Vol.5.Issue.4.2017 (Oct-Dec)
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BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com
Email:editorbomsr@gmail.com

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

RESEARCH ARTICLE

ON PARAMETRIC METRIC SPACES

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ABSTRACT

The aim of this paper is to give some basic properties in parametric metric spaces defined in literature illustrating with examples.

Keywords: Parametric metric space, open set, Hausdorffness.

AMS(2010) Subject Classification: 47H10, 54H25.

1. INTRODUCTION

In 2011, the structure of parametric metric space was introduced by Hussain et al. [1] as a new generalized metric space. After that many researchers studied this new notion ([2], [3], [4]).

Definition 1.1. [1] Let X be a nonempty set. A function $P: X \times X \times (0,\infty) \to [0,\infty)$ is said to be a parametric metric function on X, if the following conditions are satisfied; for all $x, y, z \in X$ and $t > 0$,

P1. $P(x,y,t) \geq 0$,

P2. $P(x,y,t) = 0 \Leftrightarrow x = y$,

P3. $P(x,y,t) = P(y,x,t)$,

P4. $P(x,y,t) \leq P(x,z,t) + P(z,y,t)$

Then, (X,P) is called a parametric metric space.

Example 1.1. [1] Let $X = \{f \mid f:(0,\infty) \to [0,\infty)\}$. Define $P: X \times X \times (0,\infty) \to [0,\infty)$ by

$$P(f,g,t) = |f(t) - g(t)|$$

for all $f, g \in X$ and $t > 0$. In that case, (X,P) is a parametric metric space.

Example 1.2. [1] Let $X=[0,\infty)$. Define $P:X\times X\times(0,\infty)\to[0,\infty)$ by

$$P(x,y,t)=\begin{cases} t.\max\{x,y\}, & x\neq y\\ 0, & x=y\end{cases}$$

for all $x,y\in X$ and $t>0$. In that case, (X,P) is a parametric metric space.

Definition 1.2. [1] Let (X,P) be a parametric metric space, $x\in X$ and $r>0$. Then,

$$B_P(x,r)=\{y\in X:P(x,y,t)<r,\ \forall t>0\}$$

is an open ball with center x and radius r.

Definition 1.3. [1] Let P be a parametric metric on a set X. A sequence $\{x_n\}\in X$ converges to $x\in X$ if for any $\varepsilon>0$, there exists a $n_0\in N$ such that $P(x,x_n,t)<\varepsilon$ for all $n\geq n_0$, x is said to be the limit of $\{x_n\}$ and this is denoted by $\lim\limits_{n\to\infty}x_n=x$ or $x_n\to x$ as $n\to\infty$.

Definition 1.4. [1] Let P be a parametric metric on a set X. A sequence $\{x_n\}\in X$ is called Cauchy sequence if for any $\varepsilon>0$, there exists a $n_0\in N$ such that $P(x_n,x_m,t)<\varepsilon$ for all $n,m\geq n_0$.

Definition 1.5. [1] A parametric metric space (X,P) is said to be complete if every Cauchy sequence in X is convergent.

Theorem 1.1. [1] Let (X,P) be a parametric metric space, $\{x_n\}$ and $\{y_n\}$ be sequences in this space and $x,y\in X$. If $x_n\to x$ and $y_n\to y$ as $n\to\infty$, $\lim\limits_{n\to\infty}P(x_n,y_n,t)=P(x,y,t)$.

Proof: Since $\{x_n\}$ converges to x, for any $\varepsilon>0$, there exists a $n_1\in N$ such that $P(x,x_n,t)<\frac{\varepsilon}{2}$ for all $n\geq n_1$. Similarly $\{y_n\}$ converges to y, for any $\varepsilon>0$, there exists a $n_2\in N$ such that $P(y,y_n,t)<\frac{\varepsilon}{2}$ for all $n\geq n_2$. Choose $n_0=\max\{n_1,n_2\}$. Then for all $n\geq n_0$,

$$\begin{aligned} P(x_n,y_n,t) &\leq P(x_n,x,t)+P(x,y_n,t)\\ &< P(x_n,x,t)+P(x,y,t)+P(y,y_n,t)\\ &= \frac{\varepsilon}{2}+\frac{\varepsilon}{2}+P(x,y,t).\end{aligned}$$

From this $P(x_n,y_n,t)-P(x,y,t)<\varepsilon$. Similarly,

$$\begin{aligned} P(x,y,t) &\leq P(x,x_n,t)+P(x_n,y,t)\\ &< P(x,x_n,t)+P(x_n,y_n,t)+P(y_n,y,t)\\ &= \frac{\varepsilon}{2}+\frac{\varepsilon}{2}+P(x_n,y_n,t).\end{aligned}$$

From this $P(x,y,t)-P(x_n,y_n,t)<\varepsilon$. Then,

$$|P(x_n,y_n,t)-P(x,y,t)|<\varepsilon$$

Is obtained. So $\lim\limits_{n\to\infty}P(x_n,y_n,t)=P(x,y,t)$.

