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RESEARCH ARTICLE

AL-ZUGHAIR CONVOLUTION

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ABSTRACT
Our aim in this paper is to find the convolution of Al-Zughair transformation. And how we can use it to solve some linear ordinary differential equations (LODEs) .

Introduction

We will use the new idea in **[1]** to find the convolution of Al-Zughair transformation which we will define it by a new method and so it will give us the ability to find the inverse of Al-Zughair transformation for some terms more easily than using the partition method. As well as the Al-Zughair convolution is interest in solving some linear ordinary differential equations (LODEs) by simpler way without using extended partition method.

Preliminaries

Definition (1) [2]:

Let f is defined function at period (a, b) then the integral transformation for f whose it's symbol is defined as:

$$F(p) = \int_a^b k(p,x)f(x)dx$$

Where k a fixed function of two variables is called the kernel of the transformation, a and b are real numbers or $\pm\infty$, such that the above integral is convergent.

Definition (2) [3]:

Al-Zughair transformation $[Z(f(x))]$ for the function $f(x)$ where $x \in [1, e]$ is defined by the following integral:

$$Z(f(x)) = \int_1^e \frac{(\ln x)^p}{x} f(x)dx = F(p)$$

Such that this integral is convergent, $p$is positive constant.

Property (1) [3]:
Al-Zughair transformation is characterized by the linear property, that is,

$$Z[Af(x) \pm Bg(x)] = AZ[f(x)] \pm BZ[g(x)]$$

Where A and B are constants, the functions $f(x)$ and $g(x)$ are defined when $x \in [1, e]$.

Al-Zughair transform of some fundamental functions are **[3]**

ID	Function, Function, $f(x)$	$F(p) = \int_1^e \frac{(\ln x)^p}{x} f(x)dx = Z(f(x))$	Regional of convergence
1	k ; k = constant	$\frac{k}{(p+1)}$	$p > -1$
2	$(\ln x)^n$, $n \in R$	$\frac{1}{(p+(n+1))}$	$p > -(n+1)$
3	$\ln(\ln x)$	$\frac{-1}{(p+1)^2}$	$p > -1$
4	$(\ln(\ln x))^n$, $n \in z^+$	$\frac{(-1)^n n!}{(p+1)^{n+1}}$	$p > -1$
5	$sin(a\ ln(ln\ x))$	$\frac{-a}{(p+1)^2 + a^2}$	$p > -1$ a is constant
6	$cos(a\ ln(ln\ x))$	$\frac{p+1}{(p+1)^2 + a^2}$	$p > -1$ a is constant
7	$sinh(a\ ln(ln\ x))$	$\frac{-a}{(p+1)^2 - a^2}$	$\lvert p+1 \rvert > a$ a is constant
8	$cosh(a\ ln(ln\ x))$	$\frac{p+1}{(p+1)^2 - a^2}$	$\lvert p+1 \rvert > a$ a is constant

Theorem (1) [3]:
If$Z(f(x)) = F(p)$and a is constant, then$Z\left((\ln x)^a f(x)\right) = F(s+a)$.

Property (2) [3]: If $Z^{-1}(F_1(p)) = f_1(x)$, $Z^{-1}(F_2(p)) = f_2(x)$, $\ldots, Z^{-1}(F_n(p)) = f_n(x)anda_1$, a_1 , … $a_n$are constants then,

$Z^{-1}[a_1F_1(p) + a_2F_2(p) + \cdots + a_nF_n(p)] = a_1f_1(x) + a_2f_2(x) + \cdots + a_nf_n(x).$

Definition (3) [3]:

The equation

$$a_o(\ln x)^n y^{(n)}(x) + a_1(\ln x)^{n-1}y^{(n-1)}(x) + \cdots + a_{n-1}(\ln x)y'(x) + a_n y(x) = f(x)$$

Where $a_o,\ a_1,\ \ldots,\ a_n$ are constants and $f(x)$ is a function of x, is called ***Ali's Equation*.**

Definition (4) [4]:

A function$f(x)$is piecewise continuous on an interval$[a, b]$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 <\cdots< x_n = b$such that:

1.$f(x)$is continuous on each subinterval (x_i, x_{i+1}),for$i = 0\ ,1\ ,2\ , \ldots, n-1$.

