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Proceedings of UGC Sponsored Two Day National Conference on

"EMERGING TRENDS IN MATHEMATICS"

8th & 9th December, 2016

Sponsored by UGC

Organised by

THE DEPARTMENT OF MATHEMATICS

SIR C R REDDY AUTONOMOUS COLLEGE (UG & PG)

(Affiliated to Adikavi Nannaya University, Rajamahendravaram)

ELURU – 534 007, W.G.Dt., A.P.

(Thrice accredited at 'A' level by NAAC, Bangalore;
College with Potential for Excellence)

Preface

The National Seminar is intending to motivate students and young faculty to pursue and also research in Mathematics. The seminar will provide the platform for researchers, senior researchers, create an environment for exchange of creativity of empirical ideas along with the practical and experimental methods. The main object of the seminar is to inspire teachers, academicians, researchers and students of Mathematics, to view Mathematics as a dynamics, ever growing discipline and to understand its applications.

To link the present research trends in mathematics to the students and the faculty participated in the seminar, experts in the different fields of mathematics are invited from various national level institutes. **Prof. N.V. Vignesam**, Retd. Head, Orbit Dynamics Division, ISRO Satellite Centre, Bangalore is invited for the key note address. **Prof. Y.V.S.S. Sanyasiraju,** Indian Institute of Technology (Madras), Chennai is invited to deliver a lecture on ***'Numerical Computations using Meshless Schemes'***. **Prof. B. Mishra,** Birla Institute of Technology & Sciences, Hyderabad is invited to cover the research field of various cosmological mathematical models. **Prof. G. Sarojamma**, Former Vice-Chancellor, Padmavathi Mahila Viswavidyalayam, Tirupathi is invited to cover various research models on non-linearity in differential equations. **Prof. D.N.L.V. Somayajulu,** Dean of Computer Science, National Institute of Technology, Warangal is invited to explain role of mathematics in sciences. **Prof. B. Sri Padmavathi**, Dean of School of Mathematics & Statistics, University of Hyderabad, Hyderabad is invited to explain the latest research trends in mathematics. **Prof. N. Bhaskar Reddy**, HOD, Department of Mathematics, Sri Venkateswara University, Tirupathi is invited to educate the audience in his research field which is 'Study of variable Physical Parameters in nanofluids'.

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ELURU CONSTITUENCY - A.P.

सत्यमेव जयते

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MESSAGE

India is one of the fast developing countries in the world. Our country needs a lot of development in the fields of Science & Technology. A lot of research is to go into these fields. The Government of India is encouraging research in these areas and seminars, such as these are the need of the hour as we all know there cannot be any science & Technology without mathematics.

I hope this seminar will provide a platform to discuss the latest trends in Mathematics which will be of immense value to all the participants. I wish the Department of Mathematics great success in their efforts.

(MAGANTTI VENKATESWARA RAO (BABU)
Member of Parliament (Lok Sabha)
Eluru Constituency — A.P.

BADETI KOTA RAMARAO (BUJJI)
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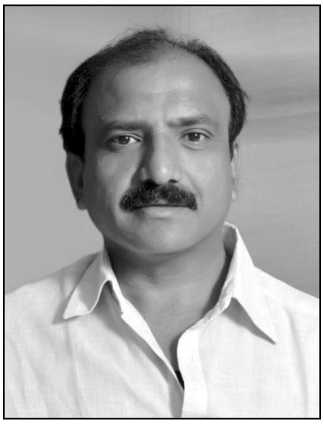

MESSAGE

A P J Abdul Kalam says we need to develop a society of knowledge. India has made its own contribution to the world in the field of Science and Mathematics. The Hindu-Arabic numeral system is India's great contribution to the world of Mathematics. We need to encourage youngsters to take interest in the field of Science & Technology and Mathematics is basic to their study.

I strongly believe this seminar will go a long away in throwing light on the emerging trends and enlighten all the lovers of Mathematics around.

(BADETI KOTA RAMARAO)
M.L.A., Eluru Assembly Constituency

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PRESIDENT Managing Committee

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MESSAGE

It is heartening to note that the Department of Mathematics is going to conduct a two-day National Seminar on "Emerging Trends in Mathematics" on 8^{th} and 9^{th} December, 2016. Mathematics is an age-old subject, as old as civilization itself. It is the heart and soul of human development on the planet earth. Whether one is educated or uneducated, he needs mathematics to live his day to day life. It is the most fundamental of all the subjects and very basic to civilization. Man needs mathematics next to food, clothes and shelter. It is one of the essentials of human life. Man cannot live as a civilized man without mathematics.

Romans killed Archimedes and impeded the progress of mathematics whereas India made a lot of contribution to mathematics even two thousand years before Pythagoras. Our Shatapatha Brahmana proves it. We are forerunners in the field of mathematics. Romans used mathematics for war and destruction but India used it for trade and commerce – for development.

I wish the department of mathematics a great success in this endeavour. I believe this seminar will throw much light on the recent trends in mathematics and help improve the knowledge of all the participants.

(KOMMAREDDY RAM BABU)
President

SIR CATTAMANCHI RAMALINGA REDDY EDUCATIONAL INSTITUTIONS, ELURU
(Regd. No. of the Society 10/1950) ADMINISTRATIVE OFFICE : `M.R.C. BHAVAN', G.N.T. Road,
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N.V.K. DURGA RAO
SECRETARY
Managing Committee

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MESSAGE

Oh mathematics! Without you how can anyone count his hearbeat, how can anyone check his blood pressure and how can anyone know his blood sugar – how can anyone tie 3 knots and walk 7 steps? Oh numbers! You make wonders. Without you how can write my door number, how can I write my date of birth, how can I measure my time? How can I get up at 5,0' clock in the morning and go to bed at 10,o' clock?

Without counting a bank cannot work – A factory cannot run – you cannot pay wages to a worker without counting – you cannot operate money without the software of numbers – you cannot talk on the telephone. There cannot be any calendar or almanac without numbers.

Numbers are India's greatest contribution to the world of knowledge. The numeral system changed the course of man's history on the planet earth.

Without mathematics – there cannot be physics, chemistry, medicine, history, economics… Mathematics is the life breath of all knowledge.

I appreciate the department of mathematics for their initiative to conduct this seminar and I wish them a resounding success.

(N.V.K. DURGA RAO)
Secretary

SIR C RAMALINGA REDDY COLLEGE, ELURU
(AIDED & AUTONOMOUS)
(College with Potential for Excellence ; Twice accredited at 'A' level by NAAC, **Bangalore**)

U.S. RAMPRASAD, B.E.
Correspondent

MESSAGE

Ancient people used symbols to represent different values. Slowly numbers evolved out of these symbols. Mario Livio, an eminent Astro – Physicist asks – Is God a Mathematician? Must be. There is mathematics in the universe. There are patterns in the stripes of a tiger, and the spirals of a sea shell – There are pattern of time in the movement of planets, sunrise and sunset. May be there is mathematics in the creation itself. There is mathematics in the intricate design of seeds on a sunflower. There is mathematics in the petals of a flower; in the meandering path of rivers; in the waves of the sea. Pythagoras said there is mathematics in music. There is 'pi' in apples. The whole gamut of computer game is nothing but mathematics. Everything is ultimately mathematics.

I think there is a mysterious connection between nature and mathematics. Mathematics is the language of the universe. It is the greatest achievement of our civilization. I am thrilled by the symbol of 'pi' and the value it represents. I often feel curious to know it history get – I am a layman to the subject and it is the job of the specialist to think of such intricate issues.

Well, I feel very glad about this seminar you are going to conduct and I wish you a grand success.

4/9/16

(U. S. RAMA PRASD)
Correspondent

SIR C RAMALINGA REDDY COLLEGE, ELURU
(AIDED & AUTONOMOUS)]
(College with Potential for Excellence ; Twice accredited at 'A' level by NAAC, Bangalore)

N. VEERRAJU CHOWDARY, Msc.,
Principal

MESSAGE

The mathematical concepts of number, magnitude and form are not unique to humans. Even some animals have this cognition. When we think of mathematics we remember the Ishango bone which was found more than twenty thousand years ago. The oldest and undisputed mathematical documents date back to Babylonian and Egyptian cultures. The ancient Babylon and Baghdad are centres of knowledge like the present day oxford and Cambridge.

Babylonian mathematics used Sexagesimal (base-60) numeral system. Form this system is derived the modern usage of 60 seconds in a minute, 60 minutes in an hour and 360 degrees in a circle etc. Moreover the Babylonians came close to a place value system where digits written in the left column represented larger values.

One really gets excited to know this highly intriguing subject and how it has evolved to the present status. Heartily encourage the department of mathematics in their endeavour to conduct this seminar – My best wishes are with them.

(N VEERRAJU CHOWDARY)
Principal

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DATE:05-12-2016

MESSAGE

It is indeed a great pleasure for me to write this short message on the occasion of two day National Seminar on Emerging Trends in Mathematics organized by the Department of Mathematics, Sir C.R. Reddy Autonomous College, Eluru, sponsored by U.G.C. during December 8th & 9th, 2016.

In this connection, I congratulate the Department of Mathematics and the Organizing Committee for conducting this seminar as it builds a platform for young Students and Researches to share their ideas and views.

I wish the National Seminar on Emerging Trends in Mathematics a grand success.

V. Uma Maheswara Rao
(V.UMA MAHESWARA RAO)
REGISTRAR

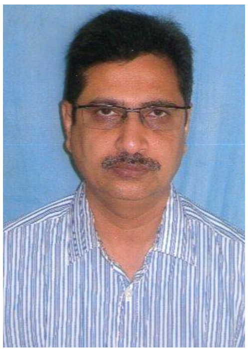
Prof.(Dr.) N.V.Vighnesam,
Dept. of Mathematics, DSCE, Bangalore
(Former Scientist; Head, Orbit Dynamics Division,
ISRO Satellite Centre, Bangalre)

MESSAGE

I am happy to note that the department of Mathematics of 'Sir C Ramalinga Reddy College, Eluru, A.P' is organizing National Seminar on "Emerging Trends in Mathematics". *Mathematics* is an indispensable subject of study. It plays an important role in forming the basis of all other sciences. Conference / Seminars are of great help in providing an opportunity to enlarge the knowledge horizon. I am sure this conference will be a forum for the academicians, researchers and industry experts to deliberate on various innovation in mathematics and its application. It also gives an opportunity for participants to interact, deliberate and debate on different fields of science, engineering and technology.

I wish the department of mathematics all the success. Wishing the very best to the organizers of the conference.

Prof. YVSS Sanyasi Raju
Professor, Department of Mathematics
IIT Madras, Chennai-36



I am happy to learn that the Department of Mathematics (UG & PG), Sir C R Reddy Autonomous College, Eluru is orgainising a UGC sponsored two Day National Seminar on "Emerging trends in Mathematics" during December 8-9, 2016. It is well known that a good knowledge of mathematical tools is very essential to understand any physical, biological or even an economical system, therefore, the proposed national seminar will serve enormously to the theorists and practitioners equally. Looking at the names of the experienced personnel from the R&D and also from the teaching institutions, in the list of speakers, I am sure it is going to be a two day exchange of ideas between the people who develop the mathematical tools and the people who actually use them in their applications. I congratulate the Organisers for providing a platform for this interaction through this national seminar. I have no doubt that the presentations of this seminar will help all the budding mathematicians in enhancing their knowledge. I wish the Conference a great success.

Sri Padmavati Mahila Visvavidyalayam

Tirupati-517502. Andhra Pradesh

Dr. G. Sarojamma
Former Vice-Chancellor
Vice-President AP Akademi of Sciences
Professor of Applied Mathematics
Tirupati

Message

I am happy to know that the Department of Mathematics, Dr. C. R. Reddy College, one the premier institutes of education is organizing a two day National Seminar on Emerging Trends in Mathematics on 8^{th} and 9^{th} of December,2016. It is heartening to note that the college has been establishing a name and fame through their yeoman service to the students. In recognition of its contribution, the college is accredited and reaccredited with **A** grade by NAAC. I am sure that this seminar would provide an exposure of the expertise of senior researchers to the students and a scope to choose a career in Mathematics. I congratulate the faculty of the department, particularly Dr. B. Jaganmohan Rao and the Principal and Chairman for encouraging the department in organizing such academic activities. I wish the seminar a grand success.

G Sarojama

(G SAROJAMMA)

Birla Institute of Technology & Scien
Hyderabad Campus

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It gives me immense pleasure to learn that Department of Mathematics, Sir C R Reddy Autonomous college, Eluru is organizing a two-day National Seminar on "Emerging Trends in Mathematics" on 8th and 9th December, 2016.

Modern science and engineering is almost entirely driven by the motion of a model. Anyproblem, which requires a quantitative answer, whether it is in industry, medicine,economics, biology, physics or mathematics, involves the formulation of the problem asa mathematical model.The ability to construct a mathematicalmodel for solving complex problem is a craft. People simulate real world phenomena bycreating models, which are either mathematical or abstract. Computational models enablethe study of complex dynamic worlds because they themselves are dynamic.Computational models prove their worth in analyzing complex situations. I hope this seminar would provide an opportunities for scientists,technologists and mathematicians to interact with each other and exchange ideas indeveloping mathematical models in different areas under various constraints.

I wish the organizer and all the participants to have a nice and productive discussions in this scientific event.

(Dr. B.Mishra)

University of Hyderabad

Prof. B. Sri Padmavati
Dean, School of Mathematics & Statistics

P.O. Central University
Hyderabad 500 046.

11th November, 2016

MESSAGE

I congratulate the organizers of the National Seminar on "Emerging Trends in Mathematics" (scheduled to be held during 8-9 December, 2016 at CR Reddy Autonomous College, Eluru) for their effort in organizing a national seminar on a very relevant / important topic that helps in motivating younger people to take up mathematics as a career option.

I also convey my best wishes for the success of the conference and hope that the interaction with researchers at this Seminar will be useful.

B.S. Padmavathi

(B. Sri Padmavati)
Dean, School of Mathematics & Statistics

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ACKNOWLEDGEMENTS

The two day National seminar on “Emerging Trends in Mathematics” has been made possible with the support of many technical experts, individuals and organizations both in man power and finance. This support is gratefully acknowledged.

I am very much grateful to our college managing committee for its constatnt encouragement in organizing this type of academic activities for the development of the college.

I owe a deep sense of gratitude to Sri Kommareddy Rambabu, President, Sir C R Reddy Educational institutions, Eluru and Chief Patron of the National Seminar for his constant encouragement, valuable guidance in organizing the seminar in most efficient way.

I am very thankful to Sri N. V. K. Durga Rao, Secretary, Sir C R Reddy Educational Institutions, Eluru for his precious support as Patron of the National Seminar.

My sincere and special thanks to Sri U S Rama Prasad, Correspondent, Sir C R Reddy College, Eluru for his encouragement, cooperation & meticulous guidance at every stage in organizing and planning the National Seminar and bring out this book.

I am thankful to Sri V. Raghu Kumar, Correspondent (PG Courses), Sir C R Reddy College, Eluru for encouragement in the success of the seminar and also I am grateful to Smt. C. Aruna Kumari, Director – PG Courses for her timely help during the seminar.

I am deeply indebted to Sri N. V. Chowdary, Principal, Sir C R Reddy College, Eluru for his constant support and having taken every responsibility for completing this task through various stages.

I profusely thank to Chief Guest Prof. V. Umamaheswara Rao, Registrar, Andhra University, Visakhapatnam for his valuable address to the gathering and make the seminar a grand success.

My sincere thanks to all the speakers, chairpersons, rapporteur of various technical sessions of the seminar have readily responded to our invitation to conduct the proceedings and to the addresses the gathering and for their kind gesture in the seminar.

I owe special thanks to Sri N.V.S.S. Patanjali, Superintendent, and his staff for their constant support throughout the seminar.

I sincerely thanks to KY Publications, Guntur for bringing out the proceedings of the seminar in book format.

I am very much grateful to my colleagues in the Department of Mathematics (PG & UG) and also organizing committee members for their continuous support in making this event successful.

Finally, I thank all the people and organizations who are directly and indirectly involved in organizing the seminar but I could not mention their names due to paucity of space.

I thank one and all.

Dr. B. Jaganmohan Rao
Organizing Secretary

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Invited Talks

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EFFECTS OF VARIABLE VISCOSITY ON MHD MIXED CONVECTIVE FLOW OF A DISSIPATIVE NANOFLUID OVER A VERTICAL MOVING SURFACE

N. Bhaskar Reddy
Dept of Mathematics, Sri Venkateswara University, Tirupati.

ABSTRACT
A steady two dimensional mixed convective MHD boundary layer flow of a nanofluid along a moving vertical surface with variable magnetic field and viscous dissipation is considered. It is assumed that the base fluid and the nanoparticles are in thermal equilibrium and no slip occurs between them. The governing non linear partial differential equations are transformed into a set of ordinary differential equations by suitable similarity transformations and the resultant equations are solved numerically using Runge-Kutta fourth order method along with shooting technique. The effects of the flow parameters on the velocity, temperature and concentration distribution of the flow field have been computed and discussed in detail. This type of problem finds applications in geophysical and astrophysical studies.

COSMOLOGICAL MODEL: AN INTRODUCTION AND MATHEMATICAL INTERPRETATION

B. Mishra
Department of Mathematics,
Birla Institute of Technology and Science-Pilani, Hyderabad Campus, Hyderabad, India-500078, E-mail:bivudutta@yahoo.com

ABSTRACT
The aim of the cosmologists is to construct mathematical models that represent the universe as a whole content rating on its large scale features. They compare these models as observed by the astronomers. Due to continued endeavour of astronomers, astrophysicists, and cosmologists, mathematicians it is possible to put together the wealth of theoretical predictions and the observational data on a unified well tested framework. Thus, cosmological model based on strong background of physical principles and mathematical techniques has been accepted as a prerequisite tool to take up experiments on cosmology and space science.

Now-a-days various cosmological and astrophysical problems are being studied by the physicists to reveal the evolution of the universe up to the present status based on Einstein's general theory of relativity. Recent cosmological observations suggest our universe is accelerating. This acceleration is explained in terms of late time acceleration of the universe and the existence of dark matter and dark energy. The simplest dark energy candidate is the cosmological constant, but it needs to be extremely fine-tuned.

HOMOTOPY ANALYSIS METHOD FOR SOLVING NON-LINEAR DIFFERENTIAL EQUATIONS

G. SAROJAMMA
Former Vice-Chancellor and
Professor of Applied Mathematics, Sri Padmavati Mahila Visvavidyalayam, Tirupati

ABSTRACT
Most of the problems in nature are nonlinear. Mathematical models for several problems are nonlinear differential equations. In general non linear differential do not contain exact solutions. We depend on numerical methods for their solutions. However, one delights in obtaining an analytical solution for the non linear differential equation. Recently the Homotopy Analysis Method introduced by Liao(2004) has become popular to solve nonlinear differential equations. This talk deals with the Homotopy Analysis Method. Application of this method is illustrated by solving simple first and second order non-linear differential equations and these solutions are compared with their corresponding exact solutions. The convergence of the HAM solutions in terms of h-curves is discussed. Further, the solutions of some partial differential equations shall be discussed.

“PRECISE POSITION ESTIMATION OF INDIAN REGIONAL NAVIGATION SATELLITE SYSTEM (IRNSS) AND IT’S APPLICATION TO USER POSITION”

Dr. N.V.VIGHNESAM
(Former Scientist, Head, Orbit Dynamics Division, ISRO Satellite Centre, Bangalore),
Professor, Department of Mathematics, Dayananda Sagar Collge of Engineering, Bangalore.

ABSTRACT
He delivered Key Note address on “Precise Position Estimation of Indian Regional Navigation Satellite System (IRNSS) and its application to User Position”. The contents on this topic include General Navigation System, Indian Regional Navigation Satellite System (IRNSS), and Methodology on IRNSS precise position estimation followed by User Position estimation. Finally specific research /Project areas of interest using IRNSS receiver data which can be carried out by Academic institutes are described. A brief description on the main features of IRNSS satellites orbit determination include Tracking systems, measurement errors, tracking data preprocessing, Satellite Ephemeris generation, Observation modeling and Estimation process is given. The Trilateration method for calculation of ‘User Position’ is explained at the end.

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NUMERICAL COMPUTATIONS USING MESHLESS SCHEMES

Y V S S SANYASIRAJU
Professor,
Department of Mathematics, IIT Madras, Chennai 36,

ABSTRACT

In this introductory lecture, an alternative method to solve interpolation and differential equation based problems in higher dimensions is introduced. Unlike the conventional schemes, these schemes are developed over a set of scattered points and insensitive to the dimension of the problem. Starting with a basic function, the basis is generated using the translates of the basic function. The translates are the data points in a general 'd' dimensional space known as centers in the terminology of the meshless schemes. Imposing the interpolation conditions results in to a linear algebraic system which has to be solved to find a solution of the interpolation problem. The well-posedness of the problems depends on the basic function and the characterization of this is depends on the nature of the basic function. Some one and two dimensional convection diffusion problems are solved using the developed meshless scheme and the accuracy is tested by comparing them with the results obtained using some of the conventional schemes. A time dependent simulation of a flow problem is presented to demonstrate the capability of the scheme.

Paper Presentations

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SOME APPLICATIONS OF MATHEMATICS IN REAL LIFE SITUATION

A.K.S. CHANDRASEKHARA RAO[1]; M.V.N. BHASKARA RAJU[2]; K.C. TAMMI RAJU[3]; P. SATYANARAYANA[4]
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ABSTRACT
The notion of mathematical induction is applied to solve some problems in real life situation, for which necessary theorems in $\mathbb{Z}$–module are stated and proved.

Key words: Integers, Ring, R–module, mathematical induction, linear combination.

1. Introduction

It is well known that the study of mathematics will not give the students of Science, Engineering or Social Science a hodgepodge of techniques, but will provide them with an understanding of basic mathematical concepts. It is a fact that the majority of the students may not be primarily interested in mathematics, particularly, algebra which interlaces the branches; number theory, geometry, real analysis, topology, and whose beautiful notions can be very useful to provide solutions to practical problems in real life situation, as the tremendous and stunning applications of algebra are not widely known. For employment, the knowledge of mathematical logic, reasoning, mathematical modelling and applicability of algebraic structures are imperative. One of the most important and oldest applications of modern algebra is the use of Boolean algebras in modelling and simplifying switching or relay circuits. The notions of linear algebra are applied to find an effective way of finding the closest code word to any received word in coding theory and cryptology.

In this paper, we provide answer to the question: How the notion of mathematical induction is applied to solve problems given in section 4, in real life situation? For, two necessary theorems in $\mathbb{Z}$–module are stated and proved. In section 2 we give some definitions, examples, properties which are useful for our purpose. In section 3 we state and prove the theorems in $\mathbb{Z}$–module. In section 4 we give the practical problems which can be solved using the theorems proved in the preceding section. For basic notions and definitions of algebra and number theory please refer to [2], [4], and [7]. In this paper, $\mathbb{Z}$ denotes the ring of integers.

2. Preliminaries

We begin with the following:

Definition 2.1 [2]: Let R be any ring; a nonempty set M is said to be a **left R–module** (or, module over R) if M is an abelian group under an operation + such that for every $r \in R$ and $m \in M$ there exists an element r m in M subject to

2.1.1 $r(a + b) = ra + rb$

2.1.2 $r(sa) = (rs)a$

2.1.3 $(r + s)a = ra + sa$ for all $a, b \in M$ and $r, s \in R$.

Remark 2.2 [2]: We make no left right distinction, it being understood that by the term R–module we mean a left R–module we mean a left R–module.

Example 2.3 [2]: Every abelian group G is a module over the ring of integers.

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Example 2.4 [2]: Take M = R. Then any ring R is an R–module over itself. It is obvious that the ring $\mathbb{Z}$ of integers is a $\mathbb{Z}$–module over itself.

Definition 2.5: Let M be an R–module and $m \in M$. If there exist elements $a_1, \ldots, a_n \in M$ such that $m = r_1a_1 + \ldots + r_na_n$, then the element m is said to be a linear combination of the elements $a_1, \ldots, a_n$.

The Principle of Mathematical Induction 2.6[4]: Let p(n) be a statement which, for each integer n, may be either true or false. Then p(n) is true for all integers $n \geq 1$, if

2.6.1 p(1) is true

2.6.2 For all $k \geq 1$, p(k) is true implies p(k + 1) is true.

3. Theorems

In this section we state and prove two theorems in $\mathbb{Z}$–module.

Theorem 3.1: Every integer n>1 in the $\mathbb{Z}$–module can be expressed as

$$n = p \times 2 + q \times 3 \text{ where } p \geq 0, q \geq 0.$$

Proof: We prove the theorem using (2.6).

Let p(n): Every integer n > 1 in the $\mathbb{Z}$–module can be expressed as $n = p \times 2 + q \times 3$ where $p \geq 0, q \geq 0$ It is obvious that p(2) is true as $2 = 1 \times 2 + 0 \times 3$.

Now assume that p(n) is true for $n \geq 2$ in $\mathbb{Z}$–module.

Then there exist two nonnegative integers p and q in $\mathbb{Z}$ such that

$$n = p \times 2 + q \times 3$$

To establish (2.6.2), there are two possibilities:

3.1.1 q = 0;

3.1.2 $q \neq 0$;

First we consider the possibility (3.1.1).

Then $n = p \times 2$.

Therefore $p \geq 1$ since $n \geq 2$. ie. $p - 1 \geq 0$.

Then
$$\begin{aligned} n + 1 &= p \times 2 + 1 \\ &= p \times 2 + (-1) \times 2 + 1 \times 3 \\ &= (p - 1) \times 2 + (q + 1) \times 3 \text{ since } q = 0 \text{ in view of (3.1.1).} \end{aligned}$$

Next, consider, the second possibility (3.1.2).

Here, $q \geq 1$. Ie. $q - 1 \geq 0$.

Then
$$\begin{aligned} n + 1 &= p \times 2 + q \times 3 + 1 \\ &= p \times 2 + q \times 3 + 2 \times 2 + (-1) \times 3 \\ &= (p + 2) \times 2 + (q - 1) \times 3. \end{aligned}$$

Thus in any case p(n + 1) is true.

In view of (2.6), p(n) is true for all integers n > 1.

This completes the proof.

Theorem 3.2: Every integer $n \geq 32$ in the $\mathbb{Z}$–module can be expressed as $n = p \times 5 + q \times 9$ where $p \geq 0, q \geq 0$

Proof: We prove the theorem using (2.6).

Let p(n): Every integer $n \geq 32$ in the $\mathbb{Z}$–module can be expressed as $n = p \times 5 + q \times 9$ where $p \geq 0, q \geq 0$.

It is obvious that p(32) is true as $32 = 1 \times 5 + 3 \times 9$.
Now assume that p(n) is true for n ≥ 32 in $\mathbb{Z}$–module.
Then there exist two nonnegative integers p and q in $\mathbb{Z}$ such that
$n = p \times 5 + q \times 9$
To establish (2.6.2), there are two possibilities:
3.2.1 q = 0;
3.2.2 q ≠ 0;
First we consider the possibility (3.2.1). Then n = p × 5.
Therefore p ≥ 7 since n ≥ 32. ie. p – 7 ≥ 0.
Then

$$\begin{aligned} n + 1 &= p \times 5 + 1 \\ &= p \times 5 + (-7) \times 5 + 4 \times 9 \\ &= (p-7) \times 5 + (q+4) \times 9 \text{ since } q = 0 \text{ in view of (3.2.1).} \end{aligned}$$

Next, consider, the second possibility (3.2.2).
Here, q ≥ 1. Ie. q – 1 ≥ 0.
Then

$$\begin{aligned} n + 1 &= p \times 5 + q \times 9 + 1 \\ &= p \times 5 + q \times 9 + 2 \times 5 + (-1) \times 9 \\ &= (p+2) \times 5 + (q-1) \times 9 \text{ and } p + 2 \geq 0,\ q - 1 \geq 0 \end{aligned}$$

Thus in any case p(n + 1) is true.
In view of (2.6), p(n) is true for all integers n ≥ 32.
This completes the proof.

