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RESEARCH ARTICLE

SOME IDENTITIES FOR THE FIBONACCI AND LUCAS SEQUENCES WITH RATIONAL SUBSCRIPT VIA MATRIX METHODS

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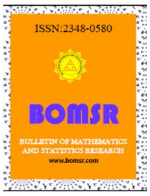

ABSTRACT

In this study, some identities for the Fibonacci and Lucas numbers with rational subscripts are established via taken the general techniques from matrix theory. For these aims, the two well-known Fibonacci matrices are considered, and special functions of the Fibonacci matrices are achieved by using certain scalar complex functions. Some identities involving terms of the Fibonacci and Lucas numbers with rational subscripts are given by these functions of the Fibonacci matrices.

Keywords: Fibonacci Numbers, Lucas Numbers, Matrix Functions.

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1. Introduction

The Fibonacci and Lucas numbers are the sequence of numbers $\{F_n\}_{n=0}^{\infty}$ and $\{L_n\}_{n=0}^{\infty}$ defined by the linear recurrence equation $F_{n+2} = F_{n+1} + F_n$ and $L_{n+2} = L_{n+1} + L_n$ with $F_1 = 1$, $F_0 = 0$ and $L_1 = 1$, $L_0 = 2$ [3], [11]. Also, the Binet's formulas for the F_n and L_n is given by $F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$, $L_n = \alpha^n + \beta^n$ where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of equation $x^2 - x - 1 = 0$. Using Binet's formulas, the definition of the F_n and L_n can be extended to negative integers n according to

$$F_{-n} = (-1)^{n+1}F_n, \textit{ and } L_{-n} = (-1)^{n+1}L_n, n \in \mathbb{Z}^+.$$

In literature authors have described a lot of methods that may be used to define the Fibonacci and the Lucas numbers with real or complex subscripts [6], [7], [8], [10], [11]. First of them, which Halsey has observed, is that the Gamma functions extends the notion of Fibonacci sequence to rational numbers. Halsey has developed a function which give Fibonacci numbers F_n for any integer n, and which also give real Fibonacci numbers F_u for any rational number u [6]. But, $F_{u+2} = F_{u+1} + F_u$,

$u \geq 0$ as known the Fibonacci recurrence identity is not valid for the real Fibonacci numbers given with this function. However, Parker has revised the recurrence identity and the restriction that u must be rational by using a function $F(x)$ defined for any real number x in the form

$$F(x) = \frac{\alpha^x - cos(\pi x)\alpha^{-x}}{\sqrt{5}},$$

where $\alpha = (1 + \sqrt{5})/2$ is the golden ratio [7]. The well known Fibonacci identity $F(2x) = F(x)L(x)$ is destroyed for the Parker's function $F(x)$. Later, Horadam and Shannaon [10] defined the following Fibonacci and the Lucas curves with complex notation:

$$F(x) = \frac{\alpha^x - e^{x\pi i}\alpha^{-x}}{\sqrt{5}}, L(x) = \alpha^x + e^{x\pi i}\alpha^{-x}.$$

The functions $F(x)$ and $L(x)$ defined in (1) hold for analogous identities of the classical Fibonacci and the Lucas numbers [11] and can be called as generalized Binet's formula for the Fibonacci $F(x) = F_x$ and the Lucas $L(x) = L_x$ numbers with real subscripts, respectively.

The Fibonacci and the Lucas numbers are generated by the matrices as well as the recurrence relations and the Binet's formulas [2], [3], [5], [11]. Matrices provide a different way of producing properties of the Fibonacci and the Lucas numbers via multiplying of its different powers and taking successive integer powers of a matrix.

The purpose of this study is to establish the Fibonacci and the Lucas numbers with rational subscripts, using matrix functions of the Fibonacci matrices in [2], [3], [5], [11];

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \ and \ Q_R = \frac{1}{2}\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix},$$

via the generalized Binet's formula, i.e.,

$$F_x = \frac{\alpha^x - \beta^x}{\sqrt{5}}, L_x = \alpha^x + \beta^x, x \in \mathbb{R}, \qquad (1)$$

and the general techniques taken from matrix theory.