2.The function f has jump discontinuity atx_i,thus

$\left|lim_{x\to x_i^+} f(x)\right| < \infty, i = 0,1,2,\ldots,n-1;$

$\left|lim_{x\to x_i^-} f(x)\right| < \infty, i = 0,1,2,\ldots,n$.

Definition (5): Convolution of Al- Zughair Transform

The convolution of Al- Zughair Transform of two functions $f(\ln x)$ and $g(\ln x)$ is defined for $\ln x \in [1, e]$ by:

$$(f * g)(\ln x) = \int_x^e f(\ln u)g\left(\frac{\ln x}{\ln u}\right)\frac{d\ln u}{\ln u} = \int_x^e f(\ln u)g\left(\frac{\ln x}{\ln u}\right)\frac{du}{u\ln u}$$

Where $\ln u \neq 0, f$ and g are piecewise continuous functions on [1, e].

Theorem (2): Let $f(\ln x)$ and $g(\ln x)$ be two functions. Al-Zughairconvolution of $f(\ln x)$ and $g(\ln x)$ denoted by $Z[(f * g\ln x$is given by the relation

$$Z[(f * g)(\ln x)] = Z[f(\ln x)].Z[g(\ln x)]$$

Proof:

$$Z[f(\ln x)].Z[g(\ln x)]$$

$$= \left(\int_1^e \frac{(\ln u)^p}{u} f(\ln u)du\right)\left(\int_1^e \frac{(\ln v)^p}{v} g(\ln v)dv\right)$$

$$= \left(\int_1^e (\ln u)^p f(\ln u)\frac{du}{u}\right)\left(\int_1^e (\ln v)^p g(\ln v)\frac{dv}{v}\right)$$

$$= \left(\int_1^e (\ln u)^p f(\ln u)d\ln u\right)\left(\int_1^e (\ln v)^p g(\ln v)d\ln v\right)$$

$$= \int_1^e \left(\int_1^e (\ln u\,.\ln v)^p f(\ln u)g(\ln v)d\ln v\right)d\ln u$$

Let $\ln u\,.\ln v = \ln x$

If $v = 1 \Rightarrow \ln u\,.(0) = \ln x \Rightarrow x = e^0 = 1$

And if $v = e \Rightarrow \ln u\,(\ln e) = \ln x \Rightarrow \ln u = \ln x \Rightarrow u = x$

Where $\ln u$ is fixed in the interior integral $\Longrightarrow \ln u\,.\,d\ln v = d\ln x$

$$Z[f(\ln x)].Z[g(\ln x)] = \int_1^e \left(\int_1^u (\ln x)^p f(\ln u) g\left(\frac{\ln x}{\ln u}\right) \frac{d\ln x}{\ln u} \right) d\ln u$$

If $g(\ln x) = 0$ for $\ln x < 1 \Longrightarrow g\left(\frac{\ln x}{\ln u}\right) = 0$ for $(\ln x < \ln u) \Longrightarrow x < u$

$$\Longrightarrow Z[f(\ln x)].Z[g(\ln x)]$$

$$= \int_1^e \left(\int_1^e (\ln x)^p f(\ln u) g\left(\frac{\ln x}{\ln u}\right) d\ln x \right) \frac{d\ln u}{\ln u}$$

$$= \int_1^e \left(\int_1^e (\ln x)^p f(\ln u) g\left(\frac{\ln x}{\ln u}\right) \frac{d\ln u}{\ln u} \right) d\ln x$$

$$= \int_1^e (\ln x)^p \left(\int_x^e f(\ln u) g\left(\frac{\ln x}{\ln u}\right) \frac{d\ln u}{\ln u} \right) d\ln x$$

$$= \int_1^e \frac{(\ln x)^p}{x} \left(\int_x^e f(\ln u) g\left(\frac{\ln x}{\ln u}\right) \frac{d\ln u}{\ln u} \right) dx$$

$$= Z[(f * g)(\ln x)]$$

Proposition of the convolution are given as follows:

1) $f * g = g * f$, the convolution is commutative
2) $c(f * g) = cf * g = f * cg$, c constant;
3) $f * (g * h) = (f * g) * h$,(associative property);
4) $f * (g + h) = (f * g) + (f * h)$, (distributive property).