4. Problems

In this section we present two problems in real life situation and their solutions.

Problem 4.1: A casino has various machines to play different games, the minimum fare of any game being $ 2. A person who wants to play the game has to deposit in the machine sufficient number of coins, purchased from the Casino, for the fare of the game displayed on that machine. Find the two denominations of coins that the casino maintains which can be used to make up any number of coins to deposit.

Solution: In view of the theorem (3.1), the solution to this problem is obvious.

Problem 4.2: A transport department of a state decides to print tickets with minimum fare Rs 32 in two denominations that can be used to make up any amount of fare. Find them.

Solution: In view of the theorem (3.2), the solution to this problem is obvious.

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MULTIVARIATE STATISTICAL ANALYSIS FOR SOURCE ALLOCATION OF ENVIRONMENTAL DATA

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ABSTRACT

Application of different multivariate statistical approaches for the interpretation of heavy metal and total petroleum hydrocarbon (TPH) data obtained during comparative study of soil from automobile workshops and agricultural areas in and around Pondicherry town (India) is presented in this study. Multivariate techniques like principal component analysis (PCA), cluster analysis (CA) and discriminant analysis (DA) were used as classification techniques. Analysis showed correlations between heavy metals (Lead, Zinc and Copper) and total petroleum hydrocarbons. On the basis of factor loadings and principal component scores, different sites were grouped based on their similarity or closeness. PCA revealed the existence of three components (rotated), which explained 100% of the variance for all sites: 1. PC1 with large loadings of 43.7% originates from roads side adjacent to automobile workshops purposes. PC2 with 37.32 % loadings clearly demonstrates proximity to agricultural fields while PC3 includes control and S3. Furthermore, multivariate statistical analysis results revealed that the heavy metals and TPH in automobile workshops and agricultural soil are derived from two different anthropogenic sources, one representing the vehicular emissions, road transport and traffic emissions and the other representing agricultural activities.

Keywords: automobile workshops, cluster analysis, discriminant analysis, emissions, principal component analysis.

Introduction

Soil, an interface for rock, biology and air and water, is a critical environment (Yang *et al.,* 2014). Urban soils that are sealed, compacted and influenced by the human activity could be considered as the source and sink of various pollutants (Cai *et al.,* 2013). The road transportation affects roadside soil environment adversely in the form of heavy metal enrichment (Zhang *et al.,* 2015). Heavy metals are the most persistent soil contaminants and show long-termtoxicity (Massas *et al.,* 2013). Disproportionate influx of heavy metals into urban soil can inflict a persistent load on the biogeochemical cycle in the urban ecosystem by deteriorating soil function, altering soil properties and other environmental concerns (Chen *et al.,* 2012; Papa *et al.,* 2010; Lin *et al.,* 2011). Soil contamination assessment and location of sources of pollution represent the crux of the soil remediation (Sollitto *et al.,* 2010). Appropriate action to protect soil quality can be planned on the basis of sources of heavy metal contamination (Lu *et al.,* 2012). Discriminating between natural and anthropogenic sources of pollution is a matter of scientific interest (Zhang *et al.,* 2012).

The concentration of heavy metals in environmental samples can be evaluated by a wide range of analytical techniques including atomic absorption spectrometry (AAS), graphite furnace atomic absorption spectrometry (GFAAS), x-ray fluorescence (XRF), inductively coupled plasma-atomic emission spectrometry (ICP-AES) and neutron activation analysis (NAA) (Reza and Singh, 2010; Zamani *et al.,* 2013; Chandrasekaran *et al.,* 2015).

In order to provide a more accurate appraisal of the heavy metal pollution, a complementary approach that integrates criteria of sediment quality, chemometric analysis and geochemical approach becomes essential (Chabukdhara and Nema, 2012). Recently, several multivariate analysis methods such as correlation, principle component analysis (PCA) and cluster analysis are useful techniques to identify the sources of heavy metal contamination in soil (Lu *et al.,* 2010; Jan *et al.,* 2010; Malik *et al.,* 2010; Karim *et al.,* 2014;Lv *et al.,* 2015), water (Li et al., 2010; Varol, 2011; Viera *et al.,* 2012) and sediments (Salati and Moore, 2010). Cluster analysis and principal component analysis have been widely used to detect the source of heavy metals detected in the road dust (Carrero *et al.,* 2013; Han *et al.,* 2014). Combustion of fossil fuel, traffic emissions, wear of brake lining materials and associated industrial processes are the chief sources of thus generated metals (Al-Khashman, 2013;Nwachukwu *et al.,* 2013; Rao *et al.,* 2015). In this paper aims to determine the extent of the pollution from different sources by means of multivariate statistical tools such as correlation, principal component analysis, cluster analysis and discriminant analysis.

Materials and Methods

Sampling and analytical methods

To determine the total heavy metal content in soils, air dried 0.5 g of dry soil was sieved through 250 microns nylon sieve and digested in aqua regia using microwave-assisted digestor (USEPA 2000). Metal concentrations in the final solution were determined using Atomic Absorption Spectrometer (Khan and Kathi, 2014). Total petroleum hydrocarbon (TPH) was measured with an InfraCal Total organic Grease (TOG)/TPH Analyzer according to the USEPA Methods 418.1 (ASTM 2004).

Data Calculation and Analysis

All multivariate statistical analyses, including principal component analysis (PCA), correlation and linear discriminant analysis (DA), were conducted using SPSS 23.0 for windows (SPSS, Germany). Principal component analysis (PCA) is employed to reduce the dimensionality of a data set while attempting to preserve the relationships present in the original data (Bai *et al.,* 2011). Obtained heavy metals and TPH were tested for correlations between them using Pearson's correlation coefficient. Kaiser-Meyer-Olkin valueapplied in this study is an indicator of the goodness of fit of the principal component analysis (Tabachnick and Fidell, 2007). Discriminant Analysis is performed for statistical grouping of most significant variables (Mustapha and Aris, 2012).

CA is a method that assembles objects into aggregations based on their interdependent variables or characteristics.The resulting clusters of should exhibit within-cluster homogeneity and heterogeneity between clusters (Wang *et al.,* 2013a).The method of agglomerative hierarchical clustering was used for grouping the observed locations. In this study, Euclidean distance is used as a similarity measure and Ward's incremental sum-of squares method were used for clustering of variables (Dobsa *et al.,* 2014).

Results and Discussion

The concentration of contaminants in roadside soil adjacent to automobile workshops indicated

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polluted status but the agricultural soils studied can be considered to be unpolluted (Fig 1). Multivariate statistical analyses on 50 street dust samples from four major streets in eastern and southern Tehran reveals that petrogenic and pyrogenic sources, traffic and related activities, might be the main anthropogenic sources of heavy metals and PAHs in Tehran dust (Saeedi *et al.,* 2012).Lead was the most commonly detected heavy metal in automobile workshop sediments although the use of leaded gasoline was discouraged (Gunawardene *et al.,* 2012).

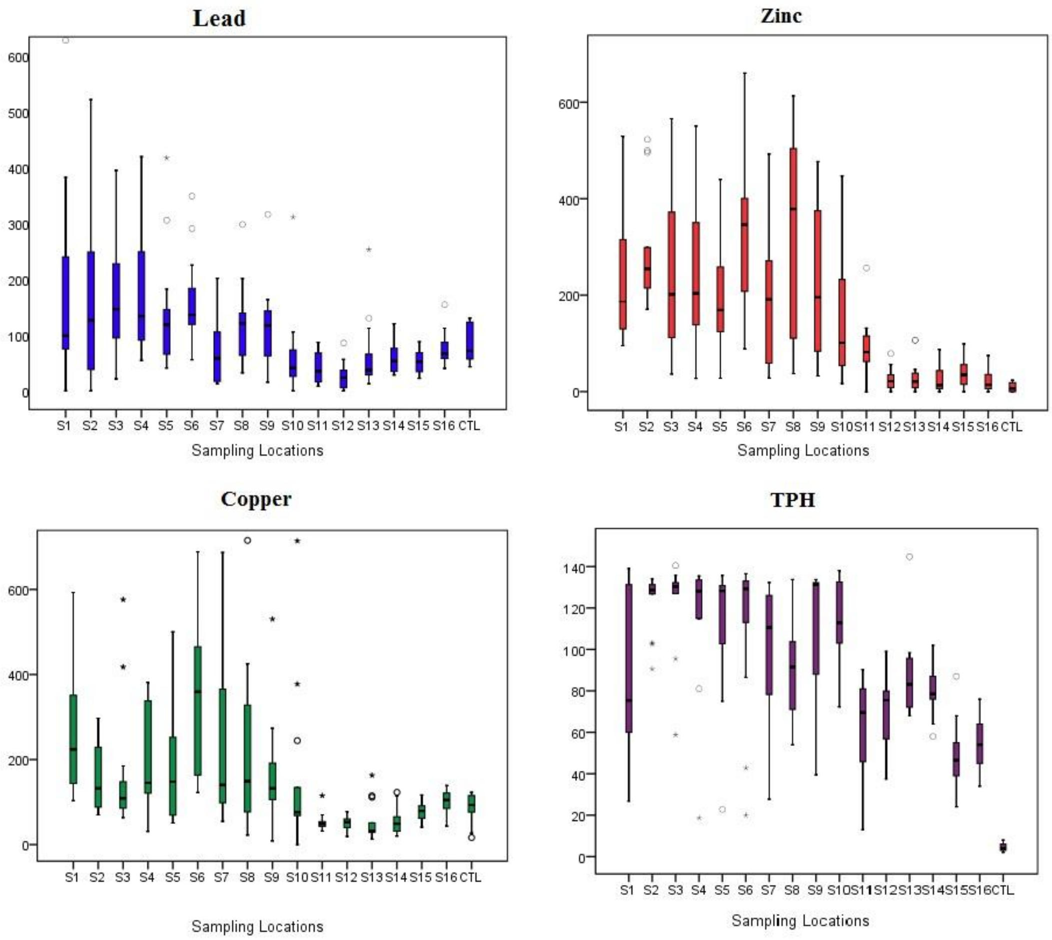

Figure 1 Concentration of heavy metals and petroleum hydrocarbons in the surface soil represented by boxplots

Multivariate statistical Analysis

The correlation study showed significant positive correlations among various metals and TPH in soil samples (Table 1). According to Pearson's coefficient, Zn, Pb and Cu are significantly correlated with TPH. Additionally, Zn showed significant strong positive correlation with the Cu (r=0.924). Similarly positive correlations are observed among all the other metals, where, Pb to Zn (r=0.797), Cu to Pb (r=0.742), TPH to Zn (r=0.762), TPH to Pb (r=0.751), TPH to Cu (r=0.684).

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Table 1: Pearson correlation among selected heavy metals and total petroleum hydrocarbons

		Zn	Pb	Cu	TPH
Zn	Pearson Correlation	1	.797**	.924**	.762**
	Sig. (2-tailed)		.000	.000	.001
	N	16	16	16	16
Pb	Pearson Correlation	.797**	1	.742**	.751**
	Sig. (2-tailed)	.000		.001	.001
	N	16	16	16	16
Cu	Pearson Correlation	.924**	.742**	1	.684**
	Sig. (2-tailed)	.000	.001		.004
	N	16	16	16	16
TPH	Pearson Correlation	.762**	.751**	.684**	1
	Sig. (2-tailed)	.001	.001	.004	
	N	16	16	16	16

****. Correlation is significant at the 0.01 level (2-tailed).**

Pearson's correlation coefficients of contaminants in soils are summarised in Table 1. Principal component analysis (PCA) was applied to simplify, assure comprehensive information and explain the actual meaning which renders the results more scientific and unbiased (Wang *et al.,* 2013c). PCA coupled with correlation between heavy metals revealed that heavy metal contamination might originate from traffic and industrial activities. To quantitatively evaluate the clustering pattern, PCA with varimax normalization is applied, and the results are given in Table 2, also illustrated in Figure 2. Three principal components were extracted for the 16 sampling locations. Principal component (PC1), which contains samples S1-S10 explains 40.45 % of the total variance. High coefficients of variation of these sampling sites indicates that the contaminants possibly originate from anthropogenic sources (Ma *et al.,* 2015). PC2 contains sampling sites between S11-S16 and explains 37% of the total Variance. The heavy metals and TPH in these sampling sites might have originated from sources other than the petrogenic vehicular emissions. PC3 contains the control taken from Pondicherry University garden. Principal components are rotated by Varimax rotation to achieve more revelatory components where loadings of variables will be high on single factor and moderate on the remaining factor (Dobsa *et al.,* 2014). The PCA results indicated that heavy metals (Pb, Zn and Cu) and TPH were affected by anthropogenic activities such as vehicular emission (Singh *et al.,* 2013).

Table 2: Rotated component matrix of sampling locations

	Component		
	1	2	3
S1	.641	-.759	.113
S2	.448	-.254	.857
S3	.251	-.486	.837
S4	.229	-.811	.538
S5	.812	-.331	.481
S6	.890	-.407	.206
S7	.970	-.125	.209

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S8	.772	-.238	.589
S9	.936	-.230	.267
S10	.904	.209	.374
S11	.345	.317	.883
S12	-.407	.899	-.160
S13	-.491	.849	-.197
S14	-.099	.991	-.087
S15	.578	.816	.019
S16	-.084	.996	.012
Control	.770	-.309	.558
Extraction Method: Principal Component Analysis. Rotation Method: Varimax with Kaiser Normalization.[a]			
a. Rotation converged in 4 iterations.			

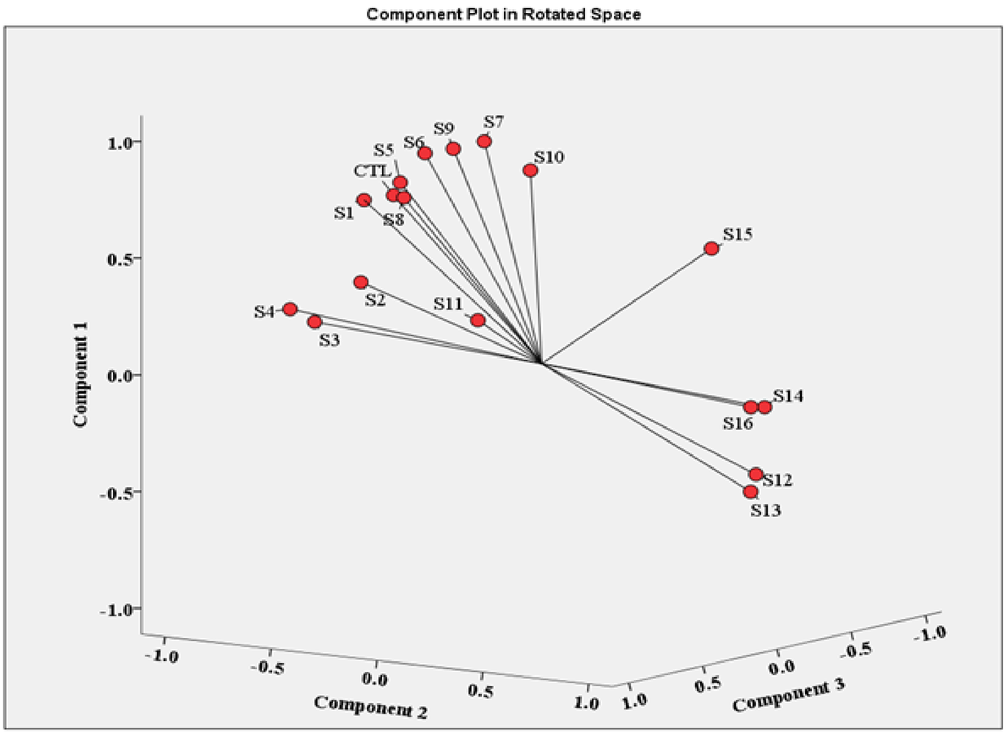

Figure 2: PCA loadings for the three principal components from the PCA for the sampling sites.

The aim of clustering of sampling locations in this study is to identify sampling sites exposed to similar levels of contamination due to heavy metals and TPH. Figure 3 shows a dendrogram obtained as result of hierarchical clustering of the observed variables. Clustering was performed by using mean values of heavy metals (Zn, Pb and Cu) and TPH as cases and 16 sampling sites as variables. The dendrogram consists of two major clusters. The first cluster contains seven variables clustered in two smaller sub-clusters. The variables contained in the first subcluster are S12, S14, S13, S15, S16 while those in the second subcluster are S11 and control. The second cluster comprises of S7, S9, S5 and S10 in one subcluster whereas S3, S4, S2, S1, S8 and S6 grouped together as second subcluster.

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From our study first cluster can be assigned to the agricultural locations and second cluster clearly represents the highly contaminated soil from automobile workshops. The distance cluster represents the degree of association between sampling locations. The lower the value on the distance cluster, the more similar is the association (Wang *et al.,* 2013b). Clustering the contaminants resulted in two different clusters. The first cluster related lead with TPH while second cluster grouped Zn with Cu (Fig. 4). Traffic appears to be the possible source of high concentration of metals such as Zn and Cu detected in the road dust of Delhi city (Rajaram *et al.,* 2014).

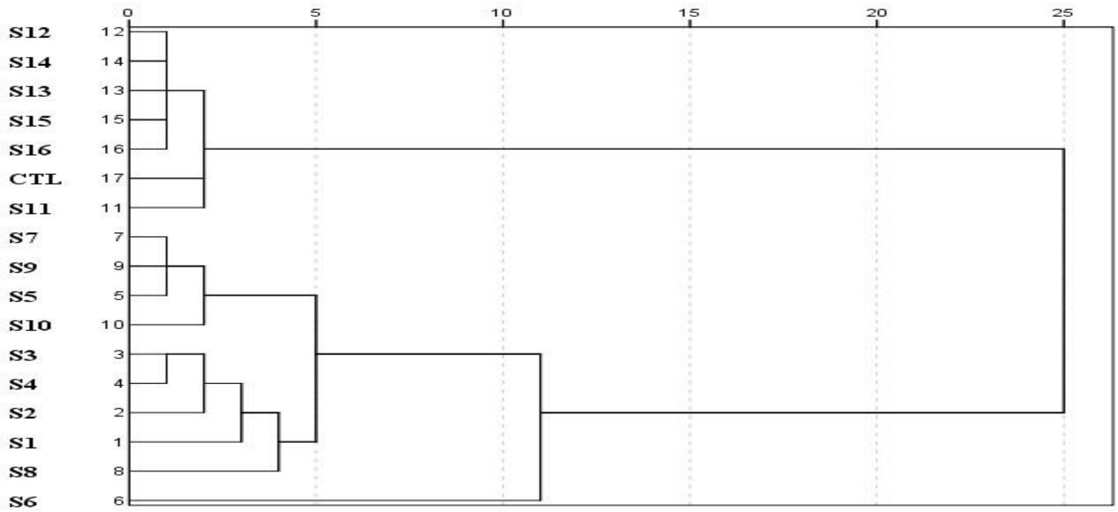

Figure 3: Dendrogram of the sampling sites obtained by hierarchical clustering

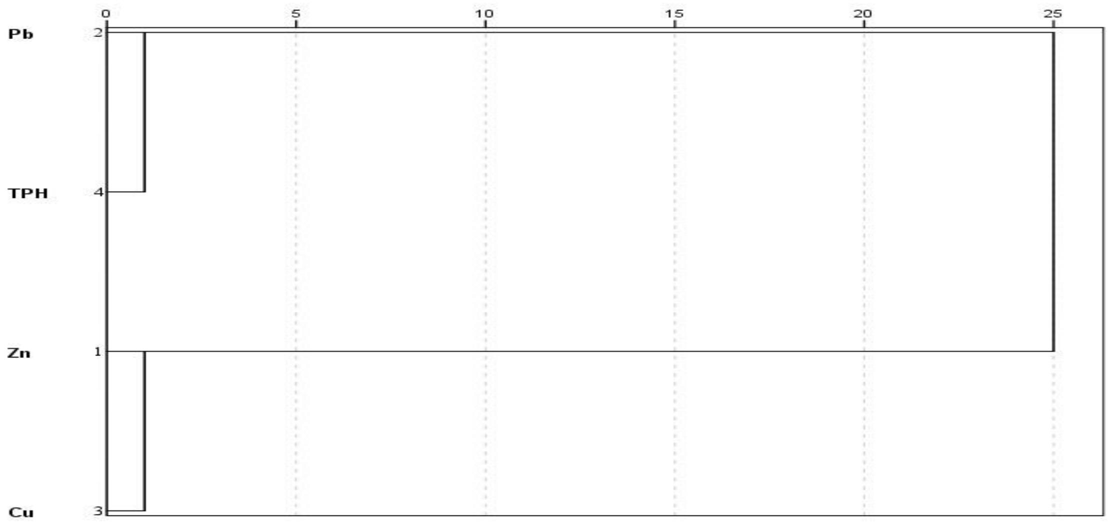

Figure 4: Dendrogram of the contaminants obtained by hierarchical clustering

Discriminant Analysis

By examining **Table 3**, it can be noted that the canonical correlation coefficients is 0.883. The lineal discriminant analysis, made by considering the sampling site as variables, demonstrated that 100 % of the samples were classified. In the present study, we could find that the value of Wilks' lambda, an indicator to measure discriminant ability is 0.22. The result of discriminant analysis shows that the discriminant function was significant at the level of $p<0.05$.Wilks' lambda is used in DA to describe the difference between groups (Wang *et al.,* 2013c). Relatively lower values of Wilks implies good discriminating power of the models (Thomas *et al.,* 2015). Similarly, canonical discriminant analysis was used to explore the effects of urbanization based on Pollution index of heavy metal contaminants in urban soils of Wien (Simon *et al.,* 2013).

Table 3: Discriminant analyses results for the two investigated sites

Variable	Canonical Correlation	Wilks Lambda statistic	Chi-squares	D.F	Significance	% of grouped cases correctly classified
Sampling Sites	0.883	0.220	1.516	2	0.469	100

Conclusions

The data obtained in the present study demonstrate that heavy metals and petroleum hydrocarbons have accumulated in roadside soil due to the influence of increasing vehicular exhaust. The results of multivariate statistic divided the sampling locations into two groups namely those contaminated by automobile workshops and the other by agricultural activities. The result of cluster analysis clearly supported the existence two different clusters of sampling sites and contaminants. The two major components from PCA suggested that PC1 was mainly affected by the use of vehicular emissions while PC2 was affected by agrochemicals. This study provides a reduction in dimensionality of large data sets and usefulness of multivariate statistical applications in deciphering sources of contamination.

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Highlights

Heavy metal sources were identified using multiple statistical analysis.

PCA and CA reveals that the sampling locations derived the contaminants from two anthropogenic sources.

Discriminating sources of contamination is crucial for planning mitigation measures.

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BALANCED FACTOR CONGRUENCES ON PRE A*- ALGEBRAS

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ABSTRACT

In this paper we prove for any element a of a Pre A*-algebra A, the set A_a is a Pre A*-algebra under the operations $\wedge$ and $\vee$ and the complementation ^{a}. We also prove that if θ is a factor congruence on A, then $\theta \cap (A_a \times A_a)$ is a balanced factor congruence of A_a for each a $\in$ A and hence there exists unique $s_a \in$ B(A_a) such that $\theta \cap (A_a \times A_a) = \theta_{s_a}$ = {(x, y) $\in A_a \times A_a$ / $s_a \wedge$ x = $s_a \wedge$ y}.

Keywords: Boolean algebra, Pre A*-algebra, congruence, centre of Pre A*-algebra, factor congruence, balanced factor congruence.
AMS subject classification (2000): 06E05, 06E25 ,06E99,06B10

INTRODUCTION

In 1948 the study of lattice theory had been made by Birkhoff [7]. In a draft paper [3], The Equational Theory of Disjoint Alternatives around 1989, E.G.Maines introduced the concept of Ada(Algebra of disjoint alternatives) $(A, \wedge, \vee, (-)', (-)_\pi, 0, 1, 2)$ which is however differ from the definition of the Ada of his later paper [4] Adas and the equational theory of if-then-else in 1993. While the Ada of the earlier draft seems to be based on extending the If-Then –Else concept more on the basis of Boolean algebra and the later concept is based on C-algebra $(A, \wedge, \vee, ')$ introduced by Fernando Guzman and Craig.C. Squir[1].

In 1994, P. Koteswara Rao[2] first introduced the concept A*-Algebra $(A, \wedge, \vee, *, (\tilde{-}), (-)_\pi, 0, 1, 2)$ not only studied the equivalence with Ada, C-algebra, Ada's connection with **3-** Ring, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra. . In 2000, J. Venkateswara Rao [5] introduced the concept of Pre A*-algebra $(A, \wedge, \vee, (\tilde{-}))$ as the variety generated by the 3-element algebra A = {0,1,2} which is an algebraic form of three valued conditional logic.In [8] A.Satyanarayana et al. generated Semilattice structure on Pre A*-Algebras .In [10], A.Satyanarayana defined a partial ordering on a Pre A*-algebra A and the properties of A as a poset are studied. In [9] A.Satyanarayana. et.all derive necessary and sufficient conditions for pre A*-algebra A to become a Boolean algebra in terms of the partial ordering**.**

In this paper we prove for any element a of a Pre A*-algebra A, the set A_a is a Pre A*-algebra under the operations $\wedge$ and $\vee$ induced by those of A and the complementation ^{a}. We also prove that if θ is a factor congruence on A ,then $\theta \cap (A_a \times A_a)$ is a balanced factor congruence of A_a for each a $\in$ A and hence there exists unique $s_a \in$ B(A_a) such that $\theta \cap (A_a \times A_a) = \theta_{s_a}$.

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1. PRELIMINARIES:

1.1. Definition: Boolean algebra is an algebra $(B, \vee, \wedge, (-)', 0, 1)$ with two binary operations, one unary operation (called complementation), and two nullary operations which satisfies:

(i) $(B, \vee, \wedge)$ is a distributive lattice

(ii) $x \wedge 0 = 0,\ x \vee 1 = 1$

(iii) $x \wedge x' = 0,\ x \vee x' = 1$

We can prove that $x'' = x,\ (x \vee y)' = x' \wedge y'$, $(x \wedge y)' = x' \vee y'$ for all $x, y \in B$.

In this section we concentrate on the algebraic structure of Pre A*-algebra and state some results which will be used in the later text.

1.2. Definition: An algebra $(A, \wedge, \vee, (-)^{\sim})$ where A is non-empty set with 1, $\wedge$, $\vee$ are binary operations and $(-)^{\sim}$ is a unary operation satisfying

(a) $x^{\sim\sim} = x \qquad \forall x \in A$

(b) $x \wedge x = x, \qquad \forall x \in A$

(c) $x \wedge y = y \wedge x, \qquad \forall x, y \in A$

(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim} \qquad \forall x, y \in A$

(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \qquad \forall x, y, z \in A$

(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \qquad \forall x, y, z \in A$

(g) $x \wedge y = x \wedge (x^{\sim} \vee y), \qquad \forall x, y \in A$ is called a Pre A*-algebra

1.3. Example [6]: 3 = {0, 1, 2} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

1.4. Note: The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^{\sim} = 2$

(b) $1 \wedge x = x$ for all $x \in$ **3**

(c) $0 \vee x = x$ for all $x \in$ **3**

(d) $2 \wedge x = 2 \vee x = 2$ for all $x \in$ **3**.