2. On the Fibonacci and Lucas Sequences with Rational Subscript

Since the matrices Q and Q_R possess two distinct eigenvalues α, is golden ratio $\alpha = (1 + \sqrt{5})/2$ and $\beta = 1 - \alpha = \alpha^{-1}$, it is well known that there exists any invertible matrices V and U to rewrite the matrices $Q = V\,J_Q\,V^{-1}$ and $Q = U\,J_{Q_R}\,U^{-1}$, where J_Q and J_{Q_R} designate the Jordan canonical form associated to Q and Q_R. Now, by using the scalar complex function $F^{(r)}(z) = z^r, r \in \mathbb{Z}$, the matrix functions $F^{(r)}(Q) = V\left(F^{(r)}(J_Q)\right)V^{-1} = Q^r$ and $F^{(r)}(Q_R) = U\left(F^{(r)}\left(J_{Q_R}\right)\right)U^{-1} = Q_R^r$ are known as

$$F^{(r)}(Q) = \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha^r & 0 \\ 0 & \beta^r \end{bmatrix}\begin{bmatrix} 1/(\alpha-\beta) & -\beta/(\alpha-\beta) \\ -1/(\alpha-\beta) & \alpha/(\alpha-\beta) \end{bmatrix} = \begin{bmatrix} F_{r+1} & F_r \\ F_r & F_{r-1} \end{bmatrix}, \qquad (2)$$

$$F^{(r)}(Q_R) = \begin{bmatrix} 1 & 1 \\ 1/(\alpha-\beta) & -1/(\alpha-\beta) \end{bmatrix}\begin{bmatrix} \alpha^r & 0 \\ 0 & \beta^r \end{bmatrix}\begin{bmatrix} 1/2 & (\alpha-\beta)/2 \\ 1/2 & (\beta-\alpha)/2 \end{bmatrix} = \begin{bmatrix} L_r/2 & 5F_r/2 \\ F_r/2 & L_r/2 \end{bmatrix}. \qquad (3)$$

It is well known there are the properties of the Fibonacci matrix Q^r in the (2) and the Fibonacci-Lucas matrix Q_R^r in the (3) for $r \in \mathbb{Z}$. But, as we have used the equations (2) and (3) in this study and shown any integer powers of the these matrices with the Jordan canonical form associated to Q and Q_R.

Now, let us consider that the complex function $F^{(1,s)}(z) \equiv z^{1/s} = f_k^{(1,s)}$, $s \in \mathbb{Z} \setminus \{0\}$, $k \in \{0,1,\dots,s-1\}$ gives the s^{th} roots of the complex number z. In other respects, for every nonzero complex number $z = |z|exp\,[i\,arg(z)], (\pi \leq arg(z) \leq \pi)$, let $f_k^{(1,s)}(z)$ be the k^{th} branch of the function $F^{(1,s)}(z)$, these branches may be characterized as follows

$$f_k^{(1,s)}(z) = z^{1/s} exp\left[\frac{1}{s}(i\, arg(z) + 2k\pi i)\right], k \in \{0,1,\cdots,s-1\}.$$

Also, the principal branch of $F^{(1,s)}(z)$ is denoted by $f_0^{(1,s)}(z) = z^{1/s}$, and for every nonzero z in $\mathbb{C}$ the all branches of $F^{(1,s)}(z)$ are rewritten as

$$f_k^{(1,s)}(z) = z^{1/s} exp\left(\frac{2k\pi i}{s}\right), k \in \{0,1,\cdots,s-1\}. \qquad (4)$$

In fact, using the equations (2) and (3), we have derived the matrices $F^{(1,s)}(Q^r) = V\left(F^{(1,s)}(J_{Q^r})\right)V^{-1}$ and $F^{(1,s)}(Q_R^r) = U\left(F^{(1,s)}\left(J_{Q_R^r}\right)\right)U^{-1}$ defined with the matrix functions;

$$F_{(k_1,k_2)}^{(1,s)}(Q^r) = V\begin{bmatrix} f_{k_1}^{(1,s)}(\alpha^r) & 0 \\ 0 & f_{k_2}^{(1,s)}(\beta^r)\end{bmatrix}V^{-1}, \qquad (5)$$

$$F_{(k_1,k_2)}^{(1,s)}(Q_R^r) = U\begin{bmatrix} f_{k_1}^{(1,s)}(\alpha^r) & 0 \\ 0 & f_{k_2}^{(1,s)}(\beta^r)\end{bmatrix}U^{-1}, \qquad k_1,k_2 \in \{0,1,\cdots,s-1\}, \qquad (6)$$

where the matrices U, U^{-1}, V and V^{-1} are given in (2) and (3). It is seen that the matrix functions $F^{(1,s)}(Q)$ and $F^{(1,s)}(Q_R)$ are the multivalued functions giving rise to s^2 branches [4], [9]. The functions $f_{(k_1,k_2)}^{(1,s)}(Q^r)$ and $f_{(k_1,k_2)}^{(1,s)}(Q_R^r)$ denote the s^{th} roots of the matrices Q^r and Q_R^r.