Proof(1):

$$(f * g)(\ln x) = \int_x^e f(\ln u) g\left(\frac{\ln x}{\ln u}\right) \frac{d\ln u}{\ln u}$$

$f$and$g$ are piecewise continuous on [1,e].

Now, let$\ln v = \frac{\ln x}{\ln u} \Longrightarrow \ln u = \frac{\ln x}{\ln v}$, $\ln v \neq 0$

$\ln x = \ln u\,.\ln v \Longrightarrow \ln u\,.\,d\ln v + \ln v\,.\,d\ln u = 0$

$\frac{d\ln u}{\ln u} = -\frac{d\ln v}{\ln v}, \qquad \ln u \neq 0, \qquad \ln v \neq 0$

If $u = x \Longrightarrow \ln v = \frac{\ln x}{\ln x} = 1 \Longrightarrow v = e$

$u = e \;\Longrightarrow\; \ln v = \ln x \;\Longrightarrow\; v = x$

$$\therefore (f * g)(\ln x) = \int_e^x f\left(\frac{\ln x}{\ln v}\right) g(\ln v) \frac{-d\ln v}{\ln v}, \qquad \ln v \neq 0$$

$$= \int_x^e g(\ln v).f\left(\frac{\ln x}{\ln v}\right) \frac{d\ln v}{\ln v} = (g * f)(\ln x)$$

Proof(2):

$$c(f*g)(\ln x) = c\int_x^e f(\ln u)g\left(\frac{\ln x}{\ln u}\right)\frac{d\ln u}{\ln u}$$

$$= \int_x^e cf(\ln u)g\left(\frac{\ln x}{\ln u}\right)\frac{d\ln u}{\ln u} = (cf*g)(\ln x)$$

By the same method we can prove$c(f*g)(\ln x) = f*cg(\ln x)$

Proof (3):

$$[f*(g*h)](\ln x) = \int_x^e f(\ln u)(g*h)\left(\frac{\ln x}{\ln u}\right)\frac{d\ln u}{\ln u}, \ln u \neq 0$$

$$= \int_x^e f(\ln u)\left(\int_{x/u}^e g(\ln v)h\left(\frac{\ln x/\ln u}{\ln v}\right)\frac{d\ln v}{\ln v}\right)\frac{d\ln u}{\ln u}, \ln u \neq 0, \ln v \neq 0$$

Let $\ln v = \frac{\ln \tau}{\ln u} \Longrightarrow d\ln v = \frac{d\ln \tau}{\ln u},\ \ \ln u \neq 0$

$$= \int_x^e f(\ln u)\left(\int_x^u g\left(\frac{\ln \tau}{\ln u}\right)h\left(\frac{\ln x}{\ln \tau}\right)\frac{d\ln \tau}{\ln u.\ln v}\right)\frac{d\ln u}{\ln u}$$

$$= \int_x^e\left(\int_\tau^e f(\ln u)g\left(\frac{\ln \tau}{\ln u}\right)\frac{d\ln u}{\ln u}\right)h\left(\frac{\ln x}{\ln \tau}\right)\frac{d\ln \tau}{\ln \tau}$$

$$= \int_x^e (f*g)(\ln \tau)h\left(\frac{\ln x}{\ln \tau}\right)\frac{d\ln \tau}{\ln \tau} = \big((f*g)*h\big)(\ln x)$$

Proof (4):$\big(f*(g+h)\big)(\ln x) = \int_x^e f(\ln u)(g+h)\left(\frac{\ln x}{\ln u}\right)\frac{d\ln u}{\ln u}$

$$= \int_x^e f(\ln u)\left(g\left(\frac{\ln x}{\ln u}\right)\frac{d\ln u}{\ln u} + h\left(\frac{\ln x}{\ln u}\right)\frac{d\ln u}{\ln u}\right)$$

$$= \int_x^e f(\ln u)g\left(\frac{\ln x}{\ln u}\right)\frac{d\ln u}{\ln u} + \int_x^e f(\ln u)h\left(\frac{\ln x}{\ln u}\right)\frac{d\ln u}{\ln u}$$