1.5. Example: 2 = {0, 1} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

x	$x^{\sim}$
0	1
1	0

1.6. Note:

(i) $\left(2, \vee, \wedge, (-)^{\sim}\right)$ is a Boolean algebra. So every Boolean algebra is a Pre A* algebra.

(ii) The Boolean algebra **2** = {0, 1} is an underlying set in the Pre A*-algebra **3 =** {0, 1, 2}. Further the readers can note that the Pre A*-algebra **3 =** {0, 1, 2} is not a Boolean algebra as there is a critical element 2 which is not in the Boolean algebra **2** = {0, 1}.

(iii) The identities 1.2(a) and 1.2(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.2(b) to 1.2(g).

1.7. Definition: Let A be a Pre A*-algebra. An element x $\in$A is called central element of A if $x \vee x^{\sim}=1$ and the set {x $\in$ A/ $x \vee x^{\sim}=1$} of all central elements of A is called the centre of A and it is denoted by B (A).

1.8. Theorem:[6] Let A be a Pre A*-algebra with 1, then B (A) is a Boolean algebra with the induced operations $\wedge, \vee, (-)^{\sim}$

1.9. Lemma: [10] Let $(A, \wedge, \vee, (-)^{\sim})$ be a Pre A*-algebra and let $a \in A$.Then the relation $\theta_a = \{(x, y) \in A \times A /\ a \wedge x = a \wedge y\}$ is

(a) a congruence relation as in Boolean algebra.

(b) $\theta_a \cap \theta_{a^{\sim}} = \theta_{a \vee a^{\sim}}$

(c) $\theta_a \cap \theta_b \subseteq \theta_{a \vee b}$

(d) $\theta_a \cap \theta_{a^{\sim}} \subseteq \theta_{a \wedge a^{\sim}}$

we will write $x\ \theta_a y$ to indicate $(x, y) \in \theta_a$.

1.10. Lemma: [10] Let A be a Pre A*-algebra and a, b $\in$ $B(A)$(Boolean algebra with the induced operations $\wedge, \vee, (-)^{\sim}$), then $\theta_a \cap \theta_b = \theta_{a \vee b}$

1.11. Definition: Let A be a Pre A*-algebra and α $\in$Con(A).Then α is called factor congruence if there exist β $\in$Con(A) such that $\alpha \cap \beta$ = Δ_A and $\alpha\ o\ \beta$ = A$\times$A. In this case β is called direct complement of α.

1.12. Definition: A congruence β on Pre A*-algebra A is called balanced if ($\beta \vee \theta$)$\cap$($\beta \vee \theta^{\sim}$) = β for any direct factor congrucnces θ and any of its direct comoplement $\theta^{\sim}$ on A.

2.Balanced Factor Congruences on Pre A*- algebras:

2.1. Theorem: Let A be a Pre A*-algebra and a $\in$A. Let A_a = {x$\in$A / a$\wedge$x = x } then A_a is closed under the operations $\wedge$ and $\vee$. Also for any x$\in$ A_a define, x^a =a$\wedge$ x$^{\sim}$ then (A_a, $\wedge$,$\vee$,^{a}) is a Pre A*-algebra with 1(here a is itself is the identity for $\wedge$in A_a ;that is 1 in A_a).

Proof: Let x, y $\in$ A_a .Then a$\wedge$x = x and a$\wedge$y = y.

Now a$\wedge$(x$\wedge$ y) = (a$\wedge$x)$\wedge$ y = x$\wedge$ y $\Rightarrow$ x$\wedge$ y$\in$ A_a

Also a$\wedge$(x$\vee$ y) = (a$\wedge$x) $\vee$(a$\wedge$y) = x$\vee$ y$\Rightarrow$ x$\vee$y$\in$ A_a

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Therefore A_a is closed under the operation $\wedge$ and $\vee$.

a $\wedge$ x^a = a $\wedge$ (a $\wedge$ $x^{\sim}$) = a $\wedge$ $x^{\sim}$ = $x^a \Rightarrow x^a \in A_a$

Thus A_a is closed under ^{a}.

Now for any x, y, z $\in A_a$

(1) $x^{aa} = (a \wedge x^{\sim})^a = a \wedge (a \wedge x^{\sim})^{\sim} = a \wedge (a^{\sim} \vee x) = a \wedge x = x$

(2) $x \wedge x = (a \wedge x) \wedge (a \wedge x) = a \wedge x = x$

(3) $x \wedge y = (a \wedge x) \wedge (a \wedge y) = (a \wedge y) \wedge (a \wedge x) = y \wedge x$

(4) $(x \wedge y)^a = a \wedge (x \wedge y)^{\sim} = a \wedge (x^{\sim} \vee y^{\sim})$

$= (a \wedge x^{\sim}) \vee (a \wedge y^{\sim})$

$= x^a \vee y^b$

(5) $x \wedge (y \wedge z) = (a \wedge x) \wedge \{(a \wedge y) \wedge (a \wedge z)\}$

$= a \wedge \{x \wedge (y \wedge z)\}$

$= a \wedge \{(x \wedge y) \wedge z\}$ (since x, y, z $\in$ A)

$= (x \wedge y) \wedge z$

(6) $x \wedge (y \vee z) = (a \wedge x) \wedge \{(a \wedge y) \vee (a \wedge z)\}$

$= \{(a \wedge x) \wedge (a \wedge y)\} \vee \{(a \wedge x) \wedge (a \wedge z)\}$

$= \{a \wedge (x \wedge y)\} \vee \{(a \wedge (x \wedge z)\}$

$= (x \wedge y) \vee (x \wedge z)$

(7) $x \wedge (x^a \vee y) = x \wedge \{(a \wedge x^{\sim}) \vee y\}$

$= \{x \wedge (a \wedge x^{\sim})\} \vee (x \wedge y)$

$= (x \wedge x^{\sim}) \vee (x \wedge y)$ (since $a \wedge x = x$)

$= x \wedge (x^{\sim} \vee y)$

$= x \wedge y$

Finally x $\in A_a$ implies that $a \wedge x = x = x \wedge a$. Thus (A_a, $\wedge, \vee, ^a$) is a Pre A*-algebra with a as identity for $\wedge$.

2.2. Theorem: Let θ be a congruence on A. Then $\theta \cap (A_a \times A_a)$ is a congruence on A_a, for each a $\in$ A.

Proof: Suppose θ be a congruence on A, and a $\in$ A.

Let x $\in A_a$ we have that (x, x) $\in A_a \times A_a$

Since θ be congruence we have (x, x) $\in \theta \cap (A_a \times A_a)$

Therefore the result is reflexive.

Let (x, y) $\in \theta \cap (A_a \times A_a)$

Then (x, y) $\in \theta$ and (x, y) $\in A_a \times A_a$

$\Rightarrow$ (y, x) $\in \theta$ and (y, x) $\in A_a \times A_a$

$\Rightarrow$ (y, x) $\in \theta \cap (A_a \times A_a)$

The result is Symmetric.

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Let (x, y) , (y, z) $\in \theta \cap (A_a \times A_a)$

$\Rightarrow$ (x, y) , (y, z) $\in \theta$ and (x, y) , (y, z) $\in A_a \times A_a$

$\Rightarrow$ (x, z) $\in \theta$ and x, y, z $\in A_a$ hence (x, z) $\in A_a \times A_a$

$\Rightarrow$ (x, z) $\in \theta \cap (A_a \times A_a)$

The result is Transitive.

Hence the relation is equivalence relation.

Let (x, y), (z, t) $\in \theta \cap (A_a \times A_a)$

Since x, y, z, t $\in A_a$, we have

$a \wedge x \wedge z = x \wedge z \Rightarrow x \wedge z \in A_a$

$a \wedge y \wedge t = y \wedge t \Rightarrow y \wedge t \in A_a$

$\Rightarrow (x \wedge z, y \wedge t) \in A_a \times A_a$

Since θ be congruence we have $(x \wedge z, y \wedge t) \in \theta \cap (A_a \times A_a)$

Now (x, y) $\in \theta \Rightarrow (x^{\sim}, y^{\sim}) \in \theta$

$\Rightarrow (a \wedge x^{\sim}, a \wedge y^{\sim}) \in \theta$ and $(a \wedge x^{\sim}, a \wedge y^{\sim}) \in A_a \times A_a$

$\Rightarrow (a \wedge x^{\sim}, a \wedge y^{\sim}) \in \theta \cap (A_a \times A_a)$

$\Rightarrow (x^{a}, y^{a}) \in \theta \cap (A_a \times A_a)$

Therefore $\theta \cap (A_a \times A_a)$ is compatible (closed) with the binary operation $\wedge$ and unary operation ^{a} on A_a.

Let (x, y), (z, t) $\in \theta \cap (A_a \times A_a)$

Since x, y, z, t $\in A_a$, we have

$a \wedge (x \vee z) = (a \wedge x) \vee (a \wedge y) = x \vee z \Rightarrow x \vee z \in A_a$

$a \wedge (y \vee t) = (a \wedge y) \vee (a \wedge t) = y \vee t \Rightarrow y \vee t \in A_a$

$\Rightarrow (x \vee z, y \vee t) \in A_a \times A_a$

Therefore $\theta \cap (A_a \times A_a)$ is compatible with $\vee$ also

Thus $\theta \cap (A_a \times A_a)$ is a congruence relation on A_a.

2.3. Theorem: Let θ be a factor congruence on a Pre A*-algebra A. Then $\theta \cap (A_a \times A_a)$ is a factor congruence on A_a.

Proof: Since θ be a factor congruence on A there is a congruence $\theta^{\sim}$ on A such that $\theta \cap \theta^{\sim} = \Delta_A$ and $\theta \circ \theta^{\sim} = A \times A$.

Consider $[\theta \cap (A_a \times A_a)] \cap [\theta^{\sim} \cap (A_a \times A_a)] = (\theta \cap \theta^{\sim}) \cap (A_a \times A_a)$

$= \Delta_A \cap (A_a \times A_a)$

$= \Delta_{A_a}$, the diagonal on A_a

Observe that every element in A_a is the form a $\wedge$ x for some x $\in$ A.

Now, let (a $\wedge$ x ,a $\wedge$ y) $\in A_a \times A_a$.Then (a $\wedge$ x, a $\wedge$ y) $\in$ A $\times$ A = $\theta \; o \; \theta^{\sim}$ which implies that there exist z $\in$ A such that (a $\wedge$ x ,z) $\in \theta$ and (z, a $\wedge$ y) $\in \theta^{\sim}$.

Now (a $\wedge$ x, a $\wedge$ z) $\in \theta$ and (a $\wedge$ z, a $\wedge$ y) $\in \theta^{\sim}$ and a $\wedge$ z $\in A_a$

Therefore (a $\wedge$ x, a $\wedge$ z) $\in \theta \cap (A_a \times A_a)$ and (a $\wedge$ z, a $\wedge$ y) $\in \theta^{\sim} \cap (A_a \times A_a)$ and hence (a $\wedge$ x ,a $\wedge$ y) $\in$ $[\theta^{\sim} \cap (A_a \times A_a)] \, o \, [\theta \cap (A_a \times A_a)]$

Therefore $[\theta \cap (A_a \times A_a)] \, o \, [\theta^{\sim} \cap (A_a \times A_a)] = A_a \times A_a$

Therefore $\theta \cap (A_a \times A_a)$ is a factor congruence on A_a and $\theta^{\sim} \cap (A_a \times A_a)$ is a direct complement of $\theta \cap (A_a \times A_a)$.

2.4. Theorem: Let A be a Pre A*-algebra with 1 induced by Boolean algebra , θ is a factor congruence on A and β a direct complement of θ.Then there exist unique a $\in$ A such that $\theta = \theta_a = \{(x, y) \in A \times A / \; a \wedge x = a \wedge y\}$ and $\beta = \theta_{a^{\sim}} (= \beta_a)$.

Proof: Let $1^{\sim}$ = 0. Then 1 and 0 are identities for operators $\wedge$ and $\vee$ respectively in A.

We have $\theta \cap \beta = \Delta_A$ and $\theta \; o \; \beta$ = A $\times$ A.

Then clearly $\theta \; o \; \beta = \beta \; o \; \theta$ = A $\times$ A.

Since (0, 1) $\in$ A $\times$ A = $\theta \; o \; \beta$, there exist a $\in$ A such that (0, a) $\in \beta$ and (a, 1) $\in \theta$.

First we observe that a is a unique element with the above property. If b $\in$ A also is such that (0, b) $\in \beta$ and (b, 1) $\in \theta$ then by the transitive and symmetry of β and θ we get (a, b) $\in \theta \cap \beta = \Delta_A$, the diagonal of A, and hence a = b

Thus a is unique such that (0, a) $\in \beta$ and (a, 1) $\in \theta$

Now we prove that $\theta = \theta_a$ and $\beta = \theta_{a^{\sim}}$

For any x, y $\in$ A we have

(0, a $\wedge$ x) = (0 $\wedge$ x, a $\wedge$ x) $\in \beta$ (since (0, a) $\in \beta$) and hence (a $\wedge$ x, a $\wedge$ y) $\in \beta$

Now (x, y) $\in \theta \Rightarrow$ (a $\wedge$ x, a $\wedge$ y) $\in \theta \cap \beta = \Delta_A$

$\Rightarrow$ a $\wedge$ x = a $\wedge$ y

$\Rightarrow$ (x, y) $\in \theta_a$

Therefore $\theta \subseteq \theta_a$.

On the other hand for any x $\in$ A, (a $\wedge$ x, x) = (a $\wedge$ x, 1 $\wedge$ x) $\in \theta$ (since (a, 1) $\in \theta$)

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Now (x, y) $\in \theta_a \Rightarrow$ a $\wedge$ x = a $\wedge$ y

We have (a $\wedge$ x, x) $\in \theta$, (a $\wedge$ y, y) $\in \theta$ and a $\wedge$ x = a $\wedge$ y $\Rightarrow$ (x, y) $\in \theta$

Therefore $\theta_a \subseteq \theta$.

Thus $\theta = \theta_a$.

Also from (0, a) $\in \beta$ and (a, 1) $\in \theta$ we have that (0, $a^{\sim}$) $\in \theta$ and ($a^{\sim}$, 1) $\in \beta$ and by interchanging θ and β in the above argument we get that

$$\beta = \theta_{a^{\sim}} = \beta_a = \{(x, y) \in A \times A / \ a \vee x = a \vee y\}$$

We have already proved that a is unique with this property.

2.5. Theorem: Let A be a Pre A*-algebra with 1 induced by Boolean algebra. Then any factor congruence on A is balanced.

Proof: Let β is a factor congrucnces on Pre A*-algebra A and θ another factor congruence on A and $\theta^{\sim}$ a direct complement of θ.

Then by 2.4. Theorem there exist a, b $\in$ A such that $\beta = \beta_a$ and $\theta = \theta_b = \beta_{b^{\sim}}$ and $\theta^{\sim} = \theta_{b^{\sim}}$

Now $(\beta \vee \theta) \cap (\beta \vee \theta^{\sim}) = (\beta_a \vee \beta_{b^{\sim}}) \cap (\beta_a \vee \beta_b)$

$$= \beta_{a \vee b^{\sim}} \cap \beta_{a \vee b}$$

$$= \beta_{(a \vee b^{\sim}) \wedge (a \vee b)}$$

$$= \beta_{a \vee (b^{\sim} \wedge b)}$$

$$= \beta_{a \vee 0} = \beta_a = \beta$$

Thus β is balanced.

2.6. Theorem: If θ is a factor congruence on A then $\theta \cap (A_a \times A_a)$ is a balanced factor congruence on A_a for each a $\in$ A and there exists unique $s_a \in$ B(A_a) such that $\theta \cap (A_a \times A_a) = \theta_{s_a}$ = {(x, y) $\in A_a \times A_a$ / $s_a \wedge$ x = $s_a \wedge$ y}

Proof: It follows from 2.3, 2.4 and 2.5 theorems

Conclusion

This manuscript makes it possible to identify a factor congruence and its is a unique direct complement on a Pre A*-algebra. The notion of balanced congruence was initiated. It is detected that any factor congruence on Pre A*algebra is balanced. For any element a in a Pre A*algebra, it has been derived a typical Pre A*algebra A_a. It is observed that if θ is a congruence on a Pre A* algebra A, then there is $\theta \cap (A_a \times A_a)$ is a congruence on A_a, for each $a \in$ A and if θ be a factor congruence on a Pre A*-algebra A, then $\theta \cap (A_a \times A_a)$ is a balanced factor congruence on A_a for each a $\in$ A and there exists unique $s_a \in$ B(A_a) such that $\theta \cap (A_a \times A_a) = \theta_{s_a}$ = {(x, y) $\in A_a \times A_a$ /

$S_a \wedge x = S_a \wedge y\}$

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IDEALS IN REGULAR RINGS

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ABSTRACT
Continuous geometry is a complete complemented irreducible modular Lattice in which join and meet are continuous. Von Neumann invented regular rings in order to coordinise these lattices. In this paper we present the basic properties of general regular rings, the nature and use of idempotents, the class of all principal right ideals (left ideals) as a complemented modular lattice.
Keywords: Idempotent, regular ring, right ideal, principal right ideal.

1.1 **Definition:** Suppose R is a ring with 1. A right ideal left ideal I of R is a set $I \subseteq R$ such that
(i) $x, y \in I \Rightarrow x + y \in I$
(ii) $x \in i, z \in R \Rightarrow xz \in I(zx \in I)$
1.2 **Note:** Suppose $a \in R$ then $\exists$ minimal right ideal (left ideal) exists containing $'a'$ which is called the principal right (left) ideal denoted by $(a)_r$ $[(a)_l]$ is the set of all $az(za): z \in R$
i.e., $(a)_r = \{az \,/\, z \in R\}$ and
$(a)_l = \{za \,/\, z \in R\}$

1.3 Note:
1. The set of all right ideals form a partially ordered set with respect to set theoretical inclusion $I \subseteq J$. This set has a minimum element $(0) = (0)_r$ and a maximum element $R = (1)_r$
2. For any set of right ideals $I_1, I_2, \ldots\ldots$ $\exists a$ maximal right ideal I such that $I \subset I_1, I_2, \ldots.$ and $I_1 \cap I_2 \cap \ldots\ldots$ is the maximal right ideal contained in every right ideal $I_1, I_2, \ldots\ldots$ and it is denoted by $glb\{I_1, I_2, \ldots\ldots\}$
3. For any set of right ideals $I_1, I_2, \ldots\ldots \exists a$ minimal right ideal I such that $I \supseteq I_1, I_2, \ldots\ldots$ and is denoted by $lub\{I_1, I_2, \ldots\ldots\}$

1.4 **Note:** For the right ideals $I_1, I_2; glb\{I_1, I_2\}$ is denoted by $I_1 \wedge I_2$ and $lub\,\{I_1, I_2\}$ is denoted by $I_1 \vee I_2$. Two right ideals I, J are inverses if $I \wedge J = (0)$ and $I \vee J = R$ clearly $I \vee J = \{x + y/x \in I, y \in J\}$

1.5 **Definition:** Suppose R is a ring with unit 1. An element $e \in R$ is said to be an idempotent if $e^2 = e$.

1.6 **Lemma:** e is an idempotent iff $(1 - e)$ is an idempotent

Proof:
Suppose e is an idempotent $\Rightarrow e^2 = e$
Now consider $(1 - e)$
$e(1 - e) = e.1 - e.e = e - e^2 = e - e = 0$
$(1 - e)^2 = (1 - e)(1 - e) = 1.(1 - e) - e(1 - e) = 1 - e$
$\therefore$ $(1 - e)$ is an idempotent
Conversely suppose that $(1 - e)$ is an idempotent.
$\Rightarrow 1 - (1 - e)$ is an idempotent
$\Rightarrow e$ is an idempotent

1.7 **Lemma:** For any idempotent e, $x \in (e)_r$ iff $ex = x$
Proof:

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Suppose e is an idempotent and let $x \in (e)_r \Rightarrow x = ey$ for some $y \in R$
Consider $ex = e(ey) = e^2y = ey = x$
$\therefore ex = x$
Conversely suppose $x = ex$
$\Rightarrow x \in (e)_r$
1.8 **Lemma:** I, J are inverse right ideals iff $I = (e)_r, J = (1-e)_r$ for suitably chosen idempotent e
Proof: Suppose I, J are inverse right ideals
$\Rightarrow I \wedge J = (0)$ and $I \vee J = R$
$\because 1 \in R \Rightarrow 1 \in I \vee J$
$\Rightarrow 1 = x + y$ where $x \in I, y \in J$
Let $z \in I$
$z = 1.z = (x+y)z = xz + yz$
Since $x \in I, xz \in I$ and $z \in I, yz \in I$
$\Rightarrow yz \in I \wedge J$
$\Rightarrow yz = 0$
So $z = xz \in (x)_r$
$\therefore I \leq (x)_r$
Clearly $(x)_r \leq I$
$\therefore I = (x)_r$
Similarly $J = (y)_r = (1-x)_r$
$x(1-x) = (1-x)x \in I$
Similarly $x(1-x) = (1-x)x \in J$
$\therefore x(1-x) \in I \wedge J = (0)$
$\Rightarrow x(1-x) = 0$
$\Rightarrow x = x^2$
$\therefore x$ is an idempotent
Let $x = e$. Similarly $y = 1-e$
$\therefore I = (e)_r$ and $J = (1-e)_r$
Conversely suppose $I = (e)_r$ and $J = (1-e)_r$
e is an idempotent
$x \in I \wedge J \Rightarrow x \in I, x \in J$
$\Rightarrow x \in (e)_r, x \in (1-e)_r$
$\Rightarrow ex = x, (1-e)x = x$
Consider $(1-e)x = x$
$\Rightarrow x - ex = x \Rightarrow x = 0$
$\therefore I \wedge J = (0)$
Since $1 = e + (1-e) \in I \vee J$
$\therefore I \vee J = R$
$\therefore I$ and J are inverses
1.9 Definition: A ring R has local units iff for each finite set $X \subseteq R, \exists$ an idempotent $e \in R$ such that $x \subseteq eRe$
eRe is unital ring with identify element e
1.10 Definition: A ring R with unit 1 is called a regular ring if for every $a \in R \exists x \in R$ such that

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$axa = a$

Here after R stands for a regular ring.

1.11 Note: For any regular ring R and $a \in R$, the following conditions are equivalent.

(i) $\exists$ an idempotent e such that $(a)_r = (e)_r[(a)_l = (e)_l]$

(ii) $\exists$ a right ideal (left ideal) J which is inverse of $(a)_r[(a)_l]$

1.12 Definition: A ring is called left semi simple (right semi simple) if it is direct sum of minimal left (right) ideals.

1.13 Note: If R is regular ring with unit 1, for every $a \in R(a \neq 0)\exists$ idempotent $e \ni (a)_r = (e)_r[(a)_l = (e)_l]$ such that $I \oplus J = R$ where $I = (e)_r$ and $J = (1-e)_r[I = (e)_l, J = (1-e)_l]$

1.14 Note: A ring R with unit 1 is left semi simple iff right semi simple

1.15 Notation: R is a regular ring. The set of all principal right ideals is denoted by R_R

The set of all principal left ideals of R is denoted by L_R

1.16 Definition: Suppose I is a right ideal of R then $I^l = \{x \in R / xy = 0, \forall y \in I\}$ suppose I is a left ideal of R, then $I^r = \{x \in R / yx = 0, \forall y \in I\}$

1.17 Lemma: I^l is a left ideal

Proof: Let $x, y \in I^l$

$\Rightarrow xz = 0, yz = 0\ \forall z \in I$

$\Rightarrow (x+y)z = xz + yz$

$\quad = 0 + 0 = 0, \forall z \in I$

$\therefore x + y \in I^l$

Let $x \in I^l$ and $r \in R$

$\Rightarrow xy = 0, \forall y \in I$ and $r \in R$

$(rx)y = r(xy) = r0 = 0, \forall y \in I$

$\therefore rx \in I^l$

$\therefore I^l$ is a left ideal

1.18 Lemma: $I \subseteq J \Rightarrow I^l \supseteq J^l$

Proof: Suppose $I \subseteq J$

Let $x \in J^l \Rightarrow xy = 0, \forall y \in J$

$\Rightarrow xy = 0, \forall y \in J$

$\Rightarrow x \in I^l\ (\because I \subseteq J)$

$\therefore I^l \supseteq J^l$

1.19 Lemma: $I \subseteq I^{lr}$

Proof: Let $x \in I \Rightarrow yx = 0, \forall y \in I^l$

$\Rightarrow x \in I^{lr}$

$\therefore I \subseteq I^{lr}$

1.20 Note: $I \subseteq I^{lr}$

1.21 Note: $I^l = I^{lrl}$

1.22 Note: $I^r = I^{rlr}$ for any principal left ideal.

1.23 Lemma: for every principal right ideal I of R, $\exists$ a principal left ideal J such that $I = J^r$

Proof: Suppose I is a principal right ideal

Then $\exists$ an idempotent $e \ni I = (e)_r$

Let $J = (1-e)_l$. Then J is a principal left ideal.

Claim $I = J^r$

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$x \in I \Leftrightarrow x \in (e)_r$
$\Leftrightarrow ex = x$
$\Leftrightarrow (1-e)x = 0$
$\Leftrightarrow z(1-e)x = 0, \forall z \in R$
$\Leftrightarrow yx = 0, \forall y \in (1-e)_l = J$
$\Leftrightarrow x \in J^r$

1.24 Note: For any principal left ideal I, $\exists$ a principal right ideal J such that $I = J^l$

1.25 Lemma: If I is a principal right ideal then $I = I^{lr}$

Proof: Suppose I is a principal right ideal
$\Rightarrow \exists$ a principal left ideal J such that
$I = J^r$ (by Lemma 1.23)
$I^{lr} = J^{rlr} = J^r$ (from the note 1.22)
$= I$

1.26 Note: If I is a principal left ideal, then $I = I^{rl}$

1.27 Lemma: If I is a principal right ideal then I^l is a principal left ideal

Proof: Suppose I is a principal right ideal.
$\Rightarrow I = J^r$ for some principal left ideal J
$I^l = J^{rl} = J$
$\therefore I^l$ is a principal left ideal.

1.28 Note: Every principal left ideal is generated by an idempotent element.

1.29 Theorem: $I \to I^l$ is a one to one mapping of $R_R \to L_R, I \to I^r$ is a one to one mapping from L_R to R_R. They are inverse to each other and anti monotonic.

Proof: Define $f: R_R \to L_R$ by $f(I) = I^l$
Suppose $f(I_1) = f(I_2)$
$\Rightarrow I_1^l = I_2^l$
$\Rightarrow I_1^{lr} = I_2^{lr}$
$\Rightarrow I_1 = I_2$
$\therefore f$ is one – one
Similarly $g: L_R \to R_r$ by $g(I) = I^r$ is one-one.
and $gof =$ identity on R_R and
$fog =$ identity on L_R
For $(gof)(I) = g\big(f(I)\big) = g\big(I^l\big) = I^{lr} = I$
Similarly $(fog)(I) = I$
Supose $I, J \in R_R$ and $I \subseteq J$
$\Rightarrow I^l \supseteq J^l \Rightarrow f(I) \supseteq f(J)$
$\therefore$ These mapping are inverses to each other and anti – monotonic.

1.30 Lemma: If I, J are principal right ideals, then $\exists$ two idempotents e, f such that $ef = fe = 0$ and $I \vee J = (e)_r \vee (f)_r$.