Theorem 1 *Let $\frac{r}{s}$ be an irreducible fraction, $r \in \mathbb{Z}\setminus\{0\}$ and $s \in \mathbb{N}\setminus\{0\}$. Then,*

$$f_{(k_1,k_2)}^{(1,s)}(Q^r) = \begin{bmatrix} \frac{C_1+C_2}{2}F_{\frac{r}{s}+1} + \frac{C_1-C_2}{2\sqrt{5}}L_{\frac{r}{s}+1} & \frac{C_1+C_2}{2}F_{\frac{r}{s}} + \frac{C_1-C_2}{2\sqrt{5}}L_{\frac{r}{s}} \\ \frac{C_1+C_2}{2}F_{\frac{r}{s}} + \frac{C_1-C_2}{2\sqrt{5}}L_{\frac{r}{s}} & \frac{C_1+C_2}{2}F_{\frac{r}{s}-1} + \frac{C_1-C_2}{2\sqrt{5}}L_{\frac{r}{s}-1}\end{bmatrix},$$

and

$$f_{(k_1,k_2)}^{(1,s)}(Q_R^r) = \frac{1}{2}\begin{bmatrix} \sqrt{5}\frac{C_1-C_2}{2}F_{\frac{r}{s}} + \frac{C_1+C_2}{2}L_{\frac{r}{s}} & \frac{C_1+C_2}{2}F_{\frac{r}{s}} + \frac{C_1-C_2}{2\sqrt{5}}L_{\frac{r}{s}} \\ 5\frac{C_1+C_2}{2}F_{\frac{r}{s}} + \sqrt{5}\frac{C_1-C_2}{2}L_{\frac{r}{s}} & \sqrt{5}\frac{C_1-C_2}{2}F_{\frac{r}{s}} + \frac{C_1+C_2}{2}L_{\frac{r}{s}}\end{bmatrix},$$

where $C_1 = exp\left(\frac{2k_1r\pi i}{s}\right)$ and $C_1 = exp\left(\frac{2k_2r\pi i}{s}\right)$ and $k_1,k_2 \in \{0,1,\cdots,s-1\}$.

Proof. The matrix functions $F^{(1,s)}(Q^r)$ are computed by using the equation (5) in forms

$$f_{(k_1,k_2)}^{(1,s)}(Q^r) = \begin{bmatrix}\alpha & \beta \\ 1 & 1\end{bmatrix}\begin{bmatrix} f_{k_1}^{(1,s)}(\alpha^r) & 0 \\ 0 & f_{k_2}^{(1,s)}(\beta^r)\end{bmatrix}\begin{bmatrix} 1/(\alpha-\beta) & -\beta/(\alpha-\beta) \\ -1/(\alpha-\beta) & \alpha/(\alpha-\beta)\end{bmatrix}$$

$$= \frac{1}{\alpha-\beta}\begin{bmatrix} exp\left(\frac{2k_1r\pi i}{s}\right)\alpha^{\frac{r}{s}+1} - exp\left(\frac{2k_2r\pi i}{s}\right)\beta^{\frac{r}{s}+1} & exp\left(\frac{2k_1r\pi i}{s}\right)\alpha^{\frac{r}{s}} - exp\left(\frac{2k_2r\pi i}{s}\right)\beta^{\frac{r}{s}} \\ exp\left(\frac{2k_1r\pi i}{s}\right)\alpha^{\frac{r}{s}} - exp\left(\frac{2k_2r\pi i}{s}\right)\beta^{\frac{r}{s}} & exp\left(\frac{2k_1r\pi i}{s}\right)\alpha^{\frac{r}{s}-1} - exp\left(\frac{2k_2r\pi i}{s}\right)\beta^{\frac{r}{s}-1}\end{bmatrix}$$

$$f_{(k_1,k_2)}^{(1,s)}(Q^r) = \begin{cases} q_{11} = \frac{C_1+C_2}{2\sqrt{5}}\left(\alpha^{\frac{r}{s}+1} - \beta^{\frac{r}{s}+1}\right) + \frac{C_1-C_2}{2\sqrt{5}}\left(\alpha^{\frac{r}{s}+1} + \beta^{\frac{r}{s}+1}\right) \\ q_{12} = q_{21} = \frac{C_1+C_2}{2\sqrt{5}}\left(\alpha^{\frac{r}{s}} - \beta^{\frac{r}{s}}\right) + \frac{C_1-C_2}{2\sqrt{5}}\left(\alpha^{\frac{r}{s}} + \beta^{\frac{r}{s}}\right) \\ q_{22} = \frac{C_1+C_2}{2\sqrt{5}}\left(\alpha^{\frac{r}{s}-1} - \beta^{\frac{r}{s}-1}\right) + \frac{C_1-C_2}{2\sqrt{5}}\left(\alpha^{\frac{r}{s}-1} + \beta^{\frac{r}{s}-1}\right)\end{cases}$$

where $C_1 = exp\left(\frac{2k_1r\pi i}{s}\right)$ and $C_2 = exp\left(\frac{2k_2r\pi i}{s}\right)$ and $k_1,k_2 \in \{0,1,\cdots,s-1\}$. By using the generalized Binet's formula, we have achieved the matrix functions $f_{(k_1,k_2)}^{(1,s)}(Q^r)$.