$= (f*g)(\ln x) + (f*h)(\ln x).$

Example (1): To find $Z^{-1}\left[\frac{k}{(p+1)(p+2)}\right]$

Firstly, we use the usual method

$$\frac{k}{(p+1)(p+2)} = \frac{A}{(p+1)} + \frac{B}{(p+2)} = \frac{Ap+2A+Bp+B}{(p+1)(p+2)}$$

$A + B = 0$

$2A + B = k \implies B = -k, \quad A = k$

$$\therefore Z^{-1}\left[\frac{k}{(p+1)(p+2)}\right] = Z^{-1}\left[\frac{k}{(p+1)}\right] - Z^{-1}\left[\frac{k}{(p+2)}\right]$$

$= k - k\ln x$

Now, we will use the convolution method we get

$$Z^{-1}\left[\frac{k}{(p+1)(p+2)}\right] = Z^{-1}\left[\frac{k}{(p+1)}.\frac{1}{(p+2)}\right]$$

$$= k *_Z \ln x$$

$$= \int_x^e k \frac{\ln x}{\ln u} \frac{d \ln u}{\ln u} = k(\ln x) \int_x^e (\ln u)^{-2} d \ln u = k(\ln x) \left. \frac{(\ln u)^{-1}}{-1} \right|_x^e$$
$$= -k \ln x + k$$

Example (2): To find$Z^{-1}\left[\frac{1}{(p+4)(p+6)}\right]$

We note that

$$\frac{1}{(p+4)(p+6)} = \frac{A}{(p+4)} + \frac{B}{(p+6)} = \frac{Ap + 6A + Bp + 4B}{(p+4)(p+6)}$$

$A + B = 0$

$6A + 4B = 1 \implies A = \frac{1}{2}, B = \frac{-1}{2}$

$$\therefore Z^{-1}\left[\frac{1}{(p+4)(p+6)}\right] = \frac{1}{2} Z^{-1}\left[\frac{1}{(p+4)}\right] - \frac{1}{2} Z^{-1}\left[\frac{1}{(p+6)}\right]$$
$$= \frac{1}{2}(\ln x)^3 - \frac{1}{2}(\ln x)^5$$

Now, we will use the convolution method we get

$$Z^{-1}\left[\frac{1}{(p+4)(p+6)}\right] = Z^{-1}\left[\frac{1}{(p+4)} . \frac{1}{(p+6)}\right]$$
$$= (\ln x)^3 *_Z (\ln x)^5 = \int_x^e (\ln u)^3 \frac{(\ln x)^5}{(\ln u)^5} \frac{d \ln u}{\ln u}$$
$$= (\ln x)^5 \int_x^e (\ln u)^{-3} d \ln u = (\ln x)^5 \left. \frac{(\ln u)^{-2}}{-2} \right|_x^e = \frac{(\ln x)^3}{2} - \frac{(\ln x)^5}{2}$$

Example (3): To find the solution of the ordinary differential equation

$$(\ln x) y'(\ln x) + 3y(\ln x) = (\ln x)^{-2} (\ln(\ln x))^2, \qquad y(1) = 0$$

Take Z to both sides

$Z[(\ln x) y'(\ln x)] + Z[3y(\ln x)] = Z[(\ln x)^{-2} (\ln(\ln x))^2]$

$y(1) - (p+1) Z[y(\ln x)] + 3Z[y(\ln x)] = \frac{2}{(p-1)^3}$

$-(p+1) Z[y(\ln x)] + 3Z[y(\ln x)] = \frac{2}{(p-1)^3}$

$$-(p-2) Z[y(\ln x)] = \frac{2}{(p-1)^3} \implies Z[y(\ln x)] = \frac{-2}{(p-1)^3 (p-2)}$$

