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RESEARCH ARTICLE

ON BILATERAL 2 ψ_2 SUMMATION FORMULA AND SOME THETA FUNCTION IDENTITIES

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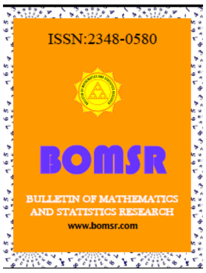

ABSTRACT

In this paper, a proof of new summation formula for $2\psi_2$ bilateral basic hypergeometric series by means of Abel's lemma on summation by parts established and also deduced sums of squares, some theta function identities, partition theoretic interpretation of identities and some identities of q - Beta and q - Gamma functions.

Keywords: Abel's lemma on summation by parts, sums of squares, theta functions

2010 Mathematics Subject Classification: Primary 33D15, Secondary 40G10,11P05..

1.Introduction

In this paper, an extension of S. Ramanujan's 1 ψ_1 bilateral basic hypergeometric summation formula (1.1) was established, in nice form. The motivation primarily come from unfulfilled claim of W. N. Bailey [4] to get $2\psi_2$bilateral basic hypergeometric summation formula. G. E. Andrews [2] has given $2\psi_2$ summation formula.

$$a^{-1}\sum_{k=0}^{\infty}\frac{\left({}^{-q}/_{a},{}^{cd}/_{ab}\right)_k}{\left({}^{-c}/_{a},{}^{-d}/_{a}\right)_{k+1}}(-b)^k-b^{-1}\sum_{k=0}^{\infty}\frac{\left({}^{-q}/_{b},{}^{cb}/_{ab}\right)_k}{\left({}^{-c}/_{b},{}^{-d}/_{b}\right)_{k+1}}(-a)^k=$$

$$(a^{-1}-b^{-1})\frac{\left(q,{}^{aq}/_{b},{}^{q,bq}/_{a}{}^{c,d,cd}/_{ab}\right)_\infty}{\left(-a,-b,{}^{-c}/_{a},{}^{-c}/_{b},{}^{-d}/_{a}{}^{-d}/_{b}\right)_\infty}\quad(1.1)$$

which is considered as an extension of Ramanujan's $1\psi_1$ summation formula (1.1).

We shall follow the notation and terminology in [8], and throughout this paper, we assume | q | < 1. The q-shifted factorial is defined by

$(a;\ q)_0 := 1$

$(a)_\infty := (a;\ q)_\infty = \prod_{K=0}^{\infty} (1 - aq^k)$,

$(a)_k := (a;\ q)_k = \frac{(a)_\infty}{(aq^k)_\infty}$, k is an integer.

The bilateral basic hypergeometric series is defined by

$$r\psi_s\left(\begin{matrix} a_1, & a_2, \dots, & a_r, \\ b_1, & b_2, \dots, & b_r, \end{matrix}\middle| q; x\right) = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n, (a_2)_n \dots, (a_r)_n,}{(b_1)_n, (b_2)_n \dots, (b_s)_n,} x^n.$$

In his 'lost' notebook [?], Ramanujan has given 'remarkable' 1 ψ_1 summation formula:

$$1\psi_1\left(\begin{matrix} a \\ b \end{matrix}\middle| q; z\right) = \sum_{k=-\infty}^{\infty} \frac{(a)_k}{(b)_k} z^k = \frac{(az)_\infty (q)_\infty (q/az)_\infty (b/a)_\infty}{(z)_\infty (b)_\infty (b/az)_\infty (q/a)_\infty}, \qquad \textbf{(1.2)}$$

where $|b/a| < |z| < 1$.

There are many of proofs of summation formula (1.1), for more details one may refer the book by B. C. Berndt [5]. Further, (1.1) has been influential in the development of Ramanujan's theory of theta and elliptic functions.