Proof: Suppose I, J are principal right ideals.
$\Rightarrow \exists$ an idempotent e such that
$I = (e)_r$ and $J = (b)_r$. Put $J_1 = \big((1-e)b\big)_r$

Claim: $I \vee J = I \vee J_1$
$I \vee J = \{eu + bv / u, v \in R\}$

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$I \vee J_1 = \{eu' + (1-e)bv \ / \ u', v \in R\}$
$\quad\quad = \{e(u' - bv) + bv \ /u', v \in R\}$
$eu' + (1-e)bv = e(u' - bv) + bv$
$u = u' - bv, u' = u + bv$
Let $x \in I \vee J$
$\Rightarrow x = eu + bv = eu + bv + ebv - ebv$
$= e(u + bv) + \ (1-e)bv \in I \vee J_1$
$\therefore I \vee J \subseteq I \vee J_1$
Similarly $I \vee J_1 \subseteq I \vee J$
$\therefore I \vee J = I \vee J_1$
$\because J_1$ is principal right ideal $\exists$ an idempotent
$f_1 \ni J_1 = (f_1)_r$
$\because f_1 \in J_1 = \big((1-e)b\big)_r \Rightarrow f_1 = (1-e)b\omega$
$ef_1 = e(1-e)b\omega = (e - e^r)b\omega$
$= (e-e)b\omega = 0$
$\therefore ef_1 = 0$
Put $f = f_1(1-e)$
Now $ff_1 = f_1(1-e)f_1 = f_1(f_1 - ef_1) = f_1f_1 = f_1^2 = f_1$
$\therefore ff_1 = f_1$
$f^2 = f.f = ff_1(1-e) = f_1(1-e) = f \ [\because f = f_1(1-e)]$
$\therefore f^2 = f$
$\therefore f$ is an idempotent.
$f = f_1(1-e) \in (f_1)_r$
$f_1 = ff_1 \in (f)_r$
$\therefore (f_1)_r = (f)_r$
$\therefore J_1 = (f)_r[\because J_1 = (f_1)_r]$
$\therefore I \vee J = I \vee J_1 = (e)_r \vee (f)_r$
$ef = ef_1(1-e) = 0(1-e) = 0 \ (\because ef_1 = 0)$
$\therefore fe = 0$
$\therefore I \vee J = (e)_r \vee (f)_r$, where $ef = fe = 0$ and $e^2 = e, f^2 = f$

1.31 Lemma: If I, J are principal right ideals, then $I \vee J$ is a principal right ideal.

Proof: Suppose I, J are principal right ideals.
By 1.30 Lemma, $\exists$ two idempotents e, f with
$ef = 0, fe = 0$ and $I \vee J = (e)_r \vee (f)_r$
$e + f \in (e)_r \vee (f)_r \Rightarrow (e+f)_r \subseteq (e)_r \vee (f)_r$ .... (1)
$\therefore (e+f)e = e^2 + fe = e^2 = e$
$\therefore e \in (e+f)_r$
$\therefore (e)_r \subseteq (e+f)_r$
Similarly $(f)_r \subseteq (e+f)_r$
$\therefore (e)_r \vee (f)_r \subseteq (e+f)_r$ .... (2)
From (1) & (2) $(e)_r \vee (f)_r = (e+f)_r$
$\therefore I \vee J = (e+f)_r$
$\therefore I \vee J$ is a principal right ideal and $e+f$ is idempotent

1.32 Lemma: For every principal right ideals $I, J, I \vee J$ and $I \wedge J$ are also principal right ideals.

Proof: Suppose I, J are principal right ideals.

By 1.31 Lemma, $I \vee J$ is a principal right ideal.

$\because I, J$ are principal right ideals, then I^l, J^l are principal left ideals.

By above note $I^l \vee J^l$ is also principal left ideal.

$\Rightarrow \left(I^l \vee J^l\right)^r$ is a principal right ideal.

$\therefore \left(I^l \vee J^l\right)^r = I^{lr} \wedge J^{lr}$

$\Rightarrow I \wedge J$ is a principal right ideal.

1.33 Note: By 1.32 Lemma, R_R is a lattice. Clearly $O = (0)_r, R = (1)_r$ the unit with the meet $I \vee J, I \wedge J$. Since for every principal right ideal $I, \exists$ a principal right ideal $J \ni I \wedge J = (0)$, $I \vee J = R$

$\therefore R_R$ is a complemented lattice.

1.34 Theorem: R_R is a modular complemented lattice. For suppose I, J, K are principal right ideals and $I \subseteq K$, then $(I \vee J) \wedge K = I \vee (J \wedge K)$

Proof: $I \subseteq I \vee J \Rightarrow I \wedge K \subseteq (I \vee J) \wedge K$

$\Rightarrow I \subseteq (I \vee J) \wedge K \; (\because I \subseteq K)$

$J \wedge K \subseteq (I \vee J) \wedge K$

$\therefore I \vee (J \wedge K) \subseteq (I \vee J) \wedge K$ ………………….. (1)

$(\because I \subseteq I \vee J, J \wedge K \subseteq K)$

$x \in (I \vee J) \wedge K \Rightarrow x \in I \vee J$ and $x \in K$

$\Rightarrow x = y + z, y \in I, z \in J$

$\because x \in K$ and $x \in I \vee J$

i.e., $x = y + z, y \in I, z \in J$

Now $x \in K, y \in I \subseteq K$

So, $z \in K$, hence $y \in J \wedge K \Rightarrow x \in I \vee (J \wedge K)$

$\therefore (I \vee J) \wedge K \subseteq I \vee (J \wedge K)$ ……. (2)

$\therefore$From (1) & (2)

$(I \vee J) \wedge K = I \vee (J \wedge K)$

1.35 Note: By the above theorem, in any regular ring, the collection of principal right ideals forms a complemented modular lattice.

Von Neumann proved a nearly complete converse with contain exactions, every complemented modular lattice is isomorphic to the lattice of principal right ideals of a regular ring, and the regular ring is unique upto isomorphism.

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MODERN APPROACH TO ELLIPTIC CURVES

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ABSTRACT

Information security nowadays is a very important subject Governments, commercial businesses, and individuals are all demanding secure information in electronic documents, which is becoming preferred over traditional documents (paper and microfilm, for example). Documents in electronic form require less storage space, its transfer is almost instantaneous, and it is accessible via simplified databases. The ability to make use of information more efficiently has resulted in a rapid increase in the value of information.

The paper gives an introduction to Elliptic curve cryptography and how it is used in the implementation of digital signature and key agreement algorithms. The paper discusses the implementation of ECC over finite fields, prime fields and binary fields. It also gives an overview of ECC implementation on different coordinate systems called the projective coordinate systems. The paper also discusses the basics of prime and binary field arithmetic.

Key words: Elliptic curve, Finite fields, Cubic curves, Cryptography.

The use of elliptic curves in cryptography was first proposed by Neil Koblitz [16]and Victor Miller [20] in 1985. Koblitz and Miller did not invent a new cryptographic algorithm but they implemented certain existing algorithms using elliptic curve arithmetic. Since its founding elliptic curve cryptography has been studied a lot in the academic world. The use of elliptic curves in cryptography is very inviting because shorter key lengths can be used than in the case of conventional cryptography e.g. RSA.

As points on an elliptic curve over $GF(2^n)$form a finite group of order $n = E(GF(2^n))$, with the point addition as a group operation. Multiplication over an elliptic curve is defined as in Section it is performed by sequentially adding a point to itself. Multiplication is the basic operation of any elliptic curve cryptosystem and many efficient algorithms to compute it have been developed. All elliptic curve cryptography (ECC) algorithms rely on the fact that calculating the point multiplication kP, where k is an integer and P is a point on an elliptic curve, is relatively easy and fast, but it is a very hard task to calculate k, if P and kP are given. The problem that must be solved, to calculate k, is called elliptic curve discrete logarithm problem and it requires an exponential time to solve.

Elliptic curve cryptography has better security with a shorter key length than any other published public-key cryptography method. Elliptic curve cryptosystem with a 173-bit key is considered as secure as RSA using a 1024-bit key and ECC with a 313-bit key is considered as secure as 4096-bit RSA . Elliptic curve cryptography is thus a very attractive alternative, especially in communication systems with limited bandwidth.

Elliptic curves have been studied by mathematicians for more than a century. An extremely rich theory has been developed around them, and in turn they have been the basis of numerous new developments in mathematics. As far as cryptography is concerned, elliptic curves have

been used for factoring and primality proving. The idea of using elliptic curves for public-key cryptosystems is due to Victor Miller

[Miller85] and Neal Koblitz [Koblitz87] in the mid-eighties. As with all cryptosystems, and especially with public-key cryptosystems, it takes years of public evaluation before a reasonable level of confidence in a new system is established. The elliptic curve public-key cryptosystems (ECPKCs) seem to have reached that level now. In the last couple of years, the first commercial

applications have appeared(email security, web security, smart cards, etc.). Before we look at how the ECPKC s work, we will give a short introduction to elliptic curves.

Definition of elliptic curves: Elliptic curves are not ellipses. They are called this because they are described by cubic equations, similar to those used for calculating the circumference of an ellipse. In general, an elliptic curve is the set of solutions of an equation of the form

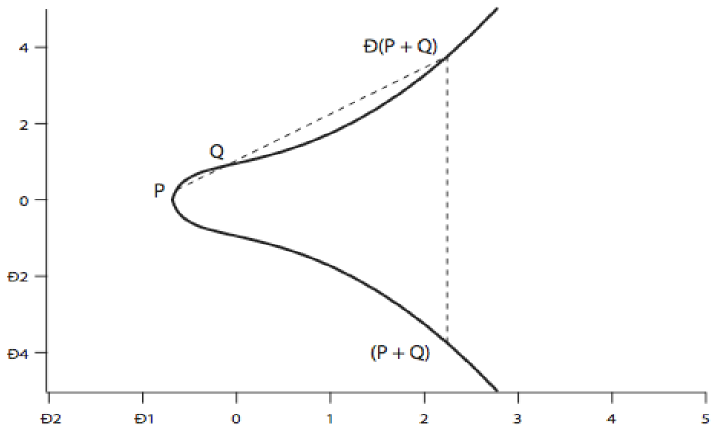
$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_5 \quad \text{………………(1)}$$

Where the coefficients ai are elements of some field (R, Z or Zp) which satisfy some Simple conditions in order to avoid singularities. Such an equation is said to be Cubic, or of degree 3, because the highest exponent it contains is 3. The Eq.1 is Called Weierstrass *equation*. Also included in the definition of any elliptic curve is a single element denoted O and called *point of infinity* or the *zero point.* An elliptic curve over real numbers may be defined as the set of points (x,y) which satisfy an elliptic curve equation of the form:

$$y^2 = x^3 + ax + b,$$

where x, y, a and b are real numbers.

Each choice of the numbers a and b yields a different elliptic curve. For example, a =1 and b =1 gives the elliptic curve with equation $y^2 = x^3 + x + 1$; the graph of this curve is shown below:
If $x^3 + ax + b$ contains no repeated factors, or equivalently if $4a^3 + 27b^2$ is not 0, then the elliptic curve $y^2 = x^3 + ax + b$ can be used to form a group. An elliptic curve group over real numbers consists of the points on the corresponding elliptic curve, together with a special point O called the Figure:
Elliptic Curve (y2 = x3+x +1) point at infinity.



(b) $y^2 = x^3 + x + 1$

1 Point addition: Elliptic Curve Addition: A Geometric Approach:

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P + Q = R is the additive property defined geometrically.

Elliptic curve groups are additive groups; that is, their basic function is addition. The addition of two points in an elliptic curve is defined geometrically.

The negative of a point P = (X_1,Y_1) is its reflection in the x-axis: the point -P is $(X_1,-Y_1)$. Notice that for each point P on an elliptic curve, the point -P is also on the curve.

Adding distinct points P and Q :The resulted point of adding two different points on the elliptic curve is computed as shown below in figure 2

When P = (X_1,Y_1) and Q = (X_2,Y_2) are not negative of each other,

$(X_1,Y_1)+ (X_2,Y_2)= (X_3,Y_3)$; where $X_1 \neq X_2$

P + Q = R where

$\lambda = (Y_2 - Y_1) / (X_2 - X_1)$

$X_3 = \lambda^2 - X_1 - X_2$ and

$$Y_3 = -Y_1 + \lambda (X_1 - X_3)$$

Note that λ is the slope of the line through P and Q.

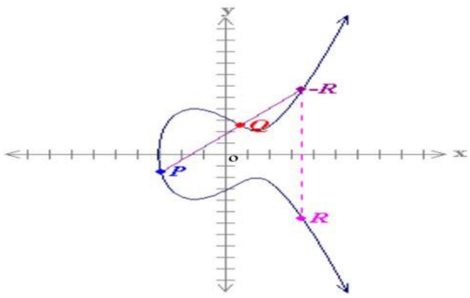


Figure 1.2

Point Addition: Suppose that P and Q are two distinct points on an elliptic curve, and the P is not -Q. To add the points P and Q, a line is drawn through the two points. This line will intersect the elliptic curve in exactly one more point, call -R. The point -R is reflected in the x-axis to the point R. The law for addition in an elliptic curve group is P + Q = R..

Point doubling :

$(X_1,Y_1)+ (X_2,Y_2)= (X_3,Y_3)$;

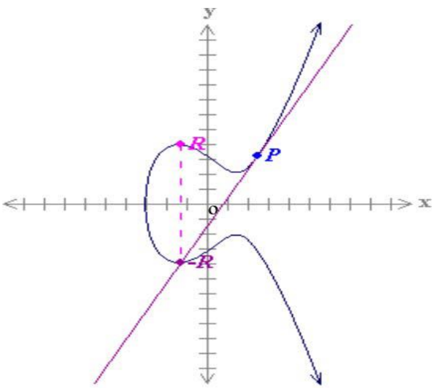
where $Y_1 \neq 0$

2P = R where

$\lambda = (3X_1^2 + a) / (2Y_1)$

$X_3 = \lambda^2 - 2X_1$ and

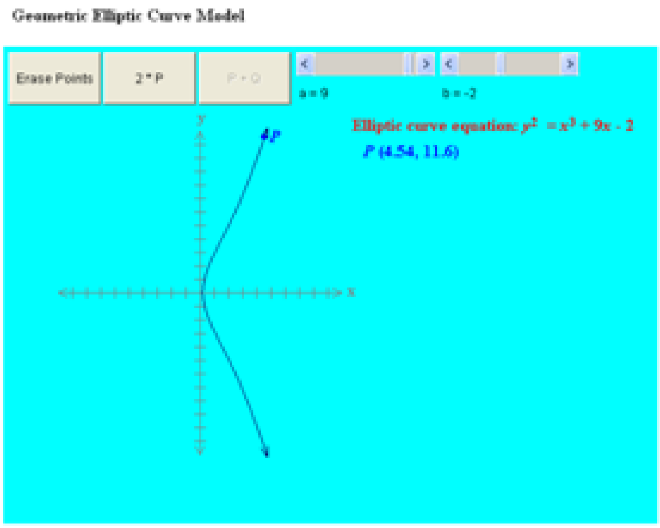
$Y_3 = \lambda (X_1 - X_3) - Y_1$



Shows how a point can be doubled graphically on the elliptic curve. Suppose we want to double a point P on the elliptic curve. A tangent line to the curve and passing by P is taken to double the point. The line must cross the curve through another point; the point is noted as -R. Then we reflect the point –R in the x-axis to the point R where R=2P.

The line through P and -P is a vertical line which does not intersect the elliptic curve at a third point; thus the points P and -P cannot be added as previously. It is for this reason that the elliptic curve group includes the point at infinity O. By definition, P + (-P) = O. As a result of this equation, P + O = P in the elliptic curve group . O is called the additive identity of the elliptic curve group; all elliptic curves have an additive identity.

Elliptic curves over real numbers: $y^2=x^3+ax+b$ with a=9,b=-2.



2.1 Elliptic curves over Finite fields F_P :

2.1.1 Introduction:

All elliptic curve operations mentioned earlier are based on real numbers. However, operations over the real numbers are inaccurate and slow, whereas cryptographic operations need to be accurate and fast. Therefore, the curve cryptography can be defined over finite fields to operate EC efficiently and accurately. A finite field is a set of a finite number of elements. Cryptographic

2.1 Elliptic curves over Finite fields FP :

2.1.1 Introduction:

All elliptic curve operations mentioned earlier are based on real numbers. However, operations over the real numbers are inaccurate and slow, whereas cryptographic operations need to be accurate and fast. Therefore, the curve cryptography can be defined over finite fields to operate EC efficiently and accurately. A finite field is a set of a finite number of elements. Cryptographic applications require fast and precise arithmetic; thus elliptic curve groups over the finite fields of Fp and F2m are used in practice.

Recall that the field Fp uses the numbers from 0 to p - 1, and computations end by taking the remainder on division by p. The number of points on E(F p) is denoted by #E(F p). The Hasse Theorem states that:

$p+1-2\sqrt{p} \leqslant \#E(F\ p) \leqslant p+1+2\sqrt{p}$.

For example, in F23 the field is composed of integers from 0 to 22, and any operation within this field will result in an integer also between 0 and 22.

An elliptic curve with the underlying field of Fp can formed by choosing the variables a and b within the field of Fp. The elliptic curve includes all points (x,y) which satisfy the elliptic curve equation modulo p (where x and y are numbers in Fp).

For example: y2 mod p = x3 + ax + b mod p has an underlying field of Fp if a and b are in Fp.

If x3 + ax + b contains no repeating factors (or, equivalently, if 4a3 + 27b2 mod p is not 0), then the elliptic curve can be used to form a group. An elliptic curve group over Fp consists of the points on the corresponding elliptic curve, together with a special point O called the point at infinity. There are finitely many points on such an elliptic curve.

2.1.2 Example of an Elliptic Curve Group over Fp:

As a very small example, consider an elliptic curve over the field F23. With a = 1 and b = 0, the elliptic curve equation is y2= x3 + x. The point (9,5) satisfies this equation since y2 mod p = x3 + x mod p

25 mod 23 = 729 + 9 mod 23

25 mod 23 = 738 mod 23

2 = 2

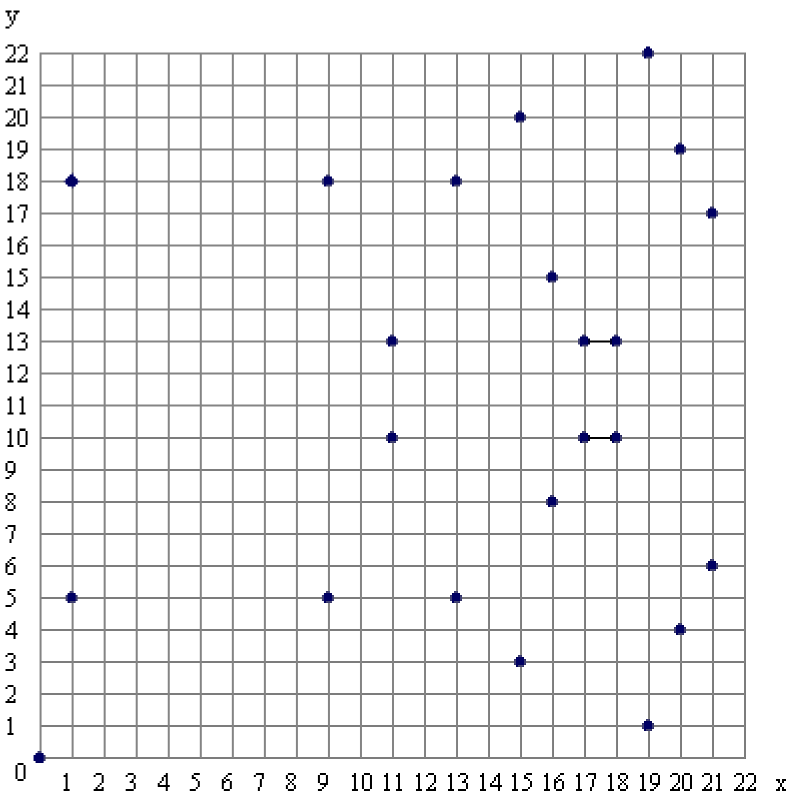
The 23 points which satisfy this equation are:

(0,0) (1,5) (1,18) (9,5) (9,18) (11,10) (11,13) (13,5)

(13,18) (15,3) (15,20) (16,8) (16,15) (17,10) (17,13) (18,10)

(18,13) (19,1) (19,22) (20,4) (20,19) (21,6) (21,17)

These points may be graphed as below:



Elliptic curve equation: $y^2 = x^3 + x$ over F_{23}

Figure 2.1.2

Note that there is two points for every x value. Even though the graph seems random, there is still

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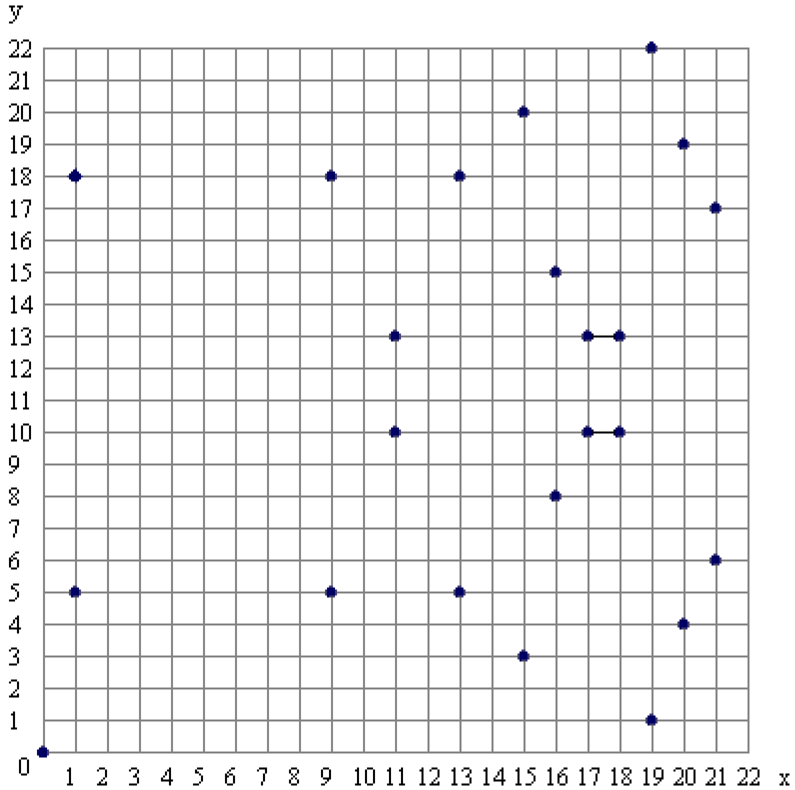
symmetry about y = 11.5. Recall that elliptic curves over real numbers, there exists a negative point for each point which is reflected through the x-axis. Over the field of F23, the negative components in the y-values are taken modulo 23, resulting in a positive number as a difference from 23. Here -P = (xP, (-yP Mod 23))

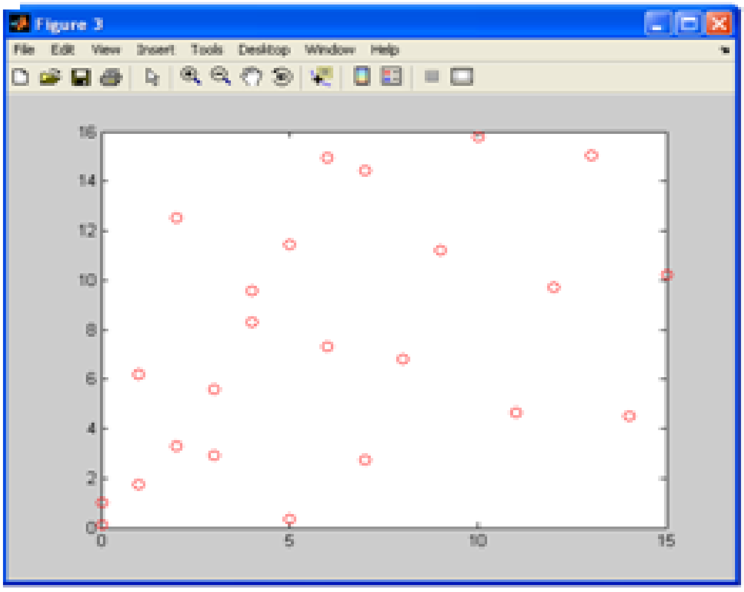


Elliptic curve equation: $y^2 = x^3 + x$ over F_{23}

2.2.1 Arithmetic in Elliptic Curve Group over F_p Point addition:

Note that these rules are exactly the same as those for elliptic curve groups over real numbers, with the exception that computations are performed modulo p.

There are several major differences between elliptic curve groups over F_p and over real numbers. Elliptic curve groups over F_p have a finite number of points, which is a desirable property for cryptographic purposes. Since these curves consist of a few discrete points, it is not clear how to "connect the dots" to make their graph look like a curve. It is not clear how geometric relationships can be applied. As a result, the geometry used in elliptic curve groups over real numbers cannot be used for elliptic curve groups over F_p. However, the algebraic rules for the arithmetic can be adapted for elliptic curves over F_p. Unlike elliptic curves over real numbers, computations over the field of F_p involve no round off error - an essential property required for a cryptosystem.



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OPTIMIZATION OF A MANPOWER MODEL WITH DOUBLE TRUNCATED NEGATIVE BINOMIAL DISTRIBUTION

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ABSTRACT

Manpower planning is the process of systematically forecasting the future demand and supply of employees through which management determines how it should move from its current manpower to its desired manpower utilization. Further human behaviour is highly variable and it is therefore essential to model manpower planning that accommodates population heterogeneity. Thus it is a prime concern for any management for human resource development to model the situation and its optimization. Mathematics and Statistics have done much more work on the development of models of manpower systems in the recent years. If the organization fails to place direct human resources in the right areas of the business at right time and right cost serious inefficiencies are likely to occur creating considerable operational difficulties or even business failure. To prevent business failure co – ordination of demand and supply is required together with the monitoring and assessment of productivity and technological changes. Different types of Manpower model plays an important role in efficient design and control of manpower system. In this paper the authors have developed and analysed a stochastic mathematical model with double truncated negative binomial distribution.While studying similar models through a thorough literature review we observed the importance of truncation parameter.This inspires us to develop the present model in which we made an attempt to minimize the truncation parameter which in turn minimizes the size of the recruitment.

Key words: *Manpower,Recruitment,Truncation parameter, Double Truncated Negative Binomial Distribution, Probability Mass Function*

1. REVIEW ON RELATED LITERATURE

Manpower planning is determination of right number and right skills of human force to suit present and future needs. Manpower planning is defined by STAINER as a "strategy for the requisition, utilization, improvement and preservation of an enterprise's human resource. It relates to establishing job specifications or the quantitative requirements of jobs determining the number of personnel required and developing sources of manpower." Manpower planning is a process determining requirements of right number and right kind of human force at right place and right time. Objectives of manpower planning are to ensure optimum use of human resources currently employed. To assess future skills requirement, to provide control measures to ensure that necessary resources are available as and when required, to determine requirement level, to anticipate redundancies and avoid unnecessary dismissals and assess training and development needs. Each organisation needs manpower planning. An organisational unit is started to accomplish certain goals. Which requires human resources with necessary qualification? These are provided through effective manpower planning. Comprehensive manpower planning helps to optimise effectiveness of human resources. In an organisation, employees who have grown old or who resign, retire, die or become incapacitated because of mental or physical ailment have to be replaced and new employees have to be recruited. This can be done through manpower planning. It is also needed for identifying surplus or shortage manpower areas and there by balancing manpower. In short manpower planning provides right size and structure of human resources which provides the basic infrastructure for smooth functioning of an organisation. It minimizes the cost of employment and nullifies the effects of disruptions in developing and utilizing the human resources.