Since the matrix Q_R^r has exactly the same eigenvalues as the matrix Q^r, the matrix functions $f_{(k_1,k_2)}^{(1,s)}(Q_R^r)$, $k_1, k_2 \in \{0,1,\dots,s-1\}$ have been obtained by similar calculation for the matrix Q_R^r and substituting Q by Q_R in the proof above.

If the matrix $F^{(1,2)}(Q_R)$ is defined for $s=2$ on the $F^{(1,s)}(Q^r)$ (and $F^{(1,s)}(Q_R^r)$), we obtain matrix functions $f_{(k_1,k_2)}^{(1,2)}(Q^r)$, which are derived from the four branches $k_1, k_2 \in \{0,1\}$, and are defined by:

Corollary 2 *The square root matrices of the matrices Q^r and Q_R^r are obtained as*

$$f_{(0,0)}^{(1,2)}(Q^r) = -f_{(1,1)}^{(1,2)}(Q^r) = \begin{bmatrix} F_{\frac{r}{2}+1} & F_{\frac{r}{2}} \\ F_{\frac{r}{2}} & F_{\frac{r}{2}-1} \end{bmatrix},$$

$$f_{(0,1)}^{(1,2)}(Q^r) = -f_{(1,0)}^{(1,2)}(Q^r) = \frac{1}{\sqrt{5}}\begin{bmatrix} L_{\frac{r}{2}+1} & L_{\frac{r}{2}} \\ L_{\frac{r}{2}} & L_{\frac{r}{2}-1} \end{bmatrix},$$

and

$$f_{(0,0)}^{(1,2)}(Q_R^r) = -f_{(1,1)}^{(1,2)}(Q_R^r) = \frac{1}{2}\begin{bmatrix} L_{\frac{r}{2}} & F_{\frac{r}{2}} \\ 5F_{\frac{r}{2}} & L_{\frac{r}{2}} \end{bmatrix},$$

$$f_{(0,1)}^{(1,2)}(Q_R^r) = -f_{(1,0)}^{(1,2)}(Q_R^r) = \frac{\sqrt{5}}{2}\begin{bmatrix} F_{\frac{r}{2}} & \frac{1}{5}L_{\frac{r}{2}} \\ L_{\frac{r}{2}} & F_{\frac{r}{2}} \end{bmatrix}.$$

In literature [1], Bicknell motions that there is only a square root matrix of the matrix Q, and by taking successive odd integer powers of this matrix, the author established

$$Q^{r/2} = \begin{bmatrix} F_{(r+2)/2} & F_{r/2} \\ F_{r/2} & F_{(r-2)/2} \end{bmatrix} = f_{(0,0)}^{(1,2)}(Q^r), \qquad (7)$$

which is established by induction, using algebraic manipulation and identities such as

$$F_{(r+2)/2}^2 + F_{r/2}^2 = F_{r+1}, \quad F_{(r+2)/2}F_{r/2} + F_{r/2}F_{(r-2)/2} = F_r,$$

obtained from the generalized Binet's formulas. Also, taking the determinant of the matrix $Q^{r/2}$ given in (7) and using matrix multiply of different power of the matrix $Q^{r/2}$, Bicknell derived some identities such as

$$F_{(r+2)/2}F_{(r-2)/2} - F_{r/2}^2 = (-1)^{r/2} = i^{r/2},$$
$$F_{(r+3)/2}F_{(r+2)/2} + F_{(r+1)/2}F_{r/2} = F_{(2r+3)/2},$$
$$F_{(2r+1)/2}L_{(2r+1)/2} = F_{2r+1},$$

for the Fibonacci and Lucas numbers whose subscripts are odd multiples of one-half.

Obviously, the property of square root matrix that the square roots of a 2×2 matrix A are those 2×2 matrices X for which $X^2 = A$, we have

$$f_{(k_1,k_2)}^{(1,2)}(Q^r) \times f_{(k_1,k_2)}^{(1,2)}(Q^r) = Q^r \qquad (8)$$

and

$$f_{(k_1,k_2)}^{(1,2)}(Q_R^r) \times f_{(k_1,k_2)}^{(1,2)}(Q_R^r) = Q_R^r. \qquad (9)$$

By using the equations (8) and (9), we can achieve some results;

Theorem 3 *For any odd integer r,*

i) $F_{\frac{r}{2}+1}F_{\frac{r}{2}-1} - F_{\frac{r}{2}}^2 = \pm e^{r\pi i/2}$,

ii) $L_{\frac{r}{2}+1}L_{\frac{r}{2}-1} - L_{\frac{r}{2}}^2 = \pm 5e^{r\pi i/2}$,

iii) $L_{\frac{r}{2}}^2 - 5F_{\frac{r}{2}}^2 = \pm 4e^{r\pi i/2}$.