Take Z^{-1} to both sides

$$y(\ln x) = Z^{-1}\left[\frac{-2}{(p-1)^3 (p-2)}\right]$$

Now we will get the solution by the convolution method

$$y(\ln x) = Z^{-1}\left[\frac{-2}{(p-1)^3} . \frac{1}{(p-2)}\right] = -(\ln x)^{-2} (\ln(\ln x))^2 *_Z (\ln x)^{-3}$$
$$y(\ln x) = \int_x^e -(\ln u)^{-2} (\ln(\ln u))^2 \left(\frac{\ln x}{\ln u}\right)^{-3} \frac{\mathrm{d} \ln u}{\ln u}$$
$$= -(\ln x)^{-3} \int_x^e (\ln(\ln u))^2 \, \mathrm{d} \ln u$$
$$= -(\ln x)^{-3} \left[-(\ln x)(\ln(\ln x))^2 - 2 \int_x^e \ln(\ln u) \, d \ln u \right]$$

$$= -(\ln x)^{-3}[-(\ln x)(\ln(\ln x))^2 + 2(\ln x)(\ln(\ln x)) + 2 - 2\ln x]$$
$$= (\ln x)^{-2}(\ln(\ln x))^2 - 2(\ln x)^{-2}(\ln(x)) - 2(\ln x)^{-3} + 2(\ln x)^{-2}$$

Example (4): To find the solution of the ordinary differential equation
$(\ln x)^2 y''(\ln x) + \ln x\, y'(\ln x) = 6(\ln(\ln x))^4, \quad y'(\ln e) = y(\ln e) = 0$
Take Z to both sides
$Z[(\ln x)^2 y''(\ln x)] + Z[\ln x\, y'(\ln x)] = Z[6(\ln(\ln x))^4]$
$$y'(1) - (p+2)y(1) + (p+2)(p+1)Z\big(y(\ln x)\big) + y(1) - (p+1)Z\big(y(\ln x)\big) = \frac{6\times 4!}{(p+1)^5}$$
$$(p+2)(p+1)Z\big(y(\ln x)\big) - (p+1)Z\big(y(\ln x)\big) = \frac{6\times 4!}{(p+1)^5}$$
$$(p+1)^2\, Z\big(y(\ln x)\big) = \frac{6\times 4!}{(p+1)^5}$$
$$Z\big(y(\ln x)\big) = \frac{6\times 4!}{(p+1)^7} = \frac{6\times 4!}{(p+1)^2(p+1)^5}$$
Take Z^{-1} to both sides
$$y(\ln x) = Z^{-1}\left[\frac{6\times 4!}{(p+1)^2(p+1)^5}\right] = 6Z^{-1}\left[\frac{1}{(p+1)^2}\cdot\frac{4!}{(p+1)^5}\right] = -6(\ln(\ln x)) *_Z (\ln(\ln x))^4$$
$$y(\ln x) = \int_x^e -6(\ln(\ln u))^4 \left(\ln\left(\frac{\ln x}{\ln u}\right)\right)\frac{d\ln u}{\ln u}$$
$$= \int_x^e -6(\ln(\ln u))^4\,(\ln(\ln x) - \ln(\ln u))\frac{d\ln u}{\ln u}$$
$$= -6(\ln(\ln x))\int_x^e (\ln(\ln u))^4\frac{d\ln u}{\ln u} + 6\int_x^e (\ln(\ln u))^5\frac{d\ln u}{\ln u}$$
$$= \frac{6}{5}(\ln(\ln x))^6 - (\ln(\ln x))^6 = \frac{1}{5}(\ln(\ln x))^6$$