2. INTRODUCTION

A manpower model is a statistical description of how change takes place in the organisation. Though British admirals has orginated the modeling of manpower system during second world war, the work of SEAL (1945) can be considered as the pioneering work in the field of manpower modeling. Manpower planning is often defined as an attempt to match the supply of people with the jobs available for them. A manpower model is a mathematical description of how change takes place in the system. Manpower modeling is the prime concern for any management of human resources development. Due to the uncertainties involved in the constant process of manpower models, stochastic modeling is needed for development and analysis of the manpower system. Various researcher have developed different manpower models with wide variation in order to suit manpower system of the different organisations.Generally manpower modeling of an organisation can be categorised into a number of mutually exclusive, exhaustive and homogeneous groups called classes. The classes may correspond to a grade, salary level, age groups or any other classification or combination of classification of interest. Flows are supposed to take place into the system as whole, between classes within the system and between the system and other sources of employment or outside the system. The stocks are the number of people in the classes and the flows are number of members who move between classes and in and out of the system. The model for the system express the relationship between the stocks and flows as the system develops in time. Sometimes, we may come across flows of people out of a manpower system called wastage, flows into system called recruitment and flows between different classes of the system may be called transfers, promotions or demotions.

Of all the flows in the manpower system, wastage is most fundamental for manpower planning. Therefore it can be said that successful manpower planning depends on the pattern of wastage. Studying the wastage rate of an organisation requires the study of Completed Length of Service (CLS) distribution of the organization.Different researcher have considered that the CLS of an employeer in an organisation is continuous and discretes having infinite and finite range by stating the assumptions under which they can apply. While fitting the CLS of an organisation with truncated distributions, the value of the truncation parameters has great importance. As the organisation always wants to minimise the number of recruits, the value of the truncation parameter which minimises the recruitment is said to be the optimal value of the truncation parameter.In this study an attempt was made to find the optimal value of the truncation parameter which signifies the minimum optimum period of service in the organisation.

MODEL DEVELOPMENT

3. MANPOWER MODELS WITH DOUBLE TRUNCATED NEGATIVE BINOMIAL DISTRIBUTION

In the manpower model with Double Truncated Negative Binomial Distribution (DTNBD), the probability mass function of the CLS of the organisation is given by

$$P(T=t)=f_t=\binom{t=n-1}{n-1}p^t \; 0\le p\le 1, t=1,2,\dots\dots A, n=1,2,\dots < A$$

Where $k=\sum_{t=1}^{A}\binom{t+n-1}{n-1}p^t$ (1)

The distribution function of the model is

$$F(T)=\sum_{t=1}^{T-1}\frac{\binom{t+n-1}{n-1}p^t}{k} \quad (2)$$

The expected length of the service of the model is

$$\mu=\sum_{t=1}^{A}\frac{t\binom{t+n-1}{n-1}p^t}{k} \quad (3)$$

The survival function of the employee in the organisation is

$$G_T=1-F_{T-1}=1-\sum_{t=1}^{T-1}\frac{\binom{t+n-1}{n-1}p^t}{k} \quad (4)$$

The approximate renewal mass function of the model is

$$h(T)=\frac{\binom{T+n-1}{n-1}pT}{k}+\frac{\left[\sum_{t=1}^{T-1}\frac{\binom{t+n-1}{n-1}p^t}{k}\right]}{\sum_{j=1}^{T-1}\left(1-\sum_{t=1}^{j-1}\frac{\binom{t+n-1}{n-1}p^t}{k}\right)} \quad (5)$$

The average number of recruits in the time period (0,T) as

$$\bar{n}_T=N\sum_{i=1}^{T}h_i$$

$$=N\sum_{i=1}^{T}\left(\frac{\binom{i+n-1}{n-1}p^i}{k}+\frac{\left(\sum_{t=1}^{i-1}\frac{\binom{t+n-1}{n-1}p^t}{k}\right)^2}{\sum_{j=1}^{i-1}\left(1-\sum_{t=1}^{j-1}\frac{\binom{t+n-1}{n-1}p^t}{k}\right)}\right) \quad (6)$$

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4. OPTIMAL RECRUITMENT POLICIES OF THE MANPOWER MODEL WITH DOUBLE TRUNCATED NEGATIVE BINOMIAL DISTRIBUTION

The optimal values of the parameters of the model can be obtained by minimizing $\overline{n}_T$ with respect to the truncation parameter A. Since the truncation parameter is discrete, one can use first order difference condition for optimizing $\overline{n}_T$ with respect to A for fixed value p.

According to the argument as discussed in section **3** the optimal values A* of A subject to the condition.

$$\Delta\overline{n}(T, A^* - 1) \leq 0 \leq \Delta\overline{n}(T, A^*) \quad (7)$$

Where Δ is the symbol for first order difference.

Using the available software packages like MATHCAD 13 and by the applications of the numerical methods the computations were made and the calculated values are tabulated in the **Table 1**.The variations in the truncation parameter A in accordance with the variations in time duration for a fixed value of p were analysed. The sensitivity analysis has been studied.

T / A	3	4	5
4	1.638 x 103	• • •	• • •
5	931.454	2.122 x 103	• • •
6	727.221	1.261 x 103	2.972 x 103
7	641.392	1.008 x 103	1.821 x 103
8	599.825	900.117	1.478 x 103
9	578.317	848.947	1.334 x 103
10	567.128	821.343	1.263 x 103
11	561.469	808.431	1.229 x 103
12	557.969	800.83	1.211 x 103
13	556.231	797.071	1.202 x 103
14	555.195	794.58	1.195 x 103
15	554.862	793.339	1.191 x 103
16	554.862	793.339	1.191 x 103
17	554.167	792.101	1. 189 x 103
18	554.167	792.101	1.189 x 103

Table:1 (p = 0.5 n = 2)

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The calculated values of the truncation parameter for different combinations of the other parameters are presented in **Table:1** and it is observed that for p = 0.5 and n = 2, the optimal value of the truncation parameter A is 16, 17 and 18 in the time duration (0,3), (0,4) and (0,5) respectively. Also as p is increasing the optimal value of the truncation parameter A is decreasing.

5. SUMMARY AND CONCLUSION

A review on the manpower modeling reveals that most of the manpower models have been analyzed through the complete length of the service (CLS) distribution of the organization. While considering CLS, the manpower systems were modeling through different discrete and continuous distributions having infinite and finite range. In case of finite range distributions, the distributions are right truncated basing on the assumption that the maximum service of an individual in the organization is finite. So, in case of manpower modeling with truncated distribution, truncation parameters has vital role in recruitment policies and hence the estimations of optimal value of the truncation parameter is necessary for the minimum recruitment level.

The inference about the optimal values of the truncation parameter in the developed manpower model with Double Truncated Negative Binomial Distribution is that the optimal values of the Truncation parameter is increasing as time duration is increasing and decreasing as p increases. The sharpness of this decrease is less as the values of the p are decreasing.

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ON IDEALS IN SEMISIMPLE SEMIGROUPS

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ABSTRACT

In this paper the terms, semisimple elements of a semigroup, semisimple semigroup are introduced. It is proved that If (1) a is idempotent element (2) a is regular(3) a is left/right regular element of a semigroup then a is semisimple, It is also proved that, If a is an element of a duo semigroup S, then 1) a is completely regular., 2) a is regular., 3) a is left regular., 4) a is right regular., 5) a is intra regular are equivalent.further it is proved that, S is semisimple simigroup if and only if every S is fully idempotent, if S is a semisimple semigroup then 1) S is globally idempotent semigroup., 2) every maximal ideal M of S is a prime ideal of S. and If S is a globally idempotent semigroup with maximal ideals then S contains semisimple elements. It is also proved that, If a duo semigroup S is semisimple, then every principal ideal of S is generated by an idempotent., if A is an ideal of a semisimple semigroup S, then 1) A is completely semiprime., 2) A is pseudo symmetric., 3) A is semipseudo symmetric are equivalent and for any semipseudo symmetric ideal A in a semigroup S, $e \neq x \in S$ such that there exists a natural number n such that $x^n \in A$, then x cannot be semisimple. It is provedthat if M is a maximal ideal in a semisimple semigroup S containing a semipseudo symmetric ideal A and $x \in S$ is an *A-potent element*. Then $x \in M$.

1. INTRODUCTION

The concept of a semigroup is very simple and plays a large role in the development of Mathematics. The theory of semigroups is similar to group theory and ring theory. The algebraic theory of semigroups was developed by CLIFFORD and PRESTON [7], [8]; PETRICH [14] and LJAPIN[12]. The ideal theory in commutative semigroups has developed by BOURNE [6], HARBANS LAL [10], SATYANARAYANA [15], [16], [17], MANNEPALLI and NAGORE [13]. The ideal theory in general semigroups has developed by ANJANEYULU [1], [2], [3], [4], [5], GIRI and WAZALWAR [9], HOEHNKE [11] and SCWARTZ [18].

2. PRILLIMINARIES :

DEFINITION 2.1: A system S = (S, .), where S is a nonempty set and . is an associative binary operation on S, is called a *semigroup.*

DEFINITION 2.2: A semigroup S is said to be *commutative* provided $ab = ba$ for all $a, b \in S$.

DEFINITION 2.3: An element a of a semigroup S is said to be *cancellative* provided a is both a left cancellative and a right cancellative element.

DEFINITION 2.4 : A nonempty subset A of a semigroup S is said to be a *left ideal* of S provided SA⊆A.

DEFINITION 2.5 : A nonempty subset A of a semigroup S is said to be a *right ideal* of S provided AS⊆A.

DEFINITION 2.6: A nonempty subset A of a semigroup S is said to be a *two sided ideal* or *ideal* of S provided it is both a left ideal and a right ideal of S.

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THEOREM 2.7 : The nonempty intersection of any family of ideals of a semigroup S is an ideal of S.
THEOREM 2.8 : The union of any family of ideals of a semigroup S is an ideal of S.
Proof : Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of ideals of S and let $A = \bigcup_{\alpha\in\Delta} A_\alpha$.
DEFINITION 2.9 : Let S be a semigroup. The intersection of all ideals of S containing a nonempty set A is called the *ideal genetated by A.* It is denoted by < A >.
DEFINITION 2.10 : An ideal A of a semigroup S is said to be a *principal ideal* provided A is an ideal generated by single element set. If an ideal A is generated by *a,* then A is denoted as <*a*> or J[*a*].
DEFINITION 2.11 : An ideal A of a semigroup S is said to be a *maximal ideal* provided A is a proper ideal of S and is not properly contained in any proper ideal of S.
DEFINITION 2. 12 : An ideal A of a semigroup S is said to be a *minimal ideal* provided A does not contain any ideal of S properly.
DEFINITION 2.13 : A semigroup S is said to be a *duo semigroup* provided S is both a left duo semigroup and a right duo semigroup.
THEOREM 2.14 : A semigroup S is a duo semigroup if and only if $xS^1=S^1x$ for all $x\in$ S.
DEFINITION 2.15 : An ideal A of a semigroup S is said to be a *globally idempotent ideal* provided A^2 = A.
DEFINITION 2.16 : A semigroup S is said to be a *globally idempotent semigroup* provided S^2 = S.
DEFINITION 2.17 : An ideal A of a semigroup S is said to be a *fully idempotent semigroup* if every ideal of S is *globally idempotent ideal* . That is A^2 = A for every ideal A of S.
DEFINITION 2.18 : An element *a* of a semigroup S is said to be an *idempotent* provided
$a^2 = a$.
DEFINITION 2.19 : An element *a* of a semigroup S is said to be a *proper idempotent* provided *a* is an idempotent which is not the identity of S if identity exists.
DEFINITION 2.20 : A semigroup S is said to be an *idempotent semigroup* or a *band* provided every element in S is an idempotent.
DEFINITION 2.21 : An element *a* of a semigroup S is said to be *regular* provided $a = axa$ for some $x\in$ S.
DEFINITION 2.22 : An element *a* of a semigroup S is said to be *left regular* provided
$a = a^2x$ for some $x\in S$.
DEFINITION 2.23 : An element *a* of semigrojup S is said to be *completely regular* provided $a = axa$ *and* $ax = xa$ for some $x\in S$.
DEFINITION 2.24: An ideal A of a semigroup S is said to be a *completely prime ideal* provided $x, y\in$ S, $xy\in$ A, implies either $x\in$ A or $y\in$ A.
DEFINITION 2.25 : An ideal A of a semigroup S is said to be a *prime ideal* provided X,Y are ideals of S, XY $\subseteq$ A implies either X $\subseteq$ A orY $\subseteq$ A.
DEFINITION 2.26: An ideal A of a semigroup S is said to be *completely semiprime* provided $x \in$S, $x^n \in$ A for some natural number *n* implies $x \in$ A.
COROLLARY 2.27 : If an ideal A of a semigroup S is completely semiprime then $x, y \in$ S, $xy \in$ A $\Longrightarrow$ < x >< y > $\subseteq$ A.
THEOREM 2.28 : If A is a completely prime ideal of a semigroup S then A is a completely semiprime ideal.
DEFINITION 2.29 : An ideal A of a semigroup S is said to be *semiprime* provided X is an ideal of S, X^n

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$\subseteq$ A for some natural number n implies X $\subseteq$ A.

DEFINITION 2.30 : An ideal A of a semigroup S is said to be *pseudo symmetric* provided $x, y \in$ S, $xy \in$ A implies $xsy \in$ A for all $s \in$ S.

THEOREM 2.31 : Every completely semiprime ideal A in a semigroup S is a pseudo symmetric ideal.

THEOREM 2.32 : Let A be any pseudo symmetric ideal in a semigroup S and $a_1, a_2 \ldots, a_n \in$ S. Then $a_1a_2 \ldots a_n \in$ A if and only if $<a_1><a_2> \ldots <a_n> \subseteq$ A.

DEFINITION 2.33 : A semigroup S is said to be *pseudo symmetric* provided every ideal in S is a pseudo symmetric ideal.

DEFINITION 2.34 : An ideal A in a semigroup S is said to be *semipseudo symmetric* provided for any natural number n, $x \in$ S, $x^n \in$ A, $\Rightarrow <x^n> \subseteq$ A.

THEOREM 2.35 : Every pseudo symmetric ideal of a semigroup is a semipseudo symmetric ideal.

DEFINITION 2.36 : A semigroup S is said to be *semipseudo symmetric* provided every ideal of S is semipseudo symmetric.

THEOREM 2.37 : If M is a maximal ideal of a semigroup S with $M_4 \neq S$, then the following are equivalent.

1) M is completely prime.
2) M is completely semiprime.
3) M is pseudo symmetric.
4) M is semipseudo symmetric.

3. SEMISIMPLE SEMIGROUPS

DEFINITION 3.1 : An element a of a semigroup S is said to be *semisimple* provided $a \in <a>^2$, that is $<a>^2 = <a>$

DEFINITION 3.2 : A semigroup S is said to be a *semisimple semigroup* provided every element in S is semisimple.

THEOREM 3.3 : Let S be a semigroup and $a \in S$. If a is idempotent then a is semisimple

Proof : Suppose that a is idempotent. Then $a = a^2 \Rightarrow < a > = <a^2>^2 = <a>^2$.

Therefore $a \in <a>^2$ and hence a is semisimple.

THEOREM 3.4 : Let S be a semigroup and $a \in S$. If a is regular then a is semisimple.

Proof : Suppose that a is regular. Then $a = axa$ for some $x \in$ S $\Rightarrow a \in <a>^2$.

Therefore a is semisimple.

THEOREM 3.5 : Let a be an element of a semigroup S. If a is left regular or right regular then a is semisimple.

Proof : Suppose a is left regular. Then $a = a^2x$ for some $x \in$ S $\Rightarrow a \in < a >^2$.

Therefore a is semisimple.

If a is right regular, then $a = xa^2$ for some $x \in$ S $\Rightarrow a \in < a >^2$. Therefore a is semisimple.

THEOREM 3.6 : Let a be an element of a semigroup S. If a is intraregular then a is semisimple.

Proof : Suppose a is intraregular. Then $a = xa^2y$ for some $x, y \in$ S $\Rightarrow a \in < a >^2$.

Therefore a is semisimple.

THEOREM 3.7 : If a is an element of a duo semigroup S, then the following are equivalent.

1. a is completely regular.
2. a is regular.
3. a is left regular.
4. a is right regular.

5. a is intra regular.
6. a is semisimple.

Proof: Since S is a duo semigroup, $aS^1=S^1a$. We have $aS^1a=a^2S^1=S^1a^2=<a^2>=<a>^2$

(1) $\Rightarrow$ (2) : Suppose a is completely regular. Then $a = axa$, $ax=xa$ for some $x\in S$. Therefore a is regular.

(2) $\Rightarrow$ (3) : Suppose a is regular. Then $a = axa$ for $x\in S$ $a \in aS^1a = a^2S^1$
$a = a^2y$ for some $y\in S^1$. Therefore a is left regular.

(3) $\Rightarrow$ (4) : Suppose a is left regular. Then $a = a^2x$ for some $x\in S \Rightarrow a\in a^2S^1 = S^1a^2$
$a=ya^2$ for some $y\in S^1$. Therefore a is right regular.

(4) $\Rightarrow$ (5) : Suppose a is right regular. Then $a = xa^2$ for some $x\in S^1 \Rightarrow aS^1a^2 = < a >^2$
$a < a >^2a = xa^2y$ for some $x, y\in S^1$. Therefore a is intra regular

(5) $\Rightarrow$ (6) : Suppose a is intraregular. Then $a = xa^2y$ for some $x, y\in S \Rightarrow a \in < a >^2$
$<a> = <a>^2$. Therefore a is semisimple.

(6) $\Rightarrow$ (1) : suppose a is semisimple. Then $a \in < a >^2 = a^2S^1 = S^1a^2$ $a = a^2x = ya^2$ for some $x, y \in S^1$. Therefore a is both left regular and right regular. Hence, a is completely regular.

THEOREM 3.8 : Let S be a semigroup, Then S is semisimple simigroup if and only if every S is fully idempotent.

Proof: Suppose that S is a semisimple semigroup, then $a \in < a >^2$ for all $a \in$ S.

Let A be an ideal of S and $a \in$ A. Now $a \in$ S and S is semisimple $\Rightarrow a \in < a >^2$ and $< a >^2 \subseteq A^2 \Rightarrow a \in A^2 \Rightarrow A \subseteq A^2$. thus We have A = A^2 .

Conversely suppose that S fully idempotent and $a \in$ S. Then $< a >$ is an ideal of S. Since S is fully idempotent, we have $< a > = < a >^2$ and $a \in < a >$. Hence $a \in < a >^2$. Therefore a is semisimple. Hence S is semisimple semigroup.

THEOREM 3.9 : If S is a semisimple semigroup then S is globally idempotent semigroup. *Proof*: Suppose that S is a semisimple semigroup, then $a \in < a >^2$ for all $a \in$ S. By theorem 3.8, we have S is fully idempotent $\Rightarrow$ every ideal of S is globally idempotent. Since S is an ideal to it self $\Rightarrow$ S $\subseteq S^2$. Therefore S = S^2 . Hence S is globally idempotent.

THEOREM 3.10 : If S is a semisimple semigroup then every maximal ideal M of S is a prime ideal of S.

Proof: Suppose S is a semisimple semigroup. By theorem 3.9, we have S is globally idempotent semigroup. Let M be a maximal ideal of S. Let A, B be two ideals of S such that AB $\subseteq$ M.

Suppose if possible A $\not\subset$ M, B $\not\subset$ M and A $\not\subset$ M $\Rightarrow$ M $\cup$ A is an ideal of S and M $\subseteq$ M $\cup$ A $\subseteq$ S. Since M is maximal, M $\cup$ A = S. Similarly B $\not\subset$ M $\Rightarrow$ M $\cup$ B = S.

Now S = S^2 = (M $\cup$ A)(M $\cup$ B) = $M^2 \cup MB \cup AM \cup AB \subseteq M \Rightarrow S \subseteq M$. Hence M = S. It is a contradiction. Therefore either A $\subseteq$ M or B $\subseteq$ M. Hence M is a prime ideal.

THEOREM 3.11 : If S is a globally idempotent semigroup with maximal ideals then S contains semisimple elements.

Proof:- Suppose S is a globally idempotent semigroup with maximal ideals. Let M be a maximal ideal in S. Then By the theorem 1.8, M is prime. Now if $a\in S\backslash M$ then $<a>^2 \not\subset M$ and hence S = M $\cup <a>$ = M $\cup <a>^2$. Therefore $a\in <a>^2$ and hence $<a> = <a>^2$. Thus a is semisimple. Therefore S contains semisimple elements.

THEOREM 3.12 : If a duo semigroup S is semisimple, then every principal ideal of S is generated by an idempotent.

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Proof : Suppose S is a semisimple semigroup and $a \in$ S . Let < a > be a principal ideal of S. Since a is semisimple, by the theorem 3.7, we have a is regular then $a = axa$ for some $x, y \in$ S. Let $ax = e$. Now $e^2 = (ax)^2 = axax = ax$. $a = axa = ea \in$ <e> $\Rightarrow$<a>$\subseteq$< e >. Now $e = ax \in$ <a> $\Rightarrow$ <e> $\subseteq$ <a>. Therefore <a> = < e >. Therefore every principal ideal is generated by an idempotent.

THEOREM 3.13 : If A is an ideal of a semisimple semigroup S, then the following are equivalent.

1) A is completely semiprime.
2) A is pseudo symmetric.
3) A is semipseudo symmetric.

Proof : (1) $\Rightarrow$(2) : Suppose that A is completely semiprime. By theorem 2.31, A is pseudo symmetric.
(2) $\Rightarrow$ (3) : Suppose that A is pseudo symmetric. By theorem 2.35, A is semipseudo symmetric.
(3) $\Rightarrow$ (1) : Suppose that A is semipseudo symmetric. Let $x \in$S, $x^2 \in$ A. Since A is semipseudo symmetric, $x^2 \in$ A $\Rightarrow$ < x >^{2} $\subseteq$A. Since S is semisimple, x is a semisimple element. Therefore $x \in$ < x >^{2} $\subseteq$A. Thus A is completely semiprime.

THEOREM 3.14 : In a semipseudo symmetric semigroup S, an element a is semisimple iff a is intraregular.

Proof : Let S be a semipseudo symmetric semigroup. Suppose an element $a \in$ S is semisimple. Then $a \in$<a>^{2}. Since S is semipseudo symmetric, <a^2> is a semipseudo symmetric ideal. Thus $a^2 \in$ <a^2> $\Rightarrow$<a>^{2} $\subseteq$<a^2> $\Rightarrow$ $a \in$<a>^{2} $\subseteq$<a^2>.

Therefore $a = sa^2t$ for some $s, t \in S^1$ and hence a is intraregular.

Conversely suppose that $a \in$S is intraregular. Then $a = xa^2y$ for some $x, y \in$ S $\Rightarrow$ $a \in$<a^2>.

Therefore, a is semisimple.

DEFINITION 7.1 : Let A be an ideal in a semigroup S. An element $x \in$S is said to be *A-potent* there exists a natural number n such that $x^n \in$A.

THEOREM 3.15 : For any semipseudo symmetric ideal A in a semigroup S, $e \neq x \in$S such that there exists a natural number n such that $x^n \in$A, then x cannot be semisimple.

Proof : Supposethat $e \neq x \in$S such that there exists a natural number n such that $x^n \in$A. Since $x^n \in$A and A is semipseudo symmetric ideal, we have <x>^{n}$\subseteq$A. If x is semisimple, then <x> = <x>^{2} and hence <x> = <x>^{n} $\subseteq$ A $\Rightarrow$ $x \in$ A. this is a contradiction. Thus x is not semisimple.

COROLLARY 3.16 : If M is a maximal ideal in a semisimple semigroup S containing a semipseudo symmetric ideal A and $x \in$S is an *A-potent element*. Then $x \in$M.

Proof : By the theorem 2.37, A is pseudo symmetric ideal. If $a \in$ S\M is A-potent, then by the theorem a cannot be semisimple. It is a contradiction. Therefore $x \in$M.

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LRS BIANCHI TYPE-I GENERALIZED GHOST PILGRIM DARK ENERGY MODEL IN SAEZ-BALLESTER THEORY WITH LINEARLY VARYING DECELERATION PARAMETER

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ABSTRACT

In this paper, we investigate LRS Bianchi type-*I* Universe filled with matter and generalized ghost pilgrim dark energy in Saez-Ballester [1986] theory of gravitation. We have used linearly varying deceleration parameter proposed by Akarsu and Dereli [2012] to obtain a deterministic solution of the field equations. Physical parameters of the model are obtained and some geometrical and kinematical properties of the model are also discussed.

Keywords: LRS Bianchi type-I space-time, Saez-Ballester theory, linearly varying deceleration parameter, Generalized ghost pilgrim dark energy.

1. INTRODUCTION

Riess et al. [1998] and Perlmutter et al. [199] and Bennett et al. [2003], through their cosmological observations suggested that the Universe is in a state of accelerated expansion. It is believed that the reason for this is an exotic type of unknown force with positive energy density and negative pressure known as dark energy. It is also suggested that more than 70% of our Universe consists of dark energy that exerts a huge negative pressure and causes the cosmic acceleration. Also 30% of the content of the Universe is gravitating matter but most of it is non – baryonic and is called 'dark matter'. Thus, our Universe is that it consists of cosmic fluid made up of dark matter and dark energy evolving independently. However, cosmic acceleration is still a cosmological mystery.

In recent years, modifying general relativity is attracting more and more attention to explain late time acceleration and existence of dark energy component in the Universe such as quintessence, phantom, tachyon and Chaplygin gas (Padmanabhan [2002,2008]; Bento et al. [2002]; Caldwell [2002]; Nojiri and Odintsov [2003] and Feng et al. [2005]).Among the various modifications of general relativity scalar tensor theories of gravity proposed by Brans and Dicke [1961], Saez and Ballester [1986] and *f(R,T)* gravity (Harko et al. [2011]) are considered to be more popular to explain late time acceleration and dark energy. Here, we focus our attention on Saez – Ballester [1986] scalar tensor theory of gravitation.

Brans and Dicke [1961] scalar tensor theory introduces a scalar field φ, in addition to the metric tensor field g_{ij}. This scalar field has the dimension of the inverse of gravitational constant and interacts equally with all forms of matter. Subsequently Saez and Ballester [1986] proposed a new scalar tensor theory of gravitation. In this theory the metric is coupled with a dimensionless scalar field in a simple manner. This coupling gives a satisfactory description of the weak fields and an antigravity regime appears in this theory. This theory also suggests a possible way to solve the 'missing matter problem' in non – flat FRW cosmologies.

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Holographic dark energy is another alternative to the solution of dark energy problem. This is based on the holographic principle. According to this principle the entropy of a system scales not with its volume, but its surface area (Susskind [1995]; Bousso [1999]). A cosmological version of this principle was proposed by Fischer and Susskind [1998] and Cohen et al. [1999]. Thus if we take the whole Universe into account, then the vacuum energy related to the holographic principle may be viewed as dark energy, usually called holographic dark energy. Hence several authors, in recent years, have investigated holographic dark energy models in Bianchi space-times. This is because of the fact that observational data suggests that the anomalies found in the cosmic microwave back ground (CMB) simulated increasing interest in anisotropic Bianchi type models. Also, these models will certainly help for a better understanding of the early stages of evolution of the Universe.