In recent paper, Somashekara, Narasimha murthy, and Shalini [12], the authors have given a new bilateral summation formula for 2ψ2 hyper geometric series.

$$2\psi_2\left(\begin{matrix} a & bc/azq \\ b & c \end{matrix} | q; z\right) = \frac{(az)_\infty (q)_\infty (q/az)_\infty (b/a)_\infty (c/a)_\infty (bc/azq)_\infty}{(z)_\infty (b)_\infty (c)_\infty (b/az)_\infty (c/az)_\infty (q/a)_\infty}, \qquad \textbf{(1.3)}$$

Where $max\left\{\left|\frac{b}{c}\right|, \left|\frac{c}{a}\right|\right\} < |z| < 1,\ |q| < 1,$

and its applications, was established on using well-known Ramanujan's 1ψ1 summation formula and the method of parameter augmentation and some applications to obtain q-gamma, q-beta function identities, eta-function identities and partition theoretic identities.

In the chapter 16 of his notebook [10], Ramanujan defines the general theta function

$f\ (a, b) := \sum_{n=-\infty}^{\infty} a^{(n+1)/2} b^{n(n-1)/2}$,

The special cases of *f (a, b)* are

$\varphi(q) :=\ f\ (q, q)\ = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}$, (1.4)

$\psi(q) :=\ f\ (q, q^3)\ = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)^2_\infty}{(q;q)_\infty}$, (1.5)

$f\ (-q) := f\ (-q, q^2) = \sum_{n=-\infty}^{\infty} (-1)^n\ q^{n(3n-1)/2} = (q; q)_\infty$, (1.6)

Ramanujan also defined the function χ(q) as

$\chi(q) = (q\ ; q^2)_\infty = \frac{(q^2;q^2)^2_\infty}{(q;q)_\infty (q^4;q^4)^2_\infty}$,**(1.7)**

Using elementary q-analysis, one can easily verify

$\psi(-q) = \frac{(q;q)_\infty}{(q^2;q^2)_\infty (q^4;q^4)_\infty}$,**(1.8)**

$\chi(-q) = \frac{(q;q)^2_\infty}{(q^2;q^2)_\infty}$, **(1.9)**

In his paper [7], W. Chu has shown how the basic hypergeometric series identities can be studied systematically on apply of modified Abel's lemma on summation by parts, which in fact, was given by Norwegian mathematician N. H. Abel [1].

For an arbitrary complex sequence {z_k}, define the backward and for- ward differences operator Q and Δ respectively as

$\nabla z_k = z_k - z_{k-1}$ *and* $\Delta z_k - z_{k+1}$

Then Abel's lemma on summation by parts may be formulated as

$\sum_{k=-\infty}^{\infty} B_k\ \ \nabla A_k = \sum_{k=-\infty}^{\infty}\ \ A_k\ \Delta B_k$,**(1.10)**

provided that the series on both sides convergent and there exists limits,

$[AB]_{\pm} := \lim_{n\to\pm\infty} A_n B_{n+1}$

For the proof one may refer paper by Chu [7]. Further, the Dedekind eta-function is defined by

$$\eta(\tau) := e^{\pi i\tau/12}\prod_{k=0}^{\infty}\left(1 - e^{2\pi ik\tau}\right) := q^{1/24}(q;q)_\infty, \mathbf{(1.11)}$$

Where q = $e^{2\pi ik\tau}$ and Im(τ) > 0.

F.H.Jackson [9] defined the q - analogue of the gamma function by

$\Gamma\, q^{(x)} = \frac{(q;q)_\infty}{(q^x q)_\infty}(1-q)^{1-x}$ 0 < q < 1. **(1.12)**

In his paper [3] on the q-gamma and q-beta function, Askey has obtained q-analogues of several classical results about the gamma function. Further, he has given the definition for q-beta function as

$B\, q^{(x,y)} = (1-q)\sum_{n=-\infty}^{\infty}(q)^{nx}\frac{(q^{n+1})_\infty}{(q^{x+y})_\infty}$, **(1.13)**

$B\, q^{(x,y)} = \frac{\Gamma\, q^{(x)}\Gamma\, q^{(y)}}{\Gamma\, q^{(x*y)}}$ **,(1.14)**

In this paper, we give proof of (1.3), on using well-known Ramanujan's 1ψ1 summation formula and Abel's lemma on summation by parts. In Section 2, we prove the summation formula (1.3). In Section 3, we show application of (1.3), namely sums of squares and some theta function identities. In Section 4, some identities deduce from (1.3) and partition theoretic interpretation of identities. In Section 5, some identities of q - Beta and q - Gamma functions.