Example (5): To find the solution of the ordinary differential equation
$(\ln x)^2 y''(\ln x) + \ln x\, y'(\ln x) + y(\ln x) = -\sin\ln(\ln x),$
$y'(\ln e) = y(\ln e) = 0$
Take Z to both sides
$Z[(\ln x)^2 y''(\ln x)] + Z[\ln x\, y'(\ln x)] + Z[y(\ln x)] = Z[-\sin\ln(\ln x)]$
$$y'(1) - (p+2)y(1) + (p+2)(p+1)Z\big(y(\ln x)\big) + y(1) - (p+1)Z\big(y(\ln x)\big) + Z\big(y(\ln x)\big)$$
$$= \frac{1}{((p+1)^2+1)}$$
$$((p+1)^2+1)Z\big(y(\ln x)\big) = \frac{1}{((p+1)^2+1)}$$
$$Z\big(y(\ln x)\big) = \frac{1}{((p+1)^2+1)^2} \Rightarrow y(\ln x) = Z^{-1}\left[\frac{1}{((p+1)^2+1).((p+1)^2+1)}\right]$$
$$= Z^{-1}\left[\frac{-1}{((p+1)^2+1)}\cdot\frac{-1}{((p+1)^2+1)}\right] = \sin\ln(\ln x) *_Z \sin\ln(\ln x)$$
$$= \int_x^e \sin\ln(\ln \mathrm{u})\sin\ln\left(\frac{\ln x}{\ln \mathrm{u}}\right)\frac{d\ln u}{\ln u} = \int_x^e \sin\ln(\ln u)\,\big(\sin(\ln(\ln x) - \ln(\ln u))\big)\frac{d\ln u}{\ln u}$$

$$= \int_x^e \sin\ln(\ln u)\,(\sin\ln(\ln x)\cos\ln(\ln u) - \cos\ln(\ln x)\sin\ln(\ln u))\frac{d\ln u}{\ln u}$$

$$= \sin\ln(\ln x)\int_x^e \sin\ln(\ln u)\cos\ln(\ln u)\frac{d\ln u}{\ln u} - \cos\ln(\ln x)\int_x^e (\sin\ln(\ln u))^2\frac{d\ln u}{\ln u}$$

$$= -\sin\ln(\ln x)\frac{\left(\cos\ln(\ln u)\right)^2}{2}\Bigg|_x^e - \cos\ln(\ln x)\int_x^e \left(\frac{1-\cos 2\ln(\ln u)}{2}\right)\frac{d\ln u}{\ln u}$$

$$= -\sin\ln(\ln x)\left[\frac{1}{2} - \frac{\left(\cos\ln(\ln x)\right)^2}{2}\right] - \frac{1}{2}\cos\ln(\ln x)\left(\ln(\ln u)|_x^e - \frac{1}{2}\sin 2\ln(\ln u)|_x^e\right)$$

$$= -\frac{1}{2}\sin\ln(\ln x) + \frac{1}{2}\ln(\ln x)\cos\ln(\ln x).$$

Example (4): To find the solution of the ordinary differential equation

$(\ln t)^2 u_{tt}(x,\ln t) + \ln t\, u_t(x,\ln t) = 3x\ln(\ln t), \quad u_t(x,\ln e) = u(x,\ln e) = 0$

Take Z to both sides

$Z[(\ln t)^2 u_{tt}(x,\ln t)] + Z[\ln t\, u_t(x,\ln t)] = Z[3x\ln(\ln t)]$

$$u_t(x,1) - (s+2)u(x,1) + (s+2)(s+1)Z\big(u(x,\ln t)\big) + u(x,1) - (s+1)Z\big(u(x,\ln t)\big) = \frac{-3x}{(s+1)^2}$$

$$(s+2)(s+1)Z\big(u(x,\ln t)\big) - (s+1)Z\big(u(x,\ln t)\big) = \frac{-3x}{(s+1)^2}$$

$$(s+1)^2 Z\big(u(x,\ln t)\big) = \frac{-3}{(s+1)^2} \Rightarrow Z\big(u(x,\ln t)\big) = \frac{-3x}{(s+1)^4}$$

Take Z^{-1} to both sides

$$u(x,\ln t) = Z^{-1}\left[\frac{-3x}{(s+1)^4}\right] = -3xZ^{-1}\left[\frac{-1}{(s+1)^2}.\frac{-1}{(s+1)^2}\right] = -3x\ln(\ln t) *_Z \ln(\ln t)$$

$$u(x,\ln t) = \int_t^e -3x\ln(\ln u)\left(\ln\left(\frac{\ln t}{\ln u}\right)\right)\frac{d\ln u}{\ln u}$$

$$= \int_t^e -3x\ln(\ln u)\,(\ln(\ln t) - \ln(\ln u))\frac{d\ln u}{\ln u}$$

$$= -3x(\ln(\ln t))\int_x^e \ln(\ln u)\frac{d\ln u}{\ln u} + 3x\int_x^e (\ln(\ln u))^2\frac{d\ln u}{\ln u}$$