Many authors have investigated several aspects of the cosmological models in Saez-Ballester theory, some of them are Rao et al. [2007, 2008], Naidu et al. [2012a, b] have discussed various aspects of Bianchi space times in Saez-Ballester [1986] scalar-tensor theory. Rao et al. [2013] have studied Bianchi type-II, VIII & IX perfect fluid dark energy cosmological model in Saez-Ballester and general theory of gravitation. Rao and Neelima [2013], Rao et al. [2014, 2015], Rao and Jyasudha [2015] have studied higher dimensional cosmological models in this theory., Mishra and Sahoo [2014] and Mete et al.[2016] have studied some Bianchi cosmological models in various theories.

Akarsu and Kilinc [2010, 2012], Yadav et al. [2011], Pradhan et al. [2012], Singh and Chaubey [2008], Saha and Yadav [2012], Adhav et al. [2011], Sarkar [2013a,b and 2014a,b] investigated holographic dark energy models in Bianchi space-times with linearly varying deceleration parameter in general relativity. Santhi et al. [2016a] have studied Some Bianchi type generalized ghost pilgrim dark energy models in general relativity. Abdul Jawad [2014] have discussed Analysis of generalized ghost pilgrim dark energy in non-flat FRW Universe. Santhi et al. [2016b] have investigated Anisotropic Generalized Ghost Pilgrim Dark Energy Model in General Relativity.

The above discussion and the investigations have motivated us to consider the generalized ghost pilgrim dark energy model in LRS Bianchi type- I space – time in the frame work of Saez – Ballester scalar tensor theory of gravitation. The plan of this paper is as follows: In section 2 we have derived Saez – Ballester field equations with the help of LRS Bianchi type- I metric in the presence of pressure less matter and generalized ghost pilgrim dark energy. Section 3 is devoted to the cosmological solution of the field equations using linearly varying deceleration parameter proposed by Akarsu and Dereli [2012]. Physical and kinematical parameters of the model are computed and discussed in section 4. The last section contains some conclusions.

2. Metric and field equations: We consider the spatially homogeneous anisotropic LRS Bianchi type-*I* space time metric of the form

$$ds^2 = dt^2 - A^2 dx^2 - B^2\left(dy^2 + dz^2\right) \tag{1}$$

where *A,B* are functions of cosmic time *t* only.

The field equations given by Saez and Ballester [1986] for the combined scalar and tensor fields are

$$R_{ij} - \frac{1}{2}Rg_{ij} - w\varphi^n\left(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}\right) = -\left(T_{ij} + \bar{T}_{ij}\right) \tag{2}$$

And the scalar field ϕ satisfies the equations

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$$2\varphi^n\varphi_{;i}^{;i} + n\varphi^{n-1}\varphi_{,k}\varphi^{,k} = 0 \tag{3}$$

Also, we have the conservation equation

$$\left(T^{ij} + \bar{T}^{ij}\right)_{;j} = 0 \tag{4}$$

where R_{ij} is the Ricci tensor, *R* is the Ricci scalar, ω and *n* are arbitrary dimensionless constants and $8\pi G = C = 1$ in the relativistic units. Also the energy momentum tensors for matter T_{ij} and for the holographic dark energy $\bar{T}_{ij}$ are defined as

$$T_{ij} = \rho_m u_i u_j$$
$$\bar{T}_{ij} = (\rho_\Lambda + p_\Lambda)u_i u_j - p_\Lambda g_{ij} \tag{5}$$

where ρ_m, ρ_Λ are the energy densities of matter and the holographic dark energy and p_Λ is the pressure of the holographic dark energy.

Now using commoving coordinates and equation (5), Saez- Ballester field equations (2) – (3) for the metric (1) take the following form

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{w}{2}\varphi^n\dot{\varphi}^2 = -\omega_\Lambda\rho_\Lambda \tag{6}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{w}{2}\varphi^n\dot{\varphi}^2 = -\omega_\Lambda\rho_\Lambda \tag{7}$$

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{w}{2}\varphi^n\dot{\varphi}^2 = \rho_m + \rho_\Lambda \tag{8}$$

$$\ddot{\varphi} + \dot{\varphi}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) + \frac{n}{2}\frac{\dot{\varphi}^2}{\varphi} = 0 \tag{9}$$

Here and in what follows and overhead dot indicates differentiation with respect to *t*. Where $\omega_\Lambda = \frac{p_\Lambda}{\rho_\Lambda}$ is equation of state (EoS) parameter of dark energy.

We can write the conservation equation of the matter and dark energy as

$$\dot{\rho}_m + \dot{\rho}_\Lambda + \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\left(\rho_m + (1+\omega_\Lambda)\rho_\Lambda\right) = 0 \tag{10}$$

3. Solutions of field equations and the cosmological model:

Now equations (6)-(9) are a system of four independent equations in six unknowns $A, B, \omega_\Lambda, \rho_\Lambda, \rho_m$ and φ. Hence we assume the following physically significant conditions to solve the above highly non-linear differential equations:

(i) The shear scalar σ^2 is proportional to scalar expansion θ so that we can take (Collins et al. 1983)

$$A = B^\mu \tag{11}$$

where $\mu \neq 1$ is a positive constant and presence the anisotropy of the space time.

(ii) we use the energy density of GGPDE defined by (Sharif and Jawad 2014)

$$\rho_\Lambda = M_{pl}^2\left(\alpha_1 H + \alpha_2 H^2\right)^\beta \tag{12}$$

where $M_{pl}^2 = 1$ is the reduced plank mass, *H* is the Hubble parameter.

(iii) A generalized linearly varying deceleration parameter (Akarsu and Dereli [2012])

$$q = \frac{-a\ddot{a}}{\dot{a}^2} = -kt + l - 1 \tag{13}$$

where $k \geq 0$ and $l \geq 0$ are constants and $k = 0$ reduces to the law of Berman [1983] which yields models with constant deceleration parameter. Here we are concerned with linearly varying deceleration parameter given by equation (13) when $k > 0$ and $l > 0$ in LRS Bianchi type-I space-time in the framework of Saez-Ballester scalar-tensor theory.

In this particular case, we have

$$a(t) = a_0 e^{\frac{2}{\sqrt{l^2-2c_1k}}\arctan h\left(\frac{kt-l}{\sqrt{l^2-2c_1k}}\right)} \quad \text{for } k>0 \text{ and } l \geq 0 \tag{14}$$

where $a_{0,}\, c_1$ are constants of integration. For convenience, in the following we consider the solution for $k > 0, l > 0$ and choose the integration constant $c_1 = 0$. The reason for considering the solution only for $k > 0, l > 0$ is not only for simplicity but also for compatibility with the observed Universe. The condition $k > 0$ means that we are dealing with increasing acceleration $\left(\dot{q} = -k < 0\right)$. With this, the equation (14) reduces to

$$a = e^{\frac{2}{l} arc\tanh\left(\frac{kt}{l}-1\right)} \tag{15}$$

Now using equations (11) and (15), we obtain the expressions for metric potentials (choosing $a_0 = 1$) as

$$A = e^{\frac{6\mu}{l(\mu+2)} arc\tanh\left(\frac{kt}{l}-1\right)}$$

$$B = e^{\frac{6}{l(\mu+2)} arc\tanh\left(\frac{kt}{l}-1\right)} \tag{16}$$

Also from equations (9) and (16) the scalar field in the model is given by

$$\varphi^{\frac{n+2}{2}} = \frac{kt\varphi_0(n+2)(kt-2l)}{12l} e^{\frac{-6}{l}\arctan h\left(\frac{kt}{l}-1\right)} \tag{17}$$

Where φ_0 is constant of integration.

Now the metric (1) with the help of equation (16) can be written as

$$ds^2 = dt^2 - e^{\frac{12\mu}{l(\mu+2)} arc\tanh\left(\frac{kt}{l}-1\right)} dx^2 - e^{\frac{12}{l(\mu+2)} arc\tanh\left(\frac{kt}{l}-1\right)} (dy^2 + dz^2) \tag{18}$$

Equation (18) along with Equation (17) represents LRS Bianchi type-I GGPDE model in Saez-Ballester theory.

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Now from equations (12) , we have the energy density of GGPDE as

$$\rho_{\Lambda} = \left[\frac{2}{t(2l-kt)}\left(\alpha_1 + \frac{2\alpha_2}{t(2l-kt)}\right)\right]^{\beta} \tag{19}$$

From the equations (8), (16), (17) and (19), we have
the energy density of matter as

$$\rho_m = (2\mu+1)\left[\frac{6}{t(2l-kt)(\mu+2)}\right]^2 + \frac{w\varphi_0^{\,2}}{2} e^{\frac{-6}{l}arc\tanh\left(\frac{kt}{l}-1\right)} - \left[\frac{2}{t(2l-kt)}\left(\alpha_1 + \frac{2\alpha_2}{t(2l-kt)}\right)\right]^{\beta} \tag{20}$$

the equation of state (EoS) parameter of GGPDE as

$$\omega_{\Lambda} = \frac{1}{6}\left[3w\varphi_0^{\,2} e^{\frac{-6}{l}arc\tan\left(\frac{kt}{l}-1\right)} - \left(\frac{6}{t(2l-kt)(\mu+2)}\right)^2\left((kt-l)(\mu^2+5\mu+6)+3(\mu^2+1)\right)\right]$$

$$X\left[\frac{2}{t(2l-kt)}\left(\alpha_1 + \frac{2\alpha_2}{t(2l-kt)}\right)\right]^{-\beta} \tag{21}$$

The overall density parameter Ω is given by

$$\Omega = \frac{\rho_{\Lambda} + \rho_m}{3H^2} = \frac{2l+1}{12k^2}\left[\left(\frac{6}{(\mu+2)}\right)^2 + \frac{w\varphi_0^{\,2}}{2} e^{\frac{-6}{n}arc\tan\left(\frac{kt}{l}-1\right)}\left(t\left(2l-kt\right)\right)^2\right] \tag{22}$$

4. Physical Discussions:

The physical and kinematical parameters of the model are the following:
Spatial volume is

$$V = AB^2 = a^3(t) = e^{\frac{6}{l}arc\tanh\left(\frac{kt}{l}-1\right)} \tag{23}$$

The average Hubble's parameter is

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right) = \frac{2}{t(2l-kt)} \tag{24}$$

The scalar expansion is

$$\theta = 3H = \frac{6}{t(2l-kt)} \tag{25}$$

The shear scalar is

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{1}{3}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)^2 = 12\left(\frac{(\mu-1)}{t(2l-kt)(\mu+2)}\right)^2 \tag{26}$$

The average anisotropy parameter is

$$A_h = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 = \frac{2(\mu-1)^2}{(\mu+2)^2} \tag{27}$$

where $H_i (i = 1,2,3)$ represent the directional Hubble parameters given by

$$H_1 = \frac{\dot{A}}{A}, \quad H_2 = \frac{\dot{B}}{B}, \quad H_3 = \frac{\dot{B}}{B} \tag{28}$$

The jerk parameter

$$j = q + q^2 - \frac{\dot{q}}{H} = \frac{1}{2}\left(l^2 - 1\right) \tag{29}$$

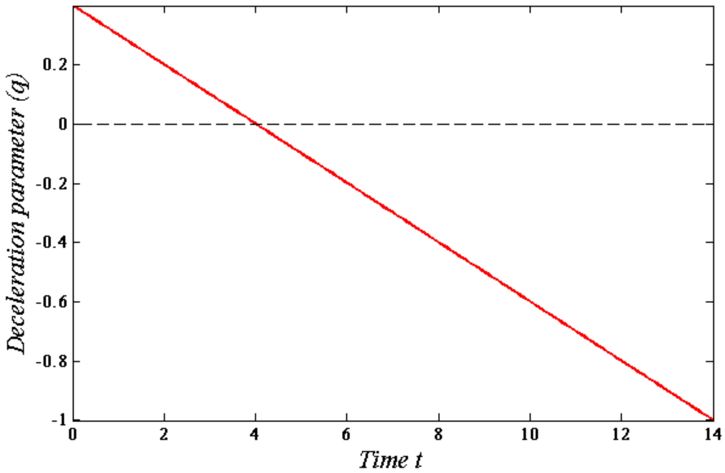

Fig.1: plot of deceleration parameter (q) versus time (t) for *k=0.1* and *l=1.4.*

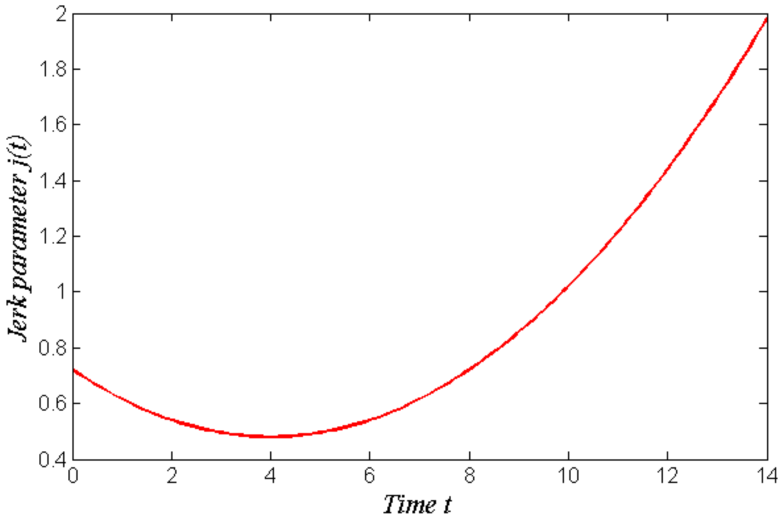

Fig.2: plot of Jerk parameter (j) versus time (t) for *k=0.1* and *l=1.4.*

The Fig. 1 shows the early deceleration and the present acceleration phase of the Universe. Observational data (Ade et al. 2013) shows that the present value of deceleration parameter lies somewhere in the range −1 <q< 0. Figure 1shows that the Universe enters into the acceleration expansion phase at t ≈ 5.15 and the present (t = 13.798) value of the deceleration parameter is q =

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−0.8384. Therefore these values are consistent with the observational results.

The Fig.2. demonstrates, the jerk parameter initially decreases with the growth of cosmic time up to t = 4.2 and then increases with time. It is clear from the figure that the jerk parameter remains in the positive domain throughout the entire history of the Universe. For low values as well as large values of cosmic time, the jerk parameter assumes larger value compared to that of Λ CDM model. However, it is interesting to note from the figure that, the present model becomes closer to the CDM model twice in the cosmic evolution i.e. at t ≈ 2.23 and at t ≈ 10.14.

5. Conclusions

This paper is devoted to study the exact solution of Saez-Ballester's[1986] field equations in LRS Bianchi type –I Universe filled with dark matter and anisotropic generalized ghost pilgrim dark energy. By considering the shear scalar σ^2 is proportional to scalar expansion θ and the linearly varying deceleration parameter (Akarsu and Dereli [2012]) an exact solution is found. We have computed physical and kinematical parameters of the Universe and discussed their cosmological significance. We have observed that there is a smooth transition of the model from deceleration to the accelerated phase which agrees with the present scenario of modern cosmology. Spatial volume of the Universe increases with time showing the spatial expansion and it is zero at *t=0* giving a point type singularity at *t=0*. It is also observed that the parameters H, $\theta, \sigma^2, \rho_\Lambda, \rho_m$ and φ all diverge at t=o and approach zero as $t \to \infty$ which shows inflationary scenario of the Universe. The EoS parameter mostly lies in the quintessence region throughout the evolution of the Universe which is in good agreement with the present day scenario of modern cosmology. Since the deceleration parameter lies in the range $-1 \leq q<0$ and since the jerk parameter is positive there occurs a smooth transition of the Universe from decelerated phase to accelerated phase. Thus our model represents the accelerated expansion of the Universe and is consistent with current observational data.

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RECENT TRENDS IN PRESENT AND FUTURE MATHEMATICS

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ABSTRACT
This paper deals with the study of recent trends in present and future mathematics and in the multidiscipline, In this paper we focus on the applications of mathematics at various levels such as mathematics in economics, mathematics in population studies, mathematics in budget preparation, mathematics in five year plans, mathematics in space technology, mathematics in bio science, mathematics in medicine and in nanotechnology. There are so many special applications to the needs of today's and tomorrow's students. First of all knowing that a successful education is one which is conducive to successful life and absolutely we know that mathematical education can and should lead to a successful life and it is a significant component of such an education. Also I would like to say mathematics is the key element in the science and technology and it plays a vital role in understanding the control and development of resources of the world around us.
Keywords: Mathematics in Economics, Population Studies, Budget Preparation, Fiver year plan, Space Technology, Bio Science, Medicine and Nanotechnology.

Introduction

Areas of Mathematics: A traditional division of mathematics is into pure mathematics and applied mathematics. Pure mathematics is studied for its intrinsic interest and applied mathematics is studied to resolve real world problems. But the division is not very clear.

Later, these divisions have come to be known as applied maths and mathematical theories which can't be applied. There has been a gradual growth in the scope of application of maths in several disciplines. Thus, the traditional division has got a new meaning. The Mathematics subject classification has been assigned with MSC subject codes and MSC divides mathematics into over 60 areas and every area is subdivided. Of late we come across divisions like RECREATIONAL mathematics, History and Biography of mathematics, mathematical logic and foundation including set theory, Proof theory, Model theory, and constructive mathematics, the list extends further viz – arithmetic, algebra, order theory, number theory, field theory and polynomials, competitive rings and combinatories.

Subdivisions in geometry surprise scholars in general and mathematicians in particular.

Sub-divisions in geometry

- Converse geometry and discrete geometry.
- Differential geometry.
- Algebraic geometry.

Division in topology

- General topology.
- Algebraic topology.

Now a close and intensive study of application of maths in several disciplines.

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Mathematical economics

Economics and mathematics are complementary disciplines. Indeed, much of current economic theory is expressed in terms of mathematic models. Most branches of economics use mathematical, statistical and computational concepts extensively. Any advances in mathematics have been motivated by problems from economics. Faculty of mathematic and the department of economics are closely related in the faculty of arts.

Students with a strong interest in economics and economic theory will find "The mathematical economics programme", very rewarding.

Mathematical medicine and Biology

The application of mathematics to medicine is an exciting and novel area of research with the discipline of applied mathematics. The creation of mathematical models of various disease processers, taking into account the interactions between the different components of the complex bio-logical system is one aspect of the research. This research has two important goals:

- Increasing our understanding of the onset of the disease and its progress.
- This mathematical analysis provides treatment strategies that are optimal.

  This research in mathematical medicine is carried out in collaboration with physicians working in "Princess Margret Hospital" in Toronto. Their research includes the following topics.
- Mathematical modeling of ovarian cancer treatment and sequencing of surgery and chemotherapy.
- Effects of surgery radiotherapy and chemotherapy on the treatment of brain tumors.
- Mathematical modeling of brain tumors. The research is enhanced by the involvement of the faculty in the "Centre for mathematical medicine".

Current research in mathematical biology focuses on understanding biological organisms at the cellular level. The research pertains to networks of Genes and proteins interacting through bio-chemical reactions.

Replication and metabolism that under lie life are elucidated system biology (or) molecular biology is studied applying ideas from systems and central theory. A study of the behaviour and function of neurons, the building blocks of the control nervous system has gained prominence. The way how the brain co-ordinates the overall activity of an organism is the latest subject of study.

Pharmacy

Maths skills for pharmacy

- Unlocking pharmaceutical calculations in the manufacturing of medicines. An integrated approach with regard to calculations relating to both pharmaceutical science and pharmacy practice.

Mathematics and architecture

Mathematics and architecture are related as architects use it for several reasons.

Architects use geometry to define the spatial forms of a building from the Pythagoreans of the sixth century onwards. In ancient Egypt, ancient Greece, India and the Islamic world buildings including pyramids, temples, mosques, palaces and mausoleums were built with special proportions for the religious reasons. In Islamic architecture geometric shapes and patterns are used to decorate buildings both outside and inside. In renaissance architecture both symmetry and proportion were applied by architects. At the end of the 20thcentury, Vladimir Shukhov, AntoniGaudi pioneered hyperboioid structures. In the 20th century modern arctecture and Deconstructivism explored

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different geometrics to achieve intended effects. In addition to the bebrequisite use of Maths in engineering of buildings, Aretects use mathematics for several other reasons. From the time Pythogoreaus, arctects in ancient Greece, Rome, Islamic world and Renaissance have chosen the proportion of the built environment.

Hyperboloid structures were used starting towards the end of the 20^{th} century. The scope the structures aesthetically interesting and strong. The early 20^{th} century movement Modern architecture pioneered by Russian constructivism used Euclidian geometry. In the Destill movement the horizontal and vertical were supposed as constructing the universe. Contemporary architecture is extremely diverse according to 90 leading architecture who responded to a 2010 world Architecture survey. The late 20^{th} century architecture Witnessed deliberate disorder which is called random form of length complexity.

Vastu sastra is the ancient canons of Indian architecture . Examples of this type of architecture are plan of Meenakshi Ammal temple at Madhurai, Vimalaksha temple in Hampi, Khandaraya Mahadeva temple at Khujaraho. The Tajmahal exemplified the Mughal architecture.

Towards the end of the 20^{th} century functional Geometry and a periodic@ were seized upon by the modern architects to provide interesting and attractive coverings for buildings. Thus it is often observed and discovered the Mathematical application in architecture from time to time and concluded that architecture as an art of building structure would have been poorer and less evolved without its application.

Mathematics and Physical sciences

An inspirable relationship between these two branches is well known as Physical sciences cannot be studied as independent or isolated subjects without applying Mathematics principles and theories. A close look of Mechanics, Mechanics of architecture, Mechanics of Deformable solids, Fluid Mechanics, Aerodynamics or Particle Mechanics makes it certain that to name Maths and Physical sciences king and queen of sciences is no exaggeration.

Mechanics addresses what happens when a real physical object is subjected to forces. This divides the study into the study of rigid solids, deformable solids and fluids.

Mechanics of Structure

It is a field of study within applied mechanics that investigates the structures under Mechanical loads such as bending of a beam, building of columns, tension of smart and deflection of a thin small etc..

Mechanics of deformable Solids

Most real life objects are not rigid. Objects change shape when subjected to force, this subject deals with notions of stress, strain and elasticity.

Fluid Mechanics

This concept includes not liquids but flowing gases but also solids under certain situations it includes notion such as viscosity, turbulent flow, laminar flow. Aerodynamics and particle mechanics and parts of physical sciences are included in Mathematics Applications.

Mathematics and Statistics

Though Mathematics and Statistics as studied in colleges and universities as separate disciplines, it is obvious that Statistics is an off-shoot of Mathematics. It is of immense help in various fields of study. It's expansion and growth as an independent discipline helps researchers arrive at solutions, otherwise impossible.

Mathematics and Computer Science

There has been an abnormal growth in the usefulness and application of Computers in various fields. This has led to the misconception that computer programmers began to think that computer skills do no need the help of Mathematics and Mathematicians think that 'Computational skills become defunct without Mathematics. But truth lies elsewhere.

Many problems in Mathematics cannot in general be solved exactly. Numerical Analysis in the study of 'Iteration methods and Algorithms for approximately for solving problems to a specified error bound. It includes numerical differentiation, numerical integration and numerical methods i.e. scientific computing.

Computer Algebra

This is also called symbolic computation or Algebraic computation. It deals with exact computation for example with integers of arbitrary size, Polynomials and elements of finite fields. Towards the end the study, we come to know about commonplace, universally practiced governmental exercises like1. Welfare schemes,2. Annual, biennial and five year plans, 3. Preparation and submission of budgets. 'Welfare States' all over the world take up welfare measures under 'Directive Principles'. These measures though do not need intricate application of Mathematics, cannot arrive 'specific figures without the knowledge of Mathematics. In the same preparation of budget or budget making, preparation of Annual, biennial and five year plans for achieving goals by elected governments, needs help from Mathematical calculations. It is a stupendous exercise.

Mathematics teaches patience, discipline and step by step problem solving skills. For those with substantial background in Mathematics unlimited career opportunities are available. According to Jobs rated Almanac, one of the Almanac books, published in Newyork, careers that require a very strong background in Mathematics were listed as the best jobs.

They were

1. Software Engineer
2. Actuary
3. Computer System Analyst
4. Computer Programmer
5. Mathematicians.

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BIOMETRIC DATA CLUSTERING USING MINIMUM SPANNING TREE

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ABSTRACT
Clustering is a standout amongst the most regularly utilized techniques for finding concealed structure in microarray quality expression information. Most present techniques for clustering tests depend on separation measurements using all qualities. This has the impact of clouding Clustering in tests that might be obvious just when taking a gander at a subset of qualities, since clamor from unessential qualities overwhelms the flag from the important qualities out yonder count. We depict a calculation for consequently recognizing clusters of tests that are discernable just in a subset of qualities. We utilize cycle between Minimal Spanning Tree based Clustering and highlight determination to evacuate commotion qualities in a stage insightful way while all the while honing the clustering. Assessment of this calculation on manufactured information demonstrates that it settle planted clusters with high exactness despite clamor and the nearness of different groups. It likewise demonstrates a low likelihood of distinguishing spurious groups. Testing the calculation on some notable smaller scale exhibit information sets uncovers referred to organic classes and in addition novel clusters. The iterative Clustering strategy offers impressive change over clustering in all qualities. This technique can be utilized to find allotments and their natural essentialness can be dictated by contrasting and clinical associates and quality explanations.

INTRODUCTION

Clustering is a standout amongst the most widely recognized strategies for finding shrouded structure in smaller scale cluster quality expression information. Clustering of tests has been utilized to find new ailment scientific classifications [1, 2, 3]. Bunch investigation is frequently performed with progressive [4], K-implies [5] or Self-Organizing Map [6] calculations, utilizing the whole arrangement of qualities as the reason for computing pair-wise separations between tests. This gives rise to weights to the outflow of all qualities and might be powerful in situations where there is an expansive contrast between subsets of tests (e.g. contrasting examples of ordinary and destructive tissues). Numerous infections, however, are portrayed by little quantities of qualities that separate between various ailment states. Giving equivalent weight to significant and unessential qualities will cloud this distinction. Figure 1 demonstrates a case, where clustering in all qualities covers the organic contrasts between tests with BRCA1 and BRCA2 change (information from Hedenfalk et al [7])

In this article we propose an iterative calculation, where we at first do a Clustering utilizing every one of the qualities. This clustering (which gives a double parcel of the examples) is utilized to choose qualities that separate between the two groups. The Clustering is done once more, however this time, just in the arrangement of qualities that was chosen in the past emphasis. This rotation amongst clustering and highlight determination proceeds until there is no adjustment in the

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arrangement of qualities (and segment) between two cycles. The last quality set is expelled, and the procedure rehashed on the rest of the qualities to discover different parcels. The calculation produces an arrangement of double parcels, alongside comparing sets of qualities which separate the groups display in these segments.

Comparable methodologies have been utilized as a part of different calculations. Ben-Dor et al [8] utilize mimicked tempering to effectively seek the space of all double specimen segments. Xing and Karp [9] utilize a Normalized Cut calculation to limit the inquiry to just the promising segments and utilize a comparable technique for cycle amongst Clustering and highlight determination. Von Heydebreck et al [10] and Tang et al [11] show calculations that select specimen segments and relating quality sets by characterizing a measure of segment quality and after that utilizing avaricious hunt (in the previous) and reproduced tempering (in the last mentioned) to boost this measure. Emphasis between group investigation and quality determination is likewise utilized as a part of the "quality shaving" calculation of Hastie et al [12]; however their objective was clustering of qualities as opposed to tests.

Calculation

We utilize a Minimal Spanning Tree (MST) based calculation [13, 14] for Clustering alongside the Fukuyama-Sugeno clustering measure. Quality choice is done on the premise of the two-example t-measurement with pooled difference. In the following three subsections we will look in detail at the Clustering and highlight determination viewpoints before showing the formal calculation.