2. MAIN RESULTS

Definition 2.1. Let (X,P) be a parametric metric space and $F,G\subset X$. G is called an open set if for any $\varepsilon>0$, there exists a $r>0$ such that $B_P(x,r)\subset G$. If X/F is an open set, then F is called a close set.

Definition 2.1. Let (X,P) be a parametric metric space.

$\tau = \{A \subset X : \textit{for all } x \in A, \textit{there exists a } r > 0 \textit{ such that } B_P(x,r) \subset A\}$

is a topology on X.

Theorem 2.1. Let (X,P) be a parametric metric space. In this space every open ball is an open set.

Proof: Consider the open ball $B_P(x,r)$. Then $y \in B_P(x,r) \Rightarrow P(x,y,t) < r$. Assume that $r' = r - P(x,y,t)$. Take into consideration the ball $B_P(y,r')$. We claim that $B_P(y,r') \subset B_P(x,r)$. Let $z \in B_P(y,r')$. Then, $P(y,z,t) < r'$. For this reason,

$$\begin{aligned} P(x,z,t) &\leq P(x,y,t) + P(y,z,t) \\ &< P(x,y,t) + r' \\ &= P(x,y,t) + r - P(x,y,t) \\ &= r. \end{aligned}$$

Then, $z \in B_P(x,r)$. So, the proof is completed.

Theorem 2.2. Every parametric metric space is a Hausdorff space.

Proof: Let (X,P) be a parametric metric space and $x, y \in X$, $x \neq y$. Consider $P(x,y,t) = r$. Take into consideration the open sets $U = B_P(x,\frac{r}{2})$ and $V = B_P(y,\frac{r}{2})$. It is clear that $x \in U$ and $y \in V$. We claim that $U \cap V = \phi$. Assume that $U \cap V \neq \phi$. Then there exists a $z \in X$ such that $z \in U \cap V$. Then $z \in U \Rightarrow P(x,z,t) < \frac{r}{2}$ and $z \in V \Rightarrow P(y,z,t) < \frac{r}{2}$. From P4,

$$\begin{aligned} P(x,y,t) &\leq P(x,z,t) + P(z,y,t) \\ &< P(x,z,t) + P(y,z,t) \\ &= \frac{r}{2} + \frac{r}{2} \\ &= r. \end{aligned}$$

Since $P(x,y,t) = r$, $r < r$ is obtained. But this is a contradiction. Then $U \cap V = \phi$. The proof is completed.

Theorem 2.3. Let (X,P) be a parametric metric space. In this space every convergent sequence has a unique limit point.

Proof: Let $\{x_n\} \subset X$ be a sequence and $x, y \in X$, $x \neq y$. Suppose that $\{x_n\}$ converges to both x and y. Since $\{x_n\}$ converges to x, for any $\varepsilon > 0$, there exists a $n_1 \in N$ such that $P(x,x_n,t) < \frac{\varepsilon}{2}$ for all $n \geq n_1$. Similarly $\{x_n\}$ converges to y, for any $\varepsilon > 0$, there exists a $n_2 \in N$ such that $P(x,x_n,t) < \frac{\varepsilon}{2}$ for all $n \geq n_2$. Choose $n_0 = \max\{n_1, n_2\}$. Then for all $n \geq n_0$,

$$\begin{aligned} P(x,y,t) &\leq P(x,x_n,t) + P(x_n,y,t) \\ . \quad &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since ε is arbitrary, $P(x,y,t) = 0 \Leftrightarrow x = y$. Then every convergent sequence in this space has a unique limit point.

Theorem 2.4. Let (X, P) be a parametric metric space. In this space every convergent sequence is a Cauchy sequence.

Proof: Let $\{x_n\} \subset X$ be a convergent sequence. Then there exists a $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

So for any $\varepsilon > 0$, there exist $n_1, n_2 \in N$ such that $P(x, x_n, t) < \frac{\varepsilon}{2}$ for all $n \geq n_1$ and $P(x, x_n, t) < \frac{\varepsilon}{2}$ for all $m \geq n_2$. Choose $n_0 = \max\{n_1, n_2\}$. Then for all $m, n \geq n_0$,

$$\begin{aligned} P(x_n, x_m, t) &\leq P(x_n, x, t) + P(x, x_m, t) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Then $\{x_n\}$ is a Cauchy sequence.

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