2.Proof of the summation formula (1.3)

Proof. Define the two sequences by

$$\boldsymbol{A_k := \frac{\left(bc/az\right)_k}{b_k}\frac{a^k z^k}{c^k}\ and\quad B_k := \frac{(a)_k}{(c)_k}\frac{c^k}{a^k}}.$$

We can easily find two extreme values

$[AB]_+ = [AB]_- = 0$ with $max\left\{\left|\frac{b}{c}\right|,\left|\frac{c}{a}\right|\right\} < |z| < 1$, and the finite differences

$\nabla A_k := \frac{(1-c/az)}{(1-bc/azq)}\frac{\left(bc/azq\right)_k}{b_k}\frac{a^k z^k}{c^k}$ and $\Delta B_k := \frac{(1-c/a)}{(1-c)}\frac{(a)_k}{(cq)_k}\frac{c^k}{a^k}$

By means of the modified Abel's lemma on summation by parts for bilateral series, we can manipulate the bilateral - series as

$$2\psi_2\begin{pmatrix} a & bc/azq \\ b & c \end{pmatrix}|q;z\Big) = \frac{(1-bc/azq)}{(1-c/az)}\sum_{k=-\infty}^{\infty} B_k\ \nabla A_k$$

$$= \frac{(1-bc/azq)}{(1-c/az)}\sum_{k=-\infty}^{\infty} A_k\ \Delta B_k$$

$$= \frac{(1-c/a)(1-bc/azq)}{(1-c)(1-c/az)}\sum_{k=-\infty}^{\infty}\frac{(a)_k\left(bc/az\right)_k}{(b)_k(cq)_k}z^k.$$

Iterating this n-times, we find recurrence relation

$$2\psi_2\left(\begin{matrix} a & bc/azq \\ b & c \end{matrix}\,|q;z\right) = \frac{(c/a)_n\left(bc/azq\right)_n}{(c)_n(c/a)_n}2\psi_2\left(\begin{matrix} a & bcq^{n-1}/az \\ b & cq^n \end{matrix}\,|q;z\right) \quad ,(2.1)$$

Letting n → ∞ in (2.1) and then applying (1.2), we obtain (1.3).

3 Some Application of (1.3)

Corrolary 1. if | q | < 1, then

$$1 + 2\sum_{n=1}^{\infty}\frac{(q;q^2)_n}{(1+q^{2n})(-q^2;q^2)_n}q^n = \varphi(q)\,,\quad (3.1)$$

$$1 + 2\sum_{n=1}^{\infty}\frac{(-q;-q^3)_n(-q^3;-q^3)_{n-1}}{(-q^2;-q^3)_n(q^3;-q^3)_n}q^n + 2\sum_{n=1}^{\infty}(-1)^n\frac{(-q;-q^3)_n(-q^3;-q^3)_{n-1}}{(q^2;-q^3)_n(q^3;-q^3)_n}q^{3n} = \varphi^2(q)\,,\quad .....(3.2)$$

$1+2\sum_{n=1}^{\infty}\frac{(q\,;q^2)_n}{(1+q^{2n})(-q^2;q^2)_n}q^n+4\sum_{n=1}^{\infty}\frac{(-q^2\,;q^2)_{n-1}}{(1+q^{2n})(-q\,;q^2)_n}q^{2n}\;=\varphi^3(q)$ ……..(3.3)

$\sum_{n=0}^{\infty}\frac{{(q\,;q^4)_n}^2}{(q^3;q^4)_n(q^4;q^4)_n}q^n=\;\chi(q)\chi(-q),$ (3.4)

$\sum_{n=0}^{\infty}\frac{{(q^2\,;q^6)_n}^2}{(q^5;q^6)_n(q^6;q^6)_n}q^n=\;\psi(q)\psi(-q),$ **(3.5)**

Proof. Putting *a = -1, z = q, b = -* q^2, *c =* q^2 and then changing *q* to q^2 in (1.3), and on some simplification using (1.4), we obtain (3.1).