Insignificant traversing trees

Let V = {x1, x2..., x N } be an arrangement of focuses with separations d ij = d(x i ,x j) characterized between all x i and x j. A tree on V is a chart without any circles whose vertices are components of V and edge lengths are d ij . A negligible crossing tree (MST) is a tree that associates all focuses with the end goal that the whole of the length of the edges is a base. A MST can be productively processed in O(N2) time (counting separation computations) utilizing either Prim's [13] or Kruskal's [14] calculation.

Cancellation of any edge from a MST brings about two disengaged trees. Expecting the length of the erased edge to be δ and indicating the arrangements of hubs in the two trees as V1 and V2, we have the property that there are no sets of focuses (x1,x2), x1 ∈ V1, x2∈ V2such that d(x i ,x j) <δ. Characterize the littlest separation between any two focuses, one in V1 and the other in V2, as the separation between V1 and V2. At that point we have the outcome that the division is in any event δ.

The criticalness of this outcome is that by erasing an edge of length δ we are guaranteed of a segment where the two clusters have a partition of at any rate δ. This implies on the off chance that we are keen on taking a gander at all paired parcels with extensive divisions between the clusters, it is adequate to take a gander at segments got by erasing edges of the MST. Rather than taking a gander at all conceivable double parcels (which number 2N-1-1) our calculation takes a gander at segments acquired by erasing single edges from the MST (which number N-1).

Insignificant Spanning Trees were at first proposed for clustering by Zahn [15]. All the more as of late, Xu et al have utilized MST for Clustering quality expression information [16].

Clustering measure

To compare the partitions obtained by deleting different edges of the MST, we use the Fukuyama-Sugeno clustering measure. Given a partition S_1, S_2 of the sample index set S, with

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each S_k containing N_k samples, denote by μ_k the mean of the samples in S_k and μ the global mean of all samples. Also denote by x_j^k the *j*-th sample in cluster S_k. Then the Fukuyama-Sugeno (F-S) clustering measure is defined as

$$FS(S) = \sum_{k=1}^{2} \sum_{j=1}^{N_k} \left[\left\| x_j^k - \mu_k \right\|^2 - \left\| \mu_k - \mu \right\|^2 \right]$$

Little estimations of FS(S) are characteristic of tight groups with a substantial detachment between clusters.

We have considered different other clustering measures. The perfect grouping measure ought to show neighborhood minima at each feasible parcel and have great execution even with countless components. We have found the Fukuyama-Sugeno (F-S) measure to give the best execution in these two regards.

2. Feature selection

For a given parcel with two clusters, we can inquire as to whether a specific quality shows adequate differential expression between tests having a place with the distinctive groups. A quality which is diversely communicated in tests having a place with various clusters can be said to be important to the parcel or to bolster the segment. There can be numerous methods for measuring a quality's support for a parcel. Here we utilize the two example t-measurement with pooled fluctuation. The t-measurement is processed for every quality to analyze the mean expression level in the two groups. Qualities with outright t-measurement more noteworthy than a limit T sift are chosen. The percentile edge parameter P sift ∈(0,100) is utilized to process T sift . T sift is the P sift/2-th percentile of an irregular variable dispersed by t-circulation with mean zero and N-2 degrees of opportunity (N is the quantity of tests). Here we utilize the t-measurement as a heuristic measure of the commitment of every quality to the chose parcel; no factual noteworthiness is suggested.

The condition for choice of a quality gets to be distinctly stricter with every emphasis. In the main emphasis we pick qualities with supreme t-measurement more noteworthy than T sift/2. This cutoff increments straightly with the quantity of cycles until it achieves T sift . This is done as such that we don't lose any valuable qualities by putting an as well stringent choice foundation before the parcel has developed near its last shape.

2. The algorithm

Initially, an MST is created using all the genes; then each binary partition obtained by deleting an edge from the tree is considered as a putative partition. The partition with the minimum value of the F-S clustering measure is selected. The t-statistic is used to select a subset of genes that discriminate between the clusters in this partition. In the next iteration, clustering is done in this set of selected genes. This process continues until the selected gene subset converges (remains the same between two iterations), resulting in a set of genes and the final partition. Having identified a partition and the associated set of genes, these selected genes are removed from the pool of genes. This prevents the algorithm from detecting the same partition the next time. The whole process repeats in the pool of remaining genes to find other partitions.

The inputs to the algorithm are the gene expression matrix $\{x_{s,g}\}$, the maximum number of partitions to be found $MaxN_p$ and percentile threshold P_{thresh}. P_{thresh} is used to compute T_{thresh}. The outer loop of the algorithm runs as long as the number of discovered partitions is less than $MaxN_p$. The set of selected genes F is initialized to be the set of all genes $Fset$ and the cutoff t is initialized as $T_{thresh}/2$. In the inner loop, an MST is created using the genes in F, and for all partitions obtained by deleting single edges from this MST, the F-S measure is calculated. For the partition P^* with the lowest F-S measure, genes are selected from F based on the t-statistic. These selected genes form the new gene set F_{new}. If $F_{new} \neq F$, the cutoff t is increased and another iteration of the inner loop is performed. If $F_{new} = F$, this means that the gene set has remained unchanged between two iterations and the current partition P^* along with the current gene set F is output. The number of discovered partitions is increased and another iteration of the outer loop is performed.

Since this is an unsupervised method, the partitions picked might be indicative of biological differences that are relevant, irrelevant (like age or sex of patients) or unknown. We control the detection of chance partitions (*i.e.* generated due to noise and not due to any biological difference) by requiring a minimum of $2M(1 - P_{thresh}/100)$ genes in support of a partition (M is the total number of genes); the algorithm is terminated if there are fewer.

P_{thresh} plays an important part in the kind of partitions that are extracted. A value of P_{thresh} close to 100 will preferentially extract partitions that are supported by genes with large differential expression between the two clusters. A smaller value of P_{thresh} will pick up partitions that are supported by larger number of genes with lower differential expression between the clusters.

P_{thresh} cannot be interpreted as a measure of the statistical significance of the partitioning since we are doing both the partitioning and the feature selection on the same set of samples. Here we only use P_{thresh} as a parameter for selecting genes.

Algorithm 1: Algorithm for iterative clustering
Input $MaxN_p$, P_{thresh}, $x_{s,g}$;
$Fset \leftarrow \{1, 2..., n\}$;
$N_p \leftarrow 0$; /*Number of currently discovered partitions*/
Compute T_{thresh};
While $N_p < MaxN_p$ **do**
$F \leftarrow Fset$;
$T \leftarrow T_{thresh}/2$;
While 1 do
If length of $F < 2M(1 - P_{thresh}/100)$ **then**
/*Not enough genes support partitions*/
exit;
end
Create MST in feature set F with metric d;
Delete edges one at a time and calculate F-S measure for each ensuring binary partition;
Find partition P^* with the lowest F-S measure;
Compute t-statistic t_g for all genes $g \in F$ for this partition;
Set F_{new} to the set of genes $\{g: |t_g| > t\}$;
If $F_{new} = F$ **AND** $t = T_{thresh}$ **then**
/*Feature set has converged */
output P^* and F;
/*Remove genes in F from $Fset$*/
$Fset \leftarrow Fset \setminus F$;
$N_p = N_p + 1$;
break;

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else

$F \leftarrow F_{new}$;

Increase t;

end

end

3. Results and discussion

Synthetic data

We first tested the algorithm on synthetic data to compare its performance against a hierarchical clustering method at detecting planted partitions. We also estimated the probability of detection of spurious partitions created by noise (*i.e.* the false detection rate).

For both iterative clustering and hierarchical clustering, we found that the probability of detecting the true partition depended only on the Euclidean distance between the clusters in the partition, and for a fixed distance, is relatively insensitive to the number of signal genes .

Figure 1 shows the results of a logistic regression analysis of the dependence of probability of detection of the true partition on the distance between the clusters for both clustering methods. Independent of the total number of genes *N*, iterative clustering detects the planted partition when the two clusters are separated by about half the distance compared to hierarchical clustering. For genes with similar levels of differential expression, this means that the iterative clustering method will detect clusters supported by a quarter of the number of genes required for detection by hierarchical clustering.

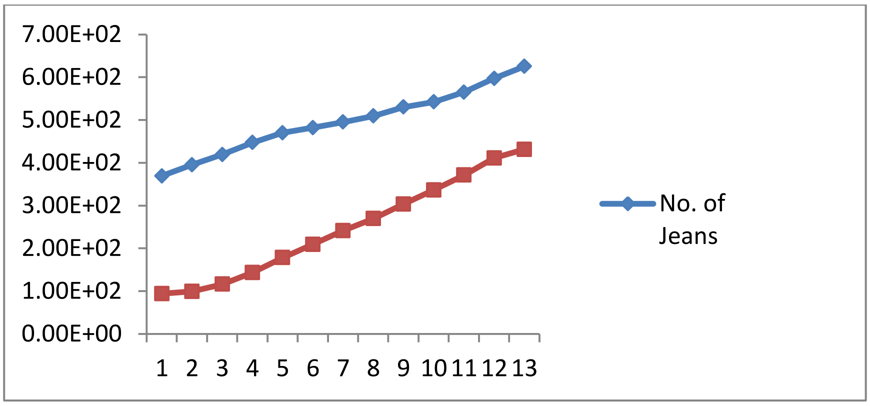

Fig-1: Relation between no. of jeans and distance between clusters.

Conclusion

We have demonstrated that our calculation works a great deal more dependably than the basic clustering calculation. Our approach consequently decides the fancied number of clusters. The target capacity is characterized to expand the general standard deviation lessening. Our calculation does not require the clients to choose and attempt different parameter blends keeping in mind the end goal to get the craved yield. Later on, we will investigate and test our proposed clustering calculations in different spaces. We will additionally concentrate the rich properties of the EMST-based grouping strategies in tackling diverse clustering issues.

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A REVIEW ON FUZZY LOGIC: THEORY AND APPLICATION

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ABSTRACT
Fuzzy logic has quickly gotten to be a standout amongst the best of today's advancements for creating complex control frameworks. The explanation behind which is exceptionally basic. Fuzzy logic addresses such applications superbly as it takes after human basic leadership with a capacity to produce exact arrangements from certain or rough data. It fills a vital crevice in building plan strategies left empty by absolutely scientific methodologies (e.g. direct control plan), and absolutely rationale based methodologies (e.g. master frameworks) in framework outline. People make descisions in view of rules. Despite the fact that, we may not know about it, all the decisions we make are altogether in view of PC like if-then articulations. In any case, the decision and the method for picking that decision are supplanted by Fuzzy logic sets and the principles are supplanted by fuzzy sets. Fuzzy sets additionally work utilizing a progression of if-then proclamations. For example, if X then An, if y then b, where An and B are all arrangements of X and Y. In this paper we have talked about the distinctive approach of Fuzzy logic in various field of science and Technology.
Key words, Fuzzy logic, Fuzzy sets, decision variables.

1. Introduction

Fuzzy logic has quickly turned out to be a standout amongst the best of today's advancements for creating modern control frameworks. The purpose behind which is exceptionally straightforward. Fuzzy logic addresses such applications impeccably as it looks like human basic leadership with a capacity to produce exact arrangements from certain data. It fills an essential hole in building plan techniques left empty by simply scientific methodologies (e.g. direct control outline), and absolutely rationale based methodologies (e.g. master frameworks) in framework outline [1].

Before delineating the systems which make Fuzzy logic machines work, it is imperative to acknowledge what Fuzzy logic really is. Fuzzy logic is a superset of conventional(Boolean) rationale that has been reached out to handle the idea of fractional truth-truth values between "totally genuine" and "totally false". As its name recommends, it is the rationale fundamental methods of thinking which are inexact as opposed to correct. The significance of Fuzzy logic gets from the way that most methods of human thinking and particularly judgment skills thinking are rough in nature [2].

The fundamental attributes of Fuzzy logic as established by Zader Lotfi are as per the following.

- In Fuzzy logic, correct thinking is seen as a constraining instance of surmised thinking.
- In Fuzzy logic everything involves degree.
- Any sensible framework can be fuzzified
- In Fuzzy logic, information is translated as a gathering of flexible or, comparably, Fuzzy logic limitation on an accumulation of factors
- Inference is seen as a procedure of spread of flexible requirements.

The third articulation thus, characterizes Boolean rationale as a subset of Fuzzy rationale.

1.1 Literature Survey

Fuzzy logic depends on the focal thought that in Fuzzy logic sets every component in the set can expect an esteem from 0 to 1, not only 0 or 1, as in exemplary set hypothesis. Along these lines, subjective attributes and numerically scaled measures can show degrees in the degree to which they have a place with the applicable sets for assessment. This level of participation of every component is a measure of the component's "having a place" to the set, and consequently of the accuracy with which it clarifies the marvel being assessed. Fuzzy logic sets can be consolidated to deliver significant conclusions, and derivations can be made, given a predetermined Fuzzy logic info work. The article shows the use of Fuzzy logic to a wage delivering property, with a subsequent Fuzzy logic set output[16]

The paper exhibits a Fuzzy logic approach that means to control the indoor air quality to give a sheltered and agreeable environment. The proposed approach assesses the present circumstance in light of introduced sensors and thus gives the best possible action(s) to keep the indoor environment protected and agreeable for living. Contributions for the Fuzzy logic controller incorporate estimations from temperature and stickiness sensors to help comfort, and that from dangerous smells sensors to upgrade wellbeing. Controller's yields incorporate signs to the cooling unit (AC), ventilation framework, and humidifier/de-humidifier, notwithstanding visual/sound cautioning alarms if there should be an occurrence of risky circumstances. To approve the convenience of the controller, different blends of developments of warmth, moistness, and smells are researched. Comes about acquired from the reenactment concentrates on show the heartiness of the proposed controller at the distinctive info scenarios.[17]

2.Conception of Fuzzy logic : Fuzzy logic is a way to deal with processing in view of "degrees of truth" instead of the standard thing "genuine or false" (1 or 0) Boolean rationale on which the current PC is based. Many basic leadership and critical thinking assignments are too intricate to ever be characterized correctly however, individuals prevail by utilizing uncertain information Fuzzy rationale looks like human thinking in its utilization of surmised data and instability to create choices. An way to deal with vulnerability that consolidates genuine qualities [0… 1] and rationale operations

Fuzzy rationale depends on the thoughts of Fuzzy logic set hypothesis and Fuzzy logic set participation frequently found in common (e.g., talked) dialect.

Case: "Youthful

Example:

Ann is 28, 0.8 in set "Youthful"

Weave is 35, 0.1 in set "Youthful"

Charlie is 23, 1.0 in set "Youthful"

Not at all like insights and probabilities, the degree is not portraying probabilities that the thing is in the set, yet rather depicts to what degree the thing is the set. We need the esteem to switch bit by bit as Young gets to be distinctly Middle and Middle gets to be distinctly Old. This is the possibility of Fuzzy logic.

3.Fuzzy set: Fuzzy logic Set Theory was formalized by Professor Lofti Zadeh at the University of California in 1965. What Zadeh proposed is particularly an outlook change that initially picked up acknowledgment in the Far East and its fruitful application has guaranteed its reception around the globe.

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Definition: A Fuzzy logic set can be characterized numerically by allocating to every conceivable individual in the universe of talk an esteem speaking to its review of enrollment in the Fuzzy logic set. For instance: a Fuzzy logic set speaking to our idea of sunny may appoint a level of participation of 1 to an overcast front of 0%, 0.8 to an overcast front of 20%, 0.4 to an overcast front of 30%, and 0 to an overcast front of 75%.

3.1 The Crisp set v.s. the Fuzzy set

The fresh set is characterized so as to segment the people in some given universe of talk into two gatherings: individuals and nonmembers.

Be that as it may, numerous characterization ideas don't show this trademark. For instance, the arrangement of tall individuals, costly autos, or sunny days.

3.2.Fuzzy set operations

- Fuzzy union ($\cup$): the union of two fuzzy sets is the maximum (MAX) of each element from two sets.

E.g.

A = {1.0, 0.20, 0.75}

B = {0.2, 0.45, 0.50}

A $\cup$ B = {MAX(1.0, 0.2), MAX(0.20, 0.45), MAX(0.75, 0.50)} = {1.0, 0.45, 0.75}

- Fuzzy intersection ($\cap$): the intersection of two fuzzy sets is just the MIN of each element from the two sets.

E.g.A $\cap$ B = {MIN(1.0, 0.2), MIN(0.20, 0.45), MIN(0.75, 0.50)} = {0.2, 0.20, 0.50}.

- The *complement* of a fuzzy variable with DOM *x* is (1-x).
- Complement ($\_^c$): The *complement* of a fuzzy set is composed of all elements' *complement.*

Example.

A^c = {1 – 1.0, 1 – 0.2, 1 – 0.75} = {0.0, 0.8, 0.25}

3.3.Crisp Relations

Ordered pairs showing connection between two sets:

(a,b): a is related to b

(2,3) are related with the relation "<"

Relations are set themselves

< = {(1,2), (2, 3), (2, 4), ….}

3.4 Fuzzy Relations

Triples showing connection between two sets:

(a,b,#): a is related to b with degree #

Fuzzy relations are set themselves

Fuzzy relations can be expressed as matrices

Fuzzy Relations Matrices

$R_1(x, y)$	unripe	semi ripe	ripe
green	1	0.5	0
yellow	0.3	1	0.4
Red	0	0.2	1

Example: Color-Ripeness relation for tomatoes

"if x is low and y is high then z is medium"

4. Building Fuzzy logic frameworks

It consist of several steps like-

(a)Fuzzification

(b)Inference

(c)Composition

(d)Defuzzification

(a)Fuzzification: Establishes the reality base of the Fuzzy logic framework. It distinguishes the info and yield of the framework, characterizes proper IF THEN principles, and utilizations crude information to determine an enrollment work.

Consider a cooling framework that decide the best flow level by testing temperature and dampness levels. The sources of info are the present temperature and dampness level. The Fuzzy logic framework yields the best air course level: "none", "low", or "high". The accompanying Fuzzy logic standards are utilized:

1. On the off chance that the room is hot, circle the air a considerable measure.
2. On the off chance that the room is cool, don't flow the air.
3. In the event that the room is cool and wet, course the air somewhat.

A information build decides participation works that guide temperatures to Fuzzy logic values and guide dampness estimations to Fuzzy logic qualities.

(b)Inference: Evaluates all principles and decides their truth values. In the event that an information does not definitely relate to an IF THEN control, fractional coordinating of the info information is utilized to introduce a reply.

Continuing the case, assume that the framework has measured temperature and dampness levels and mapped them to the Fuzzy logic estimations of .7 and .1 separately. The framework now construes reality of each Fuzzy logic run the show. To do this a straightforward technique called MAX-MIN is utilized. This strategy sets the Fuzzy logic estimation of the THEN statement to the Fuzzy logic estimation of the IF proviso. In this way, the technique gathers Fuzzy logic estimations of 0.7, 0.1, and 0.1 for standards 1, 2, and 3 individually.

(c) Compositions: Combines every single Fuzzy logic conclusion acquired by derivation into a solitary conclusion. Since various Fuzzy logic guidelines may have diverse conclusions, consider all principles.

Continuing the illustration, every surmising recommends an alternate activity

rule 1 proposes a "high" dissemination level

rule 2 proposes killing air course

rule 3 proposes a "low" flow level.

A basic MAX-MIN strategy for determination is utilized where the greatest Fuzzy logic estimation of the surmisings is utilized as the last conclusion. Along these lines, creation chooses a Fuzzy logic estimation of 0.7 since this was the most noteworthy Fuzzy logic esteem connected with the derivation conclusions.

(d) Defuzzification: Change over the Fuzzy logic esteem got from sythesis into a "fresh" esteem. This procedure is frequently perplexing since the Fuzzy logic set won't not make an interpretation of specifically into a fresh value.Defuzzification is essential, since controllers of physical frameworks require discrete signs.

Proceeding with the case, arrangement yields a Fuzzy logic estimation of 0.7. This uncertain esteem is not straightforwardly helpful since the air course levels are "none", "low", and "high". The defuzzification procedure changes over the Fuzzy logic yield of 0.7 into one of the air course levels. For this situation plainly a Fuzzy logic yield of 0.7 shows that the course ought to be set to "high".

There are numerous defuzzification strategies. Two of the more regular procedures are the centroid and most extreme strategies.

In the centroid strategy, the fresh estimation of the yield variable is processed by finding the variable estimation of the focal point of gravity of the enrollment work for the Fuzzy logic esteem.

In the most extreme strategy, one of the variable qualities at which the Fuzzy logic subset has its greatest truth esteem is picked as the fresh esteem for the yield variable.

5. Applications

The metro in Sendai, Japan utilizes a Fuzzy logic rationale control framework created by Serji Yasunobu of Hitachi.

It took 8 years to finish and was at long last put into utilization in 1987.

*Control System :*Based on tenets of rationale got from prepare drivers in order to model genuine human choices as nearly as could be expected under the circumstances

Task: Controls the speed at which the prepare takes bends and additionally the quickening and slowing mechanisms of the prepare

The consequences of the Fuzzy logic rationale controller for the Sendai metro are amazing!!

The prepare development is smoother than most different trains

Even the gifted human administrators who infrequently run the prepare can't beat the mechanized framework as far as smoothness or precision of ceasing

6. Conclusion

Deduction administers in a Fuzzy logic rationale model may encourage not just to distinguish the reason for a specific hazard additionally to outline productive and viable moderation arranges. The apparatus of Fuzzy logic data granulation, particularly as etymological factors, Fuzzy logic if-then guidelines and Fuzzy logic diagrams, has since quite a while ago assumed a noteworthy part in the utilizations of Fuzzy logic rationale. What has not been completely perceived, be that as it may, is the centrality of Fuzzy logic data granulation in human thinking and, ipso facto, its centrality in Fuzzy logic rationale. A related point is that no philosophy other than Fuzzy logic rationale gives an applied system and related methods for managing issues in which Fuzzy logic data granulation plays, or could play, a noteworthy part. With regards to such issues, the route in which people utilize Fuzzy logic data granulation to settle on sane choices in a domain of fractional information, incomplete conviction and halfway truth ought to be seen as a good example for machine insight.

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LRS BIANCHI TYPE-I GENERALISED GHOST PILGRIM DARK ENERGY MODEL IN BRANS-DICKE THEORY OF GRAVITATION

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Abstract
In this work, we investigate locally rotationally symmetric (LRS) Bianchi type-I universe filled with two interacting fields: matter and generalized ghost version of pilgrim dark energy in Brans-Dicke (BD) (Phys. Rev. 124, 925, 1961) scalar tensor theory of gravitation. The solution of the BD field equations is obtained by using following two assumptions: (i) shear scalar of the model is proportional expansion scalar, which leads to a relation between the metric potential (ii) scalar field in the BD theory is a function of average scale factor of the model. Some physical and geometrical properties are discussed for our model and which are found to be consistent with recent observation.

Keywords: LRS Bianchi type-I space-time, ghost pilgrim dark energy, Scalar tensor theory, Brans-Dicke theory.

1. INTRODUCTION

Cosmology is one of the most stimulating field in all physical sciences that deals with the study of origin and evolution of the universe. General relativity (GR) laid down the foundation of modern cosmology which leads to accelerating expansion phase of the universe (Riess et al.1998; Perlmutter et al. 1999). The force behind this phenomenon is an exotic type of energy called “dark energy” (DE) whose nature is unknown. The simplest model compatible with all cosmological observations is a ΛCDM model but it suffers issues like fine tuning and cosmic coincidence problems leading to some alternatives to investigate its description.

In most of recent dark energy models, normally we need to consider a new degree of freedom or a new parameter, to explain the cosmic acceleration of Universe. However, our prior choice to handle the dark energy problem without introducing new degrees of freedom beyond what are already known. An interesting and successful attempt, in this direction is the so called ‘‘ghost dark energy’’ (GDE), and this interesting model of dark energy, which is introduced recently, called Veneziano ghost dark (GD) energy, which supposed to exist to solve the U(1) problem in low-energy effective theory of QCD, and has attracted a lot of interests in recent years Urban and Zhitnitsky (2009a), Cai et al. (2011), Feng et al. (2012, 2013), Sadeghi et al. (2013).

It is observed that the Veneziano ghost field in QCD of the form $H+O(H^2)$ has ability in producing enough vacuum energy to explain the accelerated expansion of the universe Zhitnitsky (2012), but only leading term (i.e., H) involved in ordinary ghost DE model. It is suggested Cai et al (2012) that the contribution of the term H^2 in the ordinary ghost DE may be useful in describing the early evolution of the universe which is defined as follows $\rho_\Lambda = \alpha H + \beta H^2$ here β involves as a constant containing dimension $[energy]^2$ and corresponding energy density is called generalized ghost DE.

Its generalized version in terms of PDE is known as generalised ghost pilgrim dark energy (GGPDE) and is defined as follows (Sharif and Jawed 2014)

$$\rho_\Lambda = \left[\alpha H + \beta H^2\right]^u \tag{1.1}$$

The advantages of the GGPDE with respect to other DE models include the absence of the fine tuning and cosmic coincidence problems. Sharif and Jawed (2013a, 2013b) studied the interacting PDE in flat as well as non-flat universe models with different IR cut offs.

One of the most generalizations of the flat universe is Friedman Robertson–Walker (FRW) universe. Similarly the simplest spatially homogeneous and anisotropic flat universe is the Bianchi type-I universe. FRW universe has the same scale factor for each of the three spatial directions whereas Bianchi type-I universe has different scale factors. Near the singularity Bianchitype-I universe behave like Kanser universe. Kumar and Singh (2008), Adhav (2012), Shamir et al. (2012), Rao et al. (2011) and Rao et al. (2012a, b, c) are some of the authors who have investigated various Bianchi type-I cosmological models in alternative theories of gravitation. Recently, Santhi et al. (2016) have investigated some Bianchi type GGPDE models in general relativity.

Brans-Dicke theory of gravitation is a natural extension of general relativity which introduces an additional scalar field ϕ besides the metric tensor g_{ij} and dimensionless coupling constant ω. The Brans - Dicke field equations for combined scalar and tensor field are given by

$$G_{ij} = -8\pi\phi^{-1}T_{ij} - \omega\phi^{-2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) - \phi^{-1}(\phi_{i;j} - g_{ij}\phi_{;k}{}^{,k}) \tag{1.2}$$

and

$$\phi_{;k}{}^{,k} = 8\pi(3+2\omega)^{-1}T \tag{1.3}$$

where $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is an Einstein tensor, R is the scalar curvature, ω and n are constants, T_{ij} is the stress energy tensor of the matter and comma and semicolon denote partial and covariant differentiation respectively.

Also, we have energy – conservation equation

$$T^{ij}_{;j} = 0 \tag{1.4}$$

This equation is a consequence of the field equations (1.1) and (1.2).

Several aspects of Brans-Dicke cosmology have been extensively investigated by many authors. Rao et al. (2012) have obtained LRS Bianchi type-I dark energy cosmological model in Brans-Dicke theory of gravitation. Reddy et al. (2014) have studied Kantowski–Sachs bulk viscous string cosmological model in Brans–Dicke theory of gravitation. Rao et al. (2015) have studied FRW holographic dark energy cosmological model in Brans-Dicke theory of gravitation. Recently, Rao et al. (2016) have studied two fluid scenario for axially symmetric dark energy cosmological models in Brans-Dicke theory of gravitation

Motivated by the above investigations and discussions, in this paper we study LRS Bianchi type-I cosmological model with GGPDE and matter in Brans-Dicke theory of gravitation. The plan of the paper as follows: Sect. 1 contains metric and field equations. In Sect. 2 we find the solution of field equations and intaracting GGPDE model is obtained. In Sect. 3 we discussed some physical properties of our model. Finally we conclude our paper in Sect. 4.