Proof. It is well known that taking determinants of the matrix equations (8) and (9) give the following equations;

$$\left[det\left(f^{(1,2)}_{(k_1,k_2)}(Q^r)\right)\right]^2 = det(Q^r) = \left(det(Q)\right)^r.$$

Therefore, we obtain that

$$det\left(f^{(1,2)}_{(k_1,k_2)}(Q^r)\right) = \pm e^{r\pi i/2}, \text{ and } det\left(f^{(1,2)}_{(k_1,k_2)}(Q^r_R)\right) = \pm e^{r\pi i/2}.$$

When r is an even number, these identities are well known Cassini's identities for the Fibonacci and Lucas sequences with integer subscript.

Theorem 4 *For every integer odd number* r

i) $F_{r+1} = F^2_{\frac{r}{2}+1} + F^2_{\frac{r}{2}} = \frac{1}{5}\left(L^2_{\frac{r}{2}+1} + L^2_{\frac{r}{2}}\right)$,

ii) $2L_r = L^2_{\frac{r}{2}} + 5F^2_{\frac{r}{2}}$,

iii) $F_r = F_{\frac{r}{2}}F_{\frac{r}{2}+1} + F_{\frac{r}{2}-1}F_{\frac{r}{2}} = \frac{1}{5}\left(L_{\frac{r}{2}}L_{\frac{r}{2}+1} + L_{\frac{r}{2}-1}L_{\frac{r}{2}}\right) = F_{\frac{r}{2}}L_{\frac{r}{2}}$.

Proof. The proof is completed by equalizing of corresponding elements for the matrix equations (8) and (9).

When r is an even number, similar identities are well known identities for usual the Fibonacci and Lucas numbers.

Now, let consider the matrix equations

$$f^{(1,2)}_{(k_1,k_2)}(Q^{r_1}) \times f^{(1,2)}_{(k_1,k_2)}(Q^{r_2}) = f^{(1,2)}_{(k_1,k_2)}(Q^{r_1+r_2})$$

and

$$f^{(1,2)}_{(k_1,k_2)}(Q^{r_1}_R) \times f^{(1,2)}_{(k_1,k_2)}(Q^{r_2}_R) = f^{(1,2)}_{(k_1,k_2)}(Q^{r_1+r_2}_R), (r_1, r_2 \in \mathbb{Z} \setminus \{0\})$$

then following identities are valid;

Theorem 5 *For all integers* $r_1, r_2 \in \mathbb{Z} \setminus \{0\}$, *the following equalities are valid*:

i) $5F_{\frac{r_1+r_2}{2}+1} = L_{\frac{r_1}{2}+1}L_{\frac{r_2}{2}+1} + L_{\frac{r_1}{2}}L_{\frac{r_2}{2}}$,

ii) $5F_{\frac{r_1+r_2}{2}} = L_{\frac{r_1}{2}}L_{\frac{r_2}{2}+1} + L_{\frac{r_1}{2}-1}L_{\frac{r_2}{2}}$,

iii) $2L_{\frac{r_1+r_2}{2}} = L_{\frac{r_1}{2}}L_{\frac{r_2}{2}} + 5F_{\frac{r_1}{2}}F_{\frac{r_2}{2}}$,

iv) $2F_{\frac{r_1+r_2}{2}} = F_{\frac{r_1}{2}}L_{\frac{r_2}{2}} + L_{\frac{r_1}{2}}F_{\frac{r_2}{2}}$,

v) $L_{\frac{r_1+r_2}{2}+1} = F_{\frac{r_1}{2}+1}L_{\frac{r_2}{2}+1} + F_{\frac{r_1}{2}}L_{\frac{r_2}{2}}$,

vi) $L_{\frac{r_1+r_2}{2}} = F_{\frac{r_1}{2}}L_{\frac{r_2}{2}+1} + F_{\frac{r_1}{2}-1}L_{\frac{r_2}{2}}$,

vii) $e^{r_2\pi i/2}F_{\frac{r_1-r_2}{2}} = F_{\frac{r_1}{2}}F_{\frac{r_2}{2}-1} - F_{\frac{r_1}{2}-1}F_{\frac{r_2}{2}}, (r_1 > r_2)$

viii) $2e^{r_2\pi i/2}L_{\frac{r_1-r_2}{2}} = L_{\frac{r_1}{2}}L_{\frac{r_2}{2}} - 5F_{\frac{r_1}{2}}F_{\frac{r_2}{2}}, (r_1 > r_2)$.