Similarly putting a *= -1, z = q, b =* $-q^2$, *c =* q^3 and then changing q to q3 in (1.3), and on some simplification using (1.4), we obtain (3.2). Finally putting *a = 1, z = q, b = c = -* q^2, and then changing q to q^2 in (1.3), and on some simplification using (1.4), we obtain (3.3). Putting *a =* $q^{1/4}$ *= z, b = q, c =* $q^{3/4}$ and then changing *q to* q^4 in (1.3), using (1.7) and (1.9) with some q-analysis simplification, we obtain (3.4). Similarly, putting *a =* $q^{2/6}$, *z =* $q^{1/6,}$ *b = q, c =* $q^{5/6}$ and then changing q to q^6 in (1.3), using (1.5) and (1.8) with some q-analysis simplification, we obtain (3.5).

4 Some Partition Identities and theoretic interpretation

A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum equals *n*. For example, *n* = 3 has three partitions, namely,

3, 2 + 1, 1 + 1 + 1.

If *p(n)* denote the number of partitions of n, then *p(3) = 3*. The generating function for *p(n)* due to Euler is given by

$\sum_{n=0}^{\infty}p_r(n)q^n=\frac{1}{(q;q)_\infty}.$ ……(**4.1)**

Ramanujan [11] established following beautiful congruences for : $p_r(n)$

p(5n + 4) = 0 (mod5),

p(7n + 5) = 0 (mod7)

and

p(11n + 6) = 0 (mod11)

A part in a partition of *n* has *r* colours. For any positive integers *n* and *r,* let $p_r(n)$ denote the number of partitions of n where each part may have *r* distinct colours.

The generating function of $p_r(n)$ Berndt and Ranking [6] is given by

$\sum_{n=0}^{\infty}p_r(n)q^n=\frac{1}{(q;q)^r)_\infty}..$ **(4.2)**

For positive integers k, m and r

$\frac{1}{(q^k;q^m)_\infty^r}$ **(4.3)**

is the generating function of the number of partitions of a positive integer with parts congruent to k modulo m and each part has r colours. Similarly

$\frac{1}{(q^{k_1};q^m)_\infty^2(q^{k_2};q^m)_\infty^2}=\frac{1}{(q^{k_1},q^{k_2};q^m)_\infty^2},$ **(4.4)**

is the generating function of the number of partitions of positive integer with *parts =* k_1 *or* k_2 *(modm*) and each part has two colours.

Given a partition *π,* let *e(π)* denote the number of parts in *p*. Define $P_m(n)$ to be the set of partitions of n in which all parts are less than or equal to m. Let $q_m(n)$ be the number of partitions of *n* in which all parts are less than or equal to *m*. Define

$P_m(n)=\sum_{\pi\in p_m(n)}(-1)^{e(\pi).}$ **(4.5)**

$\sum_{n=0}^{\infty}p_m(n)q^n=\frac{1}{(-q;q)_m}..$ **(4.6)**

$\sum_{n=0}^{\infty}q_m(n)q^n=\frac{1}{(q;q)_m}..$ **(4.7)**

Define $P_o,m(n)$ to be the set of partitions of n in odd parts and all parts are less than or equal to $2m$. Let $q_{e,m}(n)$ be the number of partitions of n into even parts in which all parts are less than or equal to $2m$. Define

$P_{o,m}(n) = \sum_{\pi\in p_{o,m}(n)}(-1)^{e(\pi)}.$ **(4.8)**

so that

$\sum_{n=0}^{\infty} p_{o,m}(n)q^n = \frac{1}{(-q;q^2)_m}$ **(4.9)**

$\sum_{n=0}^{\infty} q_{e,m}(n)q^n = \frac{1}{(q^2;q^2)_m}$ **(4.10)**

We shall begin with the following definitions.