$$= -3x(\ln(\ln t))\frac{(\ln(\ln u))^2}{2}\Bigg|_t^e + 3x\frac{(\ln(\ln u))^3}{3}\Bigg|_t^e$$

$$= \frac{3x}{2}(\ln(\ln t))^3 - x(\ln(\ln t))^3 = \frac{x(\ln(\ln t))^3}{2}$$

Example (7): To find the solution of the ordinary differential equation

$(\ln t)^2 u_{tt}(x,\ln t) + \ln t\, u_t(x,\ln t) - u(x,\ln t) = -e^x\sinh\ln(\ln t),$

$u_t(x,\ln e) = u(x,\ln e) = 0$

Take Z to both sides

$Z[(\ln t)^2 u_{tt}(x,\ln t)] + Z[\ln t\, u_t(x,\ln t)] - Z[u(x,\ln t)] = Z[-e^x\sinh\ln(\ln x)]$

$$u_t(x,1)-(s+2)u(x,1)+(s+2)(s+1)Z\big(u(x,\ln t)\big)+u(x,1)-(s+1)Z\big(u(x,\ln t)\big) + Z\big(u(x,\ln t)\big)=\frac{e^x}{((s+1)^2-1)}$$

$$(s^2+2s+1-1)Z\big(u(x,\ln t)\big)=\frac{e^x}{((s+1)^2-1)}$$

$$Z\big(u(x,\ln t)\big)=\frac{e^x}{((s+1)^2+1)^2}\Longrightarrow u(x,\ln t)=Z^{-1}\left[\frac{-e^x}{((s+1)^2-1)}.\frac{-1}{((s+1)^2-1)}\right]$$

$$=e^x\sinh\ln(\ln t)*_Z\sinh\ln(\ln t)$$

$$=\int_t^e e^x\sinh\ln(\ln \mathrm{u})\sinh\ln\left(\frac{\ln t}{\ln \mathrm{u}}\right)\frac{d\ln u}{\ln u}$$

$$=\int_t^e e^x\sinh\ln(\ln u)\left(\sinh(\ln(\ln t)-\ln(\ln u))\right)\frac{d\ln u}{\ln u}$$

$$=\int_t^e e^x\sinh\ln(\ln u)\,(\sinh\ln(\ln t)\cosh\ln(\ln u)-\cosh\ln(\ln t)\sinh\ln(\ln u))\frac{d\ln u}{\ln u}$$

$$=e^x\sinh\ln(\ln t)\int_t^e\sinh\ln(\ln u)\cosh\ln(\ln u)\frac{d\ln u}{\ln u}-e^x\cosh\ln(\ln t)\int_t^e(\sinh\ln(\ln u))^2\frac{d\ln u}{\ln u}$$

$$=e^x\sinh\ln(\ln t)\left.\frac{\big(\cosh\ln(\ln u)\big)^2}{2}\right|_t^e-e^x\cosh\ln(\ln t)\int_t^e\left(\frac{\cosh 2\ln(\ln u)-1}{2}\right)\frac{d\ln u}{\ln u}$$

$$=e^x\sinh\ln(\ln t)\left[\frac{1}{2}-\frac{\big(\cosh\ln(\ln t)\big)^2}{2}\right]-\frac{1}{2}e^x\cosh\ln(\ln t)\left(\frac{1}{2}\sinh 2\ln(\ln u)|_t^e-\ln(\ln u)|_t^e\right)$$

$$=\frac{1}{2}e^x\sinh\ln(\ln t)-e^x\sinh\ln(\ln t)\frac{\big(\cosh\ln(\ln t)\big)^2}{2}+\frac{e^x}{2}\sinh\ln(\ln t)\,(\cosh\ln(\ln t))^2 -\frac{e^x}{2}\ln(\ln t)\cosh\ln(\ln t)$$

$$=\frac{e^x}{2}\sin\ln(\ln t)-\frac{e^x}{2}\ln(\ln t)\cosh\ln(\ln t).$$

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