Proof. The proof is based on method to find a Jordan canonical form of matrices, since the Q and Q_R admits to distinct eigenvalues α and β, there exists the invertible matrix V and U such that

$$f^{(1,2)}_{(k_1,k_2)}(Q^{r_1}) \times f^{(1,2)}_{(k_1,k_2)}(Q^{r_2}) = V\left(f^{(1,2)}_{(k_1,k_1)}(J^{r_1}_Q) \times f^{(1,2)}_{(k_1,k_1)}(J^{r_2}_Q)\right)V^{-1}$$

$$V\begin{bmatrix} f^{(1,2)}_{(k_1)}(\alpha^{r_1+r_2}) & 0 \\ 0 & f^{(1,2)}_{(k_2)}(\beta^{r_1+r_2}) \end{bmatrix}V^{-1} = f^{(1,2)}_{(k_1,k_2)}(Q^{r_1+r_2})$$

and

$$f^{(1,2)}_{(k_1,k_2)}(Q^{r_1}_R) \times f^{(1,2)}_{(k_1,k_2)}(Q^{r_2}_R) = U\left(f^{(1,2)}_{(k_1,k_1)}(J^{r_1}_{Q_R}) \times f^{(1,2)}_{(k_1,k_1)}(J^{r_2}_{Q_R})\right)U^{-1}$$

$$U\begin{bmatrix} f^{(1,2)}_{(k_1)}(\alpha^{r_1+r_2}) & 0 \\ 0 & f^{(1,2)}_{(k_2)}(\beta^{r_1+r_2}) \end{bmatrix}U^{-1} = f^{(1,2)}_{(k_1,k_2)}(Q^{r_1+r_2}_R)$$

where the matrices J^r_Q and $J^r_{Q_R}$ designate the Jordan canonical form associated to the matrices Q^r and Q^r_R. Consequently, by performing $f^{(1,2)}_{(0,1)}(Q^{r_1}) \times f^{(1,2)}_{(0,1)}(Q^{r_2}) = f^{(1,2)}_{(0,1)}(Q^{r_1+r_2})$ and $f^{(1,2)}_{(0,1)}(Q^{r_1}_R) \times f^{(1,2)}_{(0,1)}(Q^{r_2}_R) = f^{(1,2)}_{(0,1)}(Q^{r_1+r_2}_R)$, the desired results are established. For all integers r_1 and r_2, only two rational cases are considered:

If r_1 and r_2 are both odd integer, then r_1+r_2 is even, the equalities $(i)-(iv)$ is achieved with

$$\frac{1}{5}\begin{bmatrix} L_{\frac{r_1}{2}+1} & L_{\frac{r_1}{2}} \\ L_{\frac{r_1}{2}} & L_{\frac{r_1}{2}-1} \end{bmatrix}\begin{bmatrix} L_{\frac{r_2}{2}+1} & L_{\frac{r_2}{2}} \\ L_{\frac{r_2}{2}} & L_{\frac{r_2}{2}-1} \end{bmatrix}=\begin{bmatrix} F_{\frac{r_1+r_2}{2}+1} & F_{\frac{r_1+r_2}{2}} \\ F_{\frac{r_1+r_2}{2}} & F_{\frac{r_1+r_2}{2}-1} \end{bmatrix}$$

and

$$\frac{5}{4}\begin{bmatrix} F_{\frac{r_1}{2}} & \frac{1}{5}L_{\frac{r_1}{2}} \\ L_{\frac{r_1}{2}} & F_{\frac{r_1}{2}} \end{bmatrix}\begin{bmatrix} F_{\frac{r_2}{2}} & \frac{1}{5}L_{\frac{r_2}{2}} \\ L_{\frac{r_2}{2}} & F_{\frac{r_2}{2}} \end{bmatrix}=\frac{1}{2}\begin{bmatrix} L_{\frac{r_1+r_2}{2}} & F_{\frac{r_1+r_2}{2}} \\ 5F_{\frac{r_1+r_2}{2}} & L_{\frac{r_1+r_2}{2}} \end{bmatrix}.$$

