in$ DFN such that $[\omega]^{r} = [u]^{r} \tilde{+} [v]^{r}$ for any $r \in [0,1]$.

Hence the proof.

Theorem3.10 Let *u*, *v* ∈DFN and A_r= $[u]^r \tilde{\times} [v]^r$ for any *r* ∈ [0,1],A_r is a finite set and satisfy the conditions.

(i) There exist x_r , y_r ,(x_r , y_r) ∈ A_0 with $x_r \leq (x_r , y_r) \leq y_r$ such thatA_r = { $z \in A_0 : x_r \leq z \leq y_r$ }

(ii) $A_{r_2} \subset A_{r_1}$ for any r_1,r_2 ∈ [0,1] with $0 \leq r_1 \leq r_2 \leq 1$

(iii) For any r_0∈ [0,1],There exists real number $r_0^{/}$ with $0 < r_0^{/} < r_0$ such that $A_{r_2} = A_{r_1}$ (i.e. $A_r = A_{r_1}$ for any $r \in [r_0^{/} , r_0]$)

then there exists a unique ω∈DFN such that $[\omega]^r = [u]^r \tilde{\times} [v]^r$ for any *r* ∈ [0,1].

Proof: Owing to the similarity with the proof of theorem 3.9, we omit it.

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