Define ${P_{s,t}}^k(n)$to be the number of partitions of n in which parts are only of the form *km + s or km + t*. ${P_{s,t}}^{k,O}(n)$ denote number of partitions of n in which parts are of *n* in the form *km + s and km + t* taken together appear odd number of times. ${P_{s,t}}^{k,\varepsilon}(n)$ denote number of partitions of n in which parts are of the form *km + s* and *km + t* taken together appear even number of times.

Theorem 4.11 If ${P_{s,t}}^k(n)$ denotes the number of partitions of *n* in which parts are only of the form *km + s* or *km + t* and ${P_{s,t}}^{k,O}(n)$ and are ${P_{s,t}}^{k,\varepsilon}(n)$ as defined above, then

$P[^4_{1,3}(n) - P^{4,O}_{2,2}(n) + P^{4,\varepsilon}_{2,2}(n) = 1 + \sum_{m=1}^{n-1} P^4_{3,4}(n-m) - P^{4,O}_{1,1}(n-m) + P^{4,,\varepsilon}_{1,1}(n-m)]$ (**4.12)**

Proof. Putting a = $q^{1/4} = z$, $b = q$, $c = q^{3/4}$ and then changing q to q^4 in (1.3), we obtain

$\sum_{n=0}^{\infty} \frac{{(q\,;q^4)_n}^2}{(q^3;q^4)_n(q^4;q^4)_n} q^n = \frac{{(q^2\,;q^6)_\infty}^2}{(q;q^4)_\infty(q^3;q^4)_\infty}$ **(4.13)**

Using the definitions as defined above in (4.13), we obtain

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} P^4_{3,4}(n-m) - P^{4,O}_{1,1}(n-m) + P^{4,O}_{1,1}(n-m)q^n$$

$$= \sum_{n=0}^{\infty} P^4_{1,3}(n) - P^{4,O}_{2,2}(n) + P^{4,,\varepsilon}_{2,2}(n)\, q^n \; (\mathbf{4.14})$$

By comparing the coefficients of ***q^n^***, we obtain (4.12).

Theorem 4.15

If ${P_{s,t}}^k(n)$ *denotes the number of partitions of n in which parts are only of the form km + s or km + t and* ${P_{s,t}}^{k,O}(n)$ and are ${P_{s,t}}^{k,\varepsilon}(n)$ as defined above, then

and) are as defined above, then

1+ $\sum_{m=1}^{n-1} \; [P^6_{5,6}(n-m) - P^{6,O}_{2,2}(n-m) + P^{6,\varepsilon}_{2,2}(n-m]) = P^6_{1,5}(n) - P^{6,O}_{3,3}(n) + P^{6,,\varepsilon}_{3,3}(n)\ldots\ldots$ **(4.16)**

Proof. Putting a = $q^{2/6}$, z = $q^{1/6}$, b = q, c = $q^{5/6}$ and then changing q to q^6 in (1.3), we obtain

$$\sum_{n=0}^{\infty}\frac{(q^2;q^6)_n^2}{(q^5;q^6)_n(q^6;q^6)_n}q^n = \frac{(q^3;q^6)_\infty^2}{(q;q^6)_\infty(q^5;q^6)_\infty} \quad (4.17)$$

Using the definitions as defined above in (4.17), we obtain

$$\sum_{n=0}^{\infty}\sum_{m=1}^{n=1}[\mathcal{P}^6_{5,6}(n-m) - \mathcal{P}^{6,\mathcal{O}}_{2,2}\,(n-m) + \mathcal{P}^{6,\mathcal{E}}_{2,2}\,(n-m)]q^n$$

$\sum_{n=0}^{\infty}[\mathcal{P}^6_{5,6}(n) - \mathcal{P}^{6,\mathcal{O}}_{2,2}\,(n) + +\mathcal{P}^{6,\mathcal{E}}_{2,2}\,(n)]q^n$ ….4.18

By comparing the coefficients of q^n, we obtain (4.16).