If r_1 is even and r_2 is odd, then r_1+r_2 is odd, the equalities $(v)-(vi)$ are obtained with

$$\frac{1}{\sqrt{5}}\begin{bmatrix} F_{\frac{r_1}{2}+1} & F_{\frac{r_1}{2}} \\ F_{\frac{r_1}{2}} & F_{\frac{r_1}{2}-1} \end{bmatrix}\begin{bmatrix} L_{\frac{r_2}{2}+1} & L_{\frac{r_2}{2}} \\ L_{\frac{r_2}{2}} & L_{\frac{r_2}{2}-1} \end{bmatrix}=\frac{1}{\sqrt{5}}\begin{bmatrix} L_{\frac{r_1+r_2}{2}+1} & L_{\frac{r_1+r_2}{2}} \\ L_{\frac{r_1+r_2}{2}} & L_{\frac{r_1+r_2}{2}-1} \end{bmatrix}$$

and

$$\frac{\sqrt{5}}{4}\begin{bmatrix} L_{\frac{r_1}{2}} & F_{\frac{r_1}{2}} \\ 5F_{\frac{r_1}{2}} & L_{\frac{r_1}{2}} \end{bmatrix}\begin{bmatrix} F_{\frac{r_2}{2}} & \frac{1}{5}L_{\frac{r_2}{2}} \\ L_{\frac{r_2}{2}} & F_{\frac{r_2}{2}} \end{bmatrix}=\frac{\sqrt{5}}{2}\begin{bmatrix} F_{\frac{r_1+r_2}{2}} & \frac{1}{5}L_{\frac{r_1+r_2}{2}} \\ L_{\frac{r_1+r_2}{2}} & F_{\frac{r_1+r_2}{2}} \end{bmatrix}.$$

Since the matrices $f_{(k_1,k_2)}^{(1,2)}(Q^r)$ and $f_{(k_1,k_2)}^{(1,2)}(Q_R^r)$ are nonsingular, there are inverse of these matrices, let $f_{(k_1,k_2)}^{(-1,2)}(Q^r)$ and $f_{(k_1,k_2)}^{(-1,2)}(Q_R^r)$ denote inverse of matrices $f_{(k_1,k_2)}^{(1,2)}(Q^r)$ and $f_{(k_1,k_2)}^{(1,2)}(Q_R^r)$, which are given by

$$f_{(0,0)}^{(-1,2)}(Q^r)=-f_{(1,1)}^{(-1,2)}(Q^r)=\frac{1}{e^{r\pi i/2}}\begin{bmatrix} F_{\frac{r}{2}-1} & -F_{\frac{r}{2}} \\ -F_{\frac{r}{2}} & F_{\frac{r}{2}+1} \end{bmatrix},$$

$$f_{(0,1)}^{(-1,2)}(Q^r)=-f_{(1,0)}^{(-1,2)}(Q^r)=\frac{1}{\sqrt{5}e^{r\pi i/2}}\begin{bmatrix} -L_{\frac{r}{2}-1} & L_{\frac{r}{2}} \\ L_{\frac{r}{2}} & -L_{\frac{r}{2}+1} \end{bmatrix},$$

and

$$f_{(0,0)}^{(-1,2)}(Q_R^r)=-f_{(1,1)}^{(-1,2)}(Q_R^r)=\frac{1}{2e^{r\pi i/2}}\begin{bmatrix} L_{\frac{r}{2}} & -F_{\frac{r}{2}} \\ -5F_{\frac{r}{2}} & L_{\frac{r}{2}} \end{bmatrix},$$

$$f_{(0,1)}^{(-1,2)}(Q_R^r)=-f_{(1,0)}^{(-1,2)}(Q_R^r)=\frac{\sqrt{5}}{2e^{r\pi i/2}}\begin{bmatrix} F_{\frac{r}{2}} & \frac{-1}{5}L_{\frac{r}{2}} \\ -L_{\frac{r}{2}} & F_{\frac{r}{2}} \end{bmatrix}.$$

A lot of identities can be obtained comparing the entries (2,1) via method mentioned above using the matrix equations, the equalities $(vii)-(viii)$ are obtained with

$$f_{(0,0)}^{(1,2)}(Q^{r_1})\times f_{(0,0)}^{(-1,2)}(Q^{r_2})=f_{(0,0)}^{(1,2)}(Q^{r_1-r_2}),$$
$$f_{(0,0)}^{(1,2)}(Q_R^{r_1})\times f_{(0,0)}^{(-1,2)}(Q_R^{r_2})=f_{(0,0)}^{(1,2)}(Q_R^{r_1-r_2}).$$

In the remainder of this section, we'll focus on the $k_1=k_2=0$ principal branches of the previous matrix function:

$$f_{(0,0)}^{(1,s)}(Q^r)=\begin{bmatrix} F_{\frac{r}{s}+1} & F_{\frac{r}{s}} \\ F_{\frac{r}{s}} & F_{\frac{r}{s}-1} \end{bmatrix}, and\ f_{(0,0)}^{(1,s)}(Q_R^r)=\frac{1}{2}\begin{bmatrix} L_{\frac{r}{s}} & F_{\frac{r}{s}} \\ 5F_{\frac{r}{s}} & L_{\frac{r}{s}} \end{bmatrix}.$$

Let $r_1,r_2\in\mathbb{Z}\setminus\{0\}$, and $s\in\mathbb{N}\setminus\{0\}$. Then, it can be easily shown that:

$a)$ *If* $\frac{r_1}{s},\frac{r_2}{s}$ and $\frac{r_1+r_2}{s}$ *are all irreducible fractions, then*

$$f_{(0,0)}^{(1,s)}(Q^{r_1})\times f_{(0,0)}^{(1,s)}(Q^{r_2})=f_{(0,0)}^{(1,s)}(Q^{r_1+r_2})$$

and

$$f_{(0,0)}^{(1,s)}(Q_R^{r_1})\times f_{(0,0)}^{(1,s)}(Q_R^{r_2})=f_{(0,0)}^{(1,s)}(Q_R^{r_1+r_2}).$$

$b)$ *If* $\frac{r_1}{s}$ is any irreducible fractions and $\frac{r_2}{s}=l\in\mathbb{Z}\setminus\{0\}$, $\frac{r_1+r_2}{s}$ is any irreducible fractions, then

$$f_{(0,0)}^{(1,s)}(Q^{r_1}) \times Q^l = f_{(0,0)}^{(1,s)}(Q^{r_1+r_2})$$

and

$$f_{(0,0)}^{(1,s)}(Q_R^{r_1}) \times Q_R^l = f_{(0,0)}^{(1,s)}(Q_R^{r_1++r_2}).$$

Consequently, a number of results are obtained by performing mentioned above. For simplicity's sake, we omit the details which will appear in a similar argument below.

$a)$ If $\frac{r_1}{s}, \frac{r_2}{s}$ and $\frac{r_1+r_2}{s}$ *are all irreducible fractions, then*

i) $F_{\frac{r_1+r_2}{s}+1} = F_{\frac{r_1}{s}+1}F_{\frac{r_2}{s}+1} + F_{\frac{r_1}{s}}F_{\frac{r_2}{s}}$, *ii)* $F_{\frac{r_1+r_2}{s}-1} = F_{\frac{r_1}{s}}F_{\frac{r_2}{s}} + F_{\frac{r_1}{s}-1}F_{\frac{r_2}{s}-1}$,

iii) $F_{\frac{r_1+r_2}{s}} = F_{\frac{r_1}{s}}F_{\frac{r_2}{s}+1} + F_{\frac{r_1}{s}-1}F_{\frac{r_2}{s}}$, *iv)* $F_{\frac{r_1+r_2}{s}} = F_{\frac{r_1}{s}+1}F_{\frac{r_2}{s}} + F_{\frac{r_1}{s}}F_{\frac{r_2}{s}-1}$,

v) $F_{\frac{r_1+r_2}{s}} = \frac{1}{2}\left(L_{\frac{r_1}{s}}F_{\frac{r_2}{s}} + F_{\frac{r_1}{s}}L_{\frac{r_2}{s}}\right)$, *vi)* $L_{\frac{r_1+r_2}{s}} = \frac{1}{2}\left(L_{\frac{r_1}{s}}L_{\frac{r_2}{s}} + 5F_{\frac{r_1}{s}}F_{\frac{r_2}{s}}\right)$.

$b)$ If $\frac{r_1}{s}$ *is any irreducible fractions and* $\frac{r_2}{s} = l \in \mathbb{Z} \setminus \{0\}$, *then,*

i) $F_{\frac{r_1}{s}+l+1} = F_{\frac{r_1}{s}+1}F_{l+1} + F_{\frac{r_1}{s}}F_l$, *ii)* $F_{\frac{r_1}{s}+l} = F_{\frac{r_1}{s}+1}F_l + F_{\frac{r_1}{s}}F_{l-1}$,

iii) $F_{\frac{r_1}{s}+l} = F_{\frac{r_1}{s}}F_{l+1} + F_{\frac{r_1}{s}-1}F_l$, *iv)* $F_{\frac{r_1}{s}+l-1} = F_{\frac{r_1}{s}}F_l + F_{\frac{r_1}{s}-1}F_{l-1}$,

v) $F_{\frac{r_1}{s}+l} = \frac{1}{2}\left(L_{\frac{r_1}{s}}F_l + F_{\frac{r_1}{s}}L_l\right)$, *vi)* $L_{\frac{r_1}{s}+l} = \frac{1}{2}\left(L_{\frac{r_1}{s}}L_l + 5F_{\frac{r_1}{s}}F_l\right)$,

It seems that analogous identities of the Fibonacci and the Lucas numbers with integers subscripts hold for the Fibonacci and the Lucas numbers with rational subscripts.

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