5. Some applications of the main identity

Corollary 5. If $0 < q < 1, 0 < x, y < 1$ and $0 < x + y < 1$, then

$$B_q(x.y) = \Gamma_q(y)\Gamma_q(1\text{-}y)\sum_{k=0}^{\infty}\frac{(q^{1\text{-}x\text{-}y})_k(q^y)_k}{(q)_k^2}q^{kx} \quad \dots(5.1)$$

Proof. Putting a = q^{1-x-y}, z = q^x, and b = c = q, we obtain (3.12).

Corollary 6. If 0 < x, y < 1 and 1 < x + y < 2, then

$$B^2(x.y) = \frac{\Gamma(y\text{-}x)\Gamma(x\text{-}y+1)\Gamma(x+y+2)}{\Gamma(y)\Gamma(1+x)}\left[\sum_{k=}^{\infty}\frac{(\text{-}x)_k(x+y+1)_k}{(y)_k^2} + \sum_{k=}^{\infty}\frac{(y+1)_k^2}{(1+x)_k\,(2+x+y)_k}\right] \quad \dots.(5.2)$$

Proof. Letting $q \to 1$ in (3.9), we obtain (3.11).

Corollary 7. If $0 < x, y < 1$ and $0 < x + y < 1$, then

$$B^3(x.y) = \frac{\Gamma(1\text{-}x+y)\Gamma(x\text{-}y)}{y^2}\left[\sum_{k=}^{\infty}\frac{(1\text{-}x)_k(x+y)_k}{(1+y)_k^2} + \sum_{k=}^{\infty}\frac{(\text{-}y)_k^2}{(x)_k\,(1\text{-}x\text{-}y)_k}\right] \quad \dots.5.3$$

Proof. Letting q $\to$1 in (3.8), we obtain (3.10).

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Declarations

No Conflict of Interest: The authors declare that this research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflicts of interest

References

[1]. Abel, Niels Henrik, Untersuchungen über die Reihe: 1 + (m/1)x + m·(m - 1)/(1·2)·x²+ m·(m - 1)·(m - 2)/(1·2·3)·x³+ ... Von N. H. Abel. (1826.) Hrsg. von A. Wangerin. 1802-1829.

[2]. G. E. Andrews, On a transformation of bilateral series with applications *Proc. Amer. Math. Soc*. 25 (1970) 554 – 558

[3]. R. Askey, The q-gamma and q-beta functions, *Applicable Anal*. 8 (1978), 125–141.

[4]. W. N. Bailey, On the basic bilateral hypergeometric series $2\,\psi_2$, *Quart. J. Math. Oxford,* Ser. 21 (1950) 194 − 198

[5]. Berndt. B. C, *Ramanujan's Notebooks*, Part III, Springer-Verlag, New York 1991

[6]. B. C. Berndt and R. A. Rankin, Ramanujan: Letters and Commentary. American Mathematical Society. 1995

[7]. Chu. W, Abel's lemma on summation by parts and Ramanujan's $1\psi_1$summation, *Aeq. Math.* 72 (2006) 172 − 176.

[8]. G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics, Cambridge University press. Cambridge, 2004

[9]. F. H. Jackson, On q - definite integrals, *Quart. J. Pure. Appl. Math.,* 41(1910),193 - 203

[10]. Ramanujan. S, *Notebooks*, 2 volumes, Tata institute of Fundamental Research, Bombay, 1957.

[11]. Ramanujan, S. The Lost Notebook and Other Unpublished Papers. Narosa,New Delhi. 1998

[12]. Somashekara. D. D, Narasimha murthy. K and Shalini. S. L, On a new summation formula for $2\,\psi_2$Basic bilateral hypergeometric series and its applications, *int.jour.math.and math.sci.* V olume 2011, ArticleID 132081.