2. Metric and Energy Momentum Tensor:

The LRS Bianchi type-I line element can be taken as

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$$ds^2 = dt^2 - A^2(dx^2 + dy^2) - B^2 dz^2 \tag{2.1}$$

where *A* and *B* are functions of the cosmic time *t* only .

The energy momentum tensor for matter and the dark energy are defined as

$$T_{ij} = \rho u_i u_j \quad \text{and} \quad \bar{T}_{ij} = (\rho_\Lambda + p_\Lambda) u_i u_j - g_{ij} p_\Lambda \tag{2.2}$$

where, ρ , ρ_Λ are energy densities of matter and dark energy and p_Λ is the pressure of dark energy.

In a co moving coordinate system, we get

$$T_1^{\ 1} = T_2^{\ 2} = T_3^{\ 3} = 0,\ T_4^{\ 4} = \rho$$

$$\text{and} \quad \bar{T}_1^{\ 1} = \bar{T}_2^{\ 2} = \bar{T}_3^{\ 3} = -p_\Lambda,\ \bar{T}_4^{\ 4} = \rho_\Lambda \tag{2.3}$$

where the quantities ρ, ρ_Λ and p_Λ are functions of '*t*' only.

3. Solutions of Field equations:

The field equations (1.1) – (1.3) for the metric (2.1) with the help of equations (2.2) & (2.3) can be written as

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{\ddot{\phi}}{\phi} + \frac{\dot{A}\dot{\phi}}{A\phi} + \frac{\dot{B}\dot{\phi}}{B\phi} = -8\pi\phi^{-1} p_\Lambda \tag{3.1}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{2\dot{A}\dot{\phi}}{A\phi} + \frac{\ddot{\phi}}{\phi} = -8\pi\phi^{-1} p_\Lambda \tag{3.2}$$

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} - \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{2\dot{A}\dot{\phi}}{A\phi} + \frac{\dot{B}\dot{\phi}}{B\phi} = 8\pi\phi^{-1}(\rho + \rho_\Lambda) \tag{3.3}$$

$$\phi_{;k}^{\ ,k} = \ddot{\phi} + \dot{\phi}\left(\frac{2\dot{R}}{R} + \frac{\dot{A}}{A}\right) = 8\pi(3+2\omega)^{-1}(\rho + \rho_\Lambda - 3p_\Lambda) \tag{3.4}$$

$$\dot{\rho} + \dot{\rho}_\Lambda + 3H(\rho + \rho_\Lambda + p_\Lambda) = 0 \tag{3.5}$$

From the field equations (3.2)-(3.1) we get,

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} + \frac{\dot{A}\dot{\phi}}{A\phi} - \frac{\dot{B}\dot{\phi}}{B\phi} = 0 \tag{3.6}$$

Here the over head dot denotes differentiation with respect to 't'.

The field equations (3.1)–(3.4) are four independent equations with six unknowns ($A, B, p_\Lambda, \rho_\Lambda, \rho$ and ϕ) and the equations are highly nonlinear in nature. So, in order to get a deterministic solution we take the following possible physical conditions:

1. The shear scalar σ is proportional to scalar expansion θ, which leads to the linear relationship between the metric potentials B and A, i.e.

$$A = B^n \tag{3.7}$$

where $n \neq 0$ is an arbitrary constant

2. Relation between scalar field ϕ and the scale factor of the model $a(t)$ given by (Pimental 1985; Johri and Kalyani 1994)

$$\phi = \phi_0 a^m \tag{3.8}$$

where ϕ_0 and $m > 0$ are constants.

On using (3.7) and (3.8), in (3.6) we calculate the values of B and A as

$$B = \left[k_5\left(k_3 t + k_4\right)\right]^{\frac{1}{k_5}} \tag{3.9}$$

$$A = \left[k_5\left(k_3 t + k_4\right)\right]^{\frac{n}{k_5}} \tag{3.10}$$

where $k_1 = (n-1)$, $k_2 = \left[n^2 + \frac{m}{3}(n-1)(2n+1)\right]$, $k_5 = \frac{k_1 + k_2}{k_2}$ and k_3, k_4 are integration constants .

From equation (3.8)-(3.10), we get the scalar field as

$$\phi = \phi_0\left[k_5\left(k_3 t + k_4\right)\right]^{\frac{m(2n+1)}{3k_5}} \tag{3.11}$$

The Hubble parameter is one of the most important numbers in cosmology as it is used to estimate the size and age of the universe. It indicates the rate at which the universe is expanding.

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\frac{(2n+1)k_3}{k_5(k_3 t + k_4)} \tag{3.12}$$

We consider the interaction between GGPDE and dark matter. So, the energy densities of GGPDE and dark matter do not conserve separately. The continuity equation of the GGPDE is

$$\dot{\rho}_\Lambda + 3H(\rho_\Lambda + p_\Lambda) = -\Gamma \tag{3.13}$$

and $\Gamma = 3d^2 H\rho$ (3.14)

where Γ is interaction term and $d^2 > 0$ is a coupling constant.

The continuity equation of the matter is

$$\dot{\rho} + 3H\rho = \Gamma \tag{3.15}$$

The GGPDE equation of state (EoS) parameter is defined by

$$w_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} \tag{3.16}$$

The GGPDE is defined as

$$\rho_\Lambda = \left(\alpha H + \beta H^2\right)^u \tag{3.17}$$

where H is the Hubble parameter, α and β are constants which must satisfy the restrictions imposed by the current observational data.

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From equations (3.12)-(3.16), we get

The GGPDE energy density is

$$\rho_\Lambda = \left(\frac{\alpha(2n+1)k_3}{3k_5(k_3t+k_4)} + \frac{\beta(2n+1)^2 k_3^2}{9k_5^2(k_3t+k_4)^2} \right)^u \tag{3.18}$$

From equations (3.12), (3.14) and (3.15), we get

The energy density of matter as

$$\rho = k_6\left(k_5(k_3t+k_4)\right)^{\frac{(d^2-1)(2n+1)}{k_5}} \tag{3.19}$$

From equations (3.1), (3.2) and (3.11) we get

The pressure of GGPDE is

$$8\pi\, p_\Lambda = \begin{bmatrix} k_5\left((n+3)+\frac{2m}{3}(2n+1)\right)-(n^2+n+3)-\frac{(\omega-2)m^2}{9}(2n+1)^2 \\ -\frac{m}{3}(2n+1)(3n+1) \end{bmatrix} \frac{k_3^2\varphi_0}{2}\left[k_5(k_3t+k_4)\right]^{\frac{m(2n+1)}{3k_5}-2} \tag{3.20}$$

From equations (3.16), (3.18) and (3.20), we get

$$w_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} = -1 + \frac{uk_5}{(2n+1)} \frac{\alpha + \frac{2\beta(2n+1)k_3}{3k_5(k_3t+k_4)}}{\alpha + \frac{\beta(2n+1)k_3}{3k_5(k_3t+k_4)}} - \frac{d^2k_6\left[k_5(k_3t+k_4)\right]^{\frac{(d^2-1)(2n+1)}{k_5}}}{\left[\frac{\alpha(2n+1)k_3}{3k_5(k_3t+k_4)} + \frac{\beta(2n+1)^2k_3^2}{9k_5^2(k_3t+k_4)^2}\right]^u} \tag{3.21}$$

The coincident parameter is

$$r = \frac{\rho_\Lambda}{\rho} = \frac{1}{9k_6}\left[3\alpha\, k_3k_5(2n+1)(k_3t+k_4)+\beta(2n+1)^2k_3^2\right]\left[k_5(k_3t+k_4)\right]^{\frac{(1-d^2)(1+2n)-2k_5}{k_5}} \tag{3.22}$$

The metric (2.1), in this case can be written as

$$ds^2 = dt^2 - \left[k_5(k_3t+k_4)\right]^{\frac{2n}{k_5}}(dx^2+dy^2) - \left[k_5(k_3t+k_4)\right]^{\frac{2}{k_5}}dz^2 \tag{3.23}$$

Thus the equation (2.1) together with (3.18) - (3.22) constitutes a LRS Bianchi type-I GGPDE cosmological model in BD theory of gravitation.

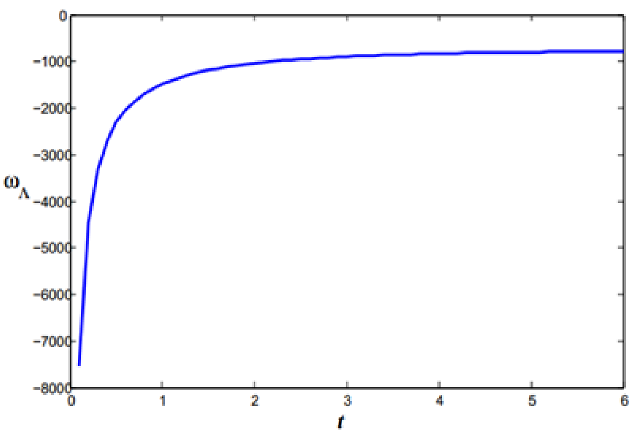

Fig.1: Plot of GGPDE EoS parameter versus time.

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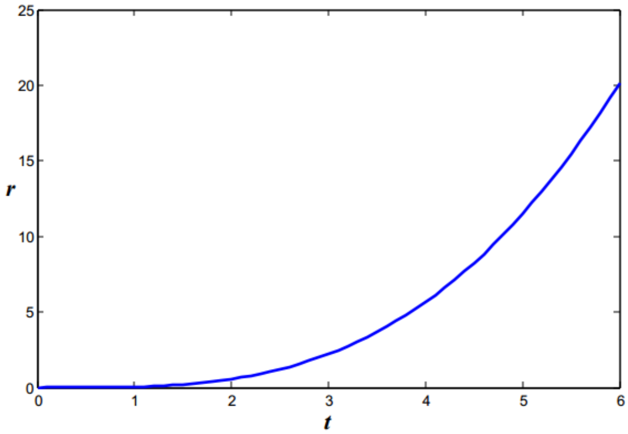

Fig. 2: Plot of coincidence parameter versus time.

Figure 1 shows the behaviour of ω_Λ as a function of the time *t*, we observed that the EoS parameter $w_\Lambda << -1$, which indicates an aggressively phantom-like behaviour. Hence, the GGPDE model in BD scalar tensor theory pertains to phantom era of the universe and hence it is consistent with the behaviour of pilgrim dark energy. Figure 2 describes the behaviour of coincidence parameter versus time *t*. It is observed that coincidence parameter increases as time increases but after some finite time it converges to a constant value. Thus the interacting term (d^2) can make the ratio of GGPDE and matter densities possible to attain a stationary value during the course of evolution and consequently can help to alleviating the coincidence problem which appears in the *ΛCDM* models.

4. Some other important properties of the model:

The spatial volume for the model is

$$V = \left(A^2 B\right) = \left[k_5\left(k_3 t + k_4\right)\right]^{\frac{n(2n+1)}{k_5}} \tag{4.1}$$

The average scale factor for the model is

$$a(t) = V^{\frac{1}{3}} = \left[k_5\left(k_3 t + k_4\right)\right]^{\frac{n(2n+1)}{3k_5}} \tag{4.2}$$

The expansion scalar θ is calculated for the flow vector u^i is given by

$$\theta = u^i_{,i} = \frac{(2n+1)k_3}{k_5(k_3 t + k_4)} \tag{4.3}$$

Look-back time-red shift: The look-back time, $\Delta t = t_0 - t(z)$ is the difference between the age of the universe at present time $Z = 0$ and the age of the universe when a particular light ray at red shift Z, the expansion scalar of the universe $a(t_z)$ is related to a_0 by $1 + Z = \frac{a_0}{a}$, where a_0 is the present scale factor. Therefore from (4.2), we get

$$1 + Z = \frac{a_0}{a} = \left[\frac{k_3 t_0 + k_4}{k_3 t + k_4}\right]^{\frac{n(2n+1)}{3k_5}} \tag{4.4}$$

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This equation can also be expressed as

$$H_0 \Delta t = \frac{2n+1}{3k_5}\left(1-[1+Z]^{\frac{-3k_5}{n(2n+1)}}\right) \tag{4.5}$$

where H_0 is the Hubble's constant.

Luminosity distance: Luminosity distance is defined as the distance which will preserve the validity of the inverse law for the fall of intensity and, is given by

$$d_L = r_1(1+Z)a_0 \tag{4.6}$$

where r_1 is the radial coordinate distance of the object at light emission and is given by

$$r_1 = \int_T^{T_0} \frac{1}{a} dT = \frac{n(2n+1)k_3}{3(k_3 t + k_4)k_5}\left[\left[k_5(k_3 t + k_4)\right]^{\frac{-n(2n+1)}{3k_5}} - \left[k_5(k_3 t_0 + k_4)\right]^{\frac{-n(2n+1)}{3k_5}}\right] \tag{4.7}$$

From equations (4.10) and (4.11), we get

The luminosity distance

$$d_L = nHZ(1+Z) \tag{4.8}$$

The distance modulus (*D*) is given by

$$D(Z) = 5\log d_L + 25 \tag{4.9}$$

From equations (4.11) and (4.12), we get

The distance modulus

$$D(Z) = 5\log nHZ(1+Z) + 25 \tag{4.10}$$

The tensor of rotation $w_{ij} = u_{i,j} - u_{j,i}$ is identically zero and hence this universe is non-rotational.

The mean anisotropy parameter A_h is given by

$$A_h = \frac{1}{4}\sum_{i=1}^{4}\left(\frac{H_i - H}{H}\right)^2 = \frac{2(n-1)^2}{(2n+1)^2} \tag{4.11}$$

where $\Delta H_i = H_i - H \quad (i = 1,2,3,4)$

The jerk parameter is given by

$$j = \frac{m^2 - 3m}{(m+3)^2} \tag{4.12}$$

and the shear scalar σ is given by

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{7}{18}\left[\frac{(2n+1)k_3}{k_5(k_3t+k_4)}\right]^2 \tag{4.13}$$

The deceleration parameter q is given by

$$q = (-3\theta^{-2})(\theta_{,i}u^i + \frac{1}{3}\theta^2) = -1 + \frac{3}{(2n+1)}\left[\frac{3(n-1)}{3n^2 + m(n-1)(2n+1)} + 1\right] \tag{4.14}$$

The deceleration parameter appears with negative sign for suitable values of $m \& n$. This implies accelerating expansion of the universe, which is consistent with the present day observations.

5. Discussion and Conclusions:

Here we investigated LRS Bianchi type-I cosmological model with GGPDE in Brans-Dicke theory. The solution of the field equations is obtained by assuming some possible physical conditions. Also, some basic geometrical and physical properties of the model are discussed. We observe that the model (3.23) has singularity at $t = -k_4/k_3$ for $m < 0$ and the spatial volume increases with the increase of time $'t'$. The expansion scalar θ, shear scalar σ and the Hubble parameter *H* decrease with the increase of time. The deceleration parameter appears with negative sign for suitable values of $m \& n$, this implies accelerating expansion of the universe, which is consistent with the present day observations. From (4.7), we can observe that $A_h \neq 0$ and this indicates that our model is anisotropic.

We have also observed that the matter energy density, the GGPDE energy density and the pressure of GGPDE are decrease with the increase of time. The GGPDE EoS parameter of our model is varying in aggressive phantom region, which is in agreement with pilgrim dark energy behaviour. The coincidence parameter is found to be an increasing function of time. We have obtained expressions for look-back time ΔT, distance modulus $D(z)$ and luminosity distance d_L versus red shift and discussed their significance.

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USE OF REDLICH–KISTER POLYNOMIAL FOR FITTING EXCESS PARAMETERS

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ABSTRACT

Excess molar volumes and deviations in adiabatic compressibility in anisic aldehyde + Methyl salicylate binary mixture at (303.15, 308.15, 313.15 and 318.15) K were calculated from experimental density and sound velocity. These experimental values were used to test the applicability of the correlative reduced Redlich–Kister equation.

INTRODUCTION

To fit the data generally polynomials such as Legendre, Leguarre, Redlich- Kister are used. But Redlich – Kister polynomial is widely used by many researchers [1-5]. The Redlich -Kister equation is a powerful mixing rule form which can take on higher order terms necessary for correlation of the composition dependence of non-ideal liquid mixtures. Furthermore, the advent of modern day densitometers and similar instrumentation, has added a large body of high precision liquid mixture thermo physical property data, also requiring multiple interaction parameters to correlate within measurement uncertainty, which can similarly be furnished by the Redlich -Kister equation.

EXPERIMENTAL

Anisic aldehyde and methyl salicylate from Merk were purified as described in the literature [6-8]. The pure chemicals were stored over activated 4Å molecular sieves to reduce water content before use.

The mixtures are prepared gravimetrically using an electronic balance (Shimadzu AY120) with an uncertainty of $\pm 1 \times 10^{-7}$kg and stored in airtight bottles. The uncertainty on mole fraction is estimated to be 1×10^{-4}.It is ensured that the mixtures are properly mixed and the measurement of the required parameters was done within one day of preparation.

The densities, ρ, of pure liquids and their mixtures are determined using a 10^{-5} m^3 double-arm pycnometer, and the values from triplicate replication at each temperature are reproducible within 2×10^{-1} kg m^3 and the uncertainty in the measurement of density is found to be 2 parts in 10^4 parts. The reproducibility in mole fractions was within ±0.0002

Temperature control for the measurement of viscosity and density is achieved by using a microprocessor assisted circulating water bath, (supplied by Mac, New Delhi) regulated to ±0.01 K, using a proportional temperature controller. Adequate precautions were taken to minimize evaporation losses during the actual measurements.

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The ultrasonic velocity of sound (U) is measured using an ultrasonic interferometer (Mittal Enterprises, New Delhi model F-05) operating at 2 MHz The measured speeds of sound have a precision of 0.8 m.sec-1 and an uncertainty less than ± 0.1 m.sec-1. The temperature stability was maintained within 0.01K.by circulating water bath around the measuring cell through a pump.

Theoretical Considerations

1. Mean molar volume (M)

The mean molar volume of a binary liquid mixture at a given mole fraction is given by

$$V = M / \rho \qquad 1$$

2.Excess Volume (V^E)

Excess molar volume defined as the difference between the volume of mixing of a real mixture and the value corresponding to the ideal mixture at the same conditions of temperature, pressure and compositions and is given by

$$V^E = V - (V_1X_1+V_2X_2) \qquad 2$$

where V_1 and V_2 are the mean molar volumes of pure liquids respectively

3.Adiabatic Compressibility (β_{ad})

The thermodynamic property, which influences the interactions between the molecules in liquid mixture, is called adiabatic compressibility. Assuming that ultrasonic absorption is negligible, adiabatic compressibility can be obtained from the density and velocity of ultrasonic sound using the relation

$$\beta_{ad} = 1/ \beta U^2 \qquad 3$$

4. Deviation in Adiabatic compressibility ($\Delta\beta_{ad}$)

The deviation in adiabatic compressibility at a given mole fraction is the difference between adiabatic compressibility and the sum of the fractional contributions of the two liquids and is given by

$$\Delta\beta_{ad} = \beta_{ad} - (\beta_{ad\,1}X_1 + \beta_{ad\,2}X_2) \qquad 4$$

where $\beta_{ad\,1}$ and $\beta_{ad\,2}$ are the individual adiabatic compressibility values of pure liquids in the binary mixtures at that temperature.

The excess thermodynamic functions (Y^E) introduced by Scatchard [9] in 1931, provided a way to represent directly the deviation of a solution from ideal behaviour. The difference between the thermodynamic function of mixing for a real system and the value corresponding to a perfect solution at the same temperature, pressure and composition is called the thermodynamic excess function. (denoted by Y^E)

Excess values for all the parameters are computed using the general formula

$$Y^E = Y_{exp} - (X_1Y_1+X_2Y_2) \qquad 5$$

The values of $\Delta\beta_{ad}$ and V^E were least squares fitted to the Redlich – Kister [10] type polynomial

$$Y^E = X(1-X)\sum_{i=1}^{n} A_i(1-2X) \qquad 6$$

The coefficients of A_i in the above equation along with the standard deviation σ (Y^E) are given in Tables. These coefficients are the adjustable parameters to get best - fit values of Y^E. The standard deviations σ (Y^E) were calculated by using the relation

$$\sigma(Y^E) = \sum[(Y^E{}_{exp} - Y^E{}_{cal})^2 / (m-n)]^{1/2} \qquad 7$$

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where m is the number of experimental data points and n is the number of coefficients considered and (n =5 in the present calculation). $Y_{cal}{}^{E}$ has been obtained from the above equation using the best - fit values of A_i.

Results and Discussion

The ultrasonic velocity(U) and density (ρ) are measured for the binary system of anisic aldehyde + methyl salicylate and these values are presented in Table 1. The ultrasonic velocity of anisic aldehyde + methyl salicylate system against the mole fraction shows the non-linear variation. The non-existence of any maximum or a dip at any intermediate concentrations of anisaldehyde and methyl salicylate mixtures indicates the absence of complex formation between the compounds [11, 12].

From the experimental data adiabatic compressibility (β_{ad}), molar volume (V), deviation in adiabatic compressibility ($\Delta\beta_{ad}$) and excess molar volume (V^E) were calculated over the whole composition range. The computed values of $\Delta\beta_{ad}$ and V^E are fitted to Redlich – Kister Polynomial equation and checked the validity. The calculated values of deviation in adiabatic compressibility ($\Delta\beta_{ad}$) and excess molar volume (V^E) along with the values calculated from Redlich – Kister Polynomial equation are presented in Table (2). The variation of deviation in adiabatic compressibility with the mole fraction of anisic aldehyde at different temperatures are represented in figures from 1-4. The variation of excess molar volumes with the mole fraction of anisic aldehyde at different temperatures are represented in figures from 5-8.From the figures it can be concluded that the computed values are almost nearer to the values calculated from Redlich – Kister polynomial. Both $\Delta\beta_{ad}$ and V^E are negative and reaches a maximum at intermediate concentration which supports the existence of strong interaction between the consecutive molecules.

Standard deviation (σ) are also calculated and presented along with the Redlich coefficients in the table 3. From the values of σ it can be observed that the values are very low which concludes that the accuracy of our data.

Table 1: Ultrasonic velocities (U) and densities (ρ) of Anisic Aldehyde + Methyl Salicylate at the temperatures 303.15,308.15,313.15 and 318.15K

Mole fraction X_1	Velocity m/s	Density gm/cm^3	Velocity m/s	Density gm/cm^3	Velocity m/s	Density gm/cm^3	Velocity m/s	Density gm/cm^3
	303.15K		**308.15K**		**313.15K**		**318.15K**	
0.0000	1389.5	1.1752	1374.4	1.1685	1358.0	1.1678	1343.40	1.1645
0.1059	1411.6	1.1708	1395.5	1.1645	1380.3	1.1641	1366.70	1.1607
0.2104	1430.2	1.1663	1413.1	1.1602	1398.9	1.1593	1385.50	1.1557
0.3135	1447.7	1.1616	1429.7	1.1557	1416.4	1.1541	1403.10	1.1500
0.4153	1463.9	1.1567	1445.5	1.1511	1432.9	1.1485	1419.80	1.1443
0.5159	1480.0	1.1515	1459.6	1.1460	1448.4	1.1426	1435.10	1.1382
0.6152	1493.7	1.1456	1471.9	1.1400	1461.4	1.1360	1447.50	1.1312
0.7132	1507.5	1.1394	1484.9	1.1336	1475.0	1.1292	1460.20	1.1239
0.8100	1520.9	1.1331	1496.9	1.1272	1488.0	1.1223	1472.40	1.1167
0.9056	1533.0	1.1268	1508.3	1.1207	1500.0	1.1150	1484.10	1.1087
1.0000	1543.5	1.1204	1517.0	1.1140	1509.4	1.1067	1492.00	1.0995

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Table 2: Adiabatic compressibility (β_{ad}) and molar volume (V) of Anisic Aldehyde + Methyl Salicylate at the temperatures 303.15, 308.15, 313.15 and 318.15K

Mole fraction X_1	Ad. Comp. $\beta_{ad}\times10^{-12}$ m^2N^{-1}	Mol.Vol. V cm^3 mol^{-1}	Ad. Comp. $\beta_{ad}\times10^{-12}$ m^2N^{-1}	Mol.Vol. V cm^3 mol^{-1}	Ad. Comp. $\beta_{ad}\times10^{-12}$ m^2N^{-1}	Mol.Vol. V cm^3 mol^{-1}	Ad. Comp. $\beta_{ad}\times10^{-12}$ m^2N^{-1}	Mol.Vol. V cm^3 mol^{-1}
	303.15K		308.15K		313.15K		318.15K	
0.0000	44.0728	129.4929	45.3049	130.2353	46.4336	130.3134	47.5828	130.6827
0.1059	42.8641	128.5300	44.0962	129.2254	45.0882	129.2698	46.1247	129.6484
0.2104	41.9176	127.5897	43.1640	128.2605	44.0789	128.3601	45.0756	128.7600
0.3135	41.0758	126.6825	42.3316	127.3292	43.1902	127.5057	44.1697	127.9603
0.4153	40.3419	125.8079	41.5768	126.4200	42.4070	126.7062	43.3517	127.1712
0.5159	39.6472	124.9765	40.9588	125.5763	41.7185	125.9499	42.6596	126.4368
0.6152	39.1237	124.2311	40.4891	124.8413	41.2177	125.2809	42.1914	125.8125
0.7132	38.6198	123.5282	40.0079	124.1600	40.7066	124.6493	41.7299	125.2316
0.8100	38.1534	122.8463	39.5925	123.4884	40.2443	124.0331	41.3059	124.6495
0.9056	37.7632	122.1722	39.2225	122.8372	39.8616	123.4684	40.9505	124.1667
1.0000	37.4640	121.5191	39.0071	122.2172	39.6608	123.0234	40.8571	123.8290

Table 3: Deviation in adiabatic compressibility (Δβad) and excess molar volume (VE) computed and of Anisic Aldehyde + Methyl Salicylate at the temperatures 303.15, 308.15, 313.15 and 318.15K

Mole fraction X1	Cal 303.15K	RK	Cal 308.15K	RK	Cal 313.15K	RK	Cal 318.15K	RK
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1059	-0.5091	-0.5091	-0.5420	-0.5420	-0.6284	-0.6284	-0.7460	-0.7460
0.2104	-0.7650	-0.7600	-0.8161	-0.8159	-0.9299	-0.9256	-1.0923	-1.0718
0.3135	-0.9250	-0.9250	-0.9988	-0.9988	-1.1201	-1.1201	-1.3044	-1.3044
0.4153	-0.9860	-1.0187	-1.1124	-1.1014	-1.2136	-1.2291	-1.4376	-1.4521
0.5159	-1.0162	-1.0162	-1.0971	-1.0971	-1.2211	-1.2211	-1.4535	-1.4535
0.6152	-0.8836	-0.9130	-0.9417	-0.9858	-1.0496	-1.0953	-1.2541	-1.2997
0.7132	-0.7398	-0.7398	-0.8056	-0.8056	-0.8968	-0.8968	-1.0562	-1.0562
0.8100	-0.5664	-0.5406	-0.6113	-0.6040	-0.7035	-0.6806	-0.8292	-0.8070
0.9056	-0.3248	-0.3248	-0.3793	-0.3793	-0.4387	-0.4387	-0.5416	-0.5416
1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4: Deviation in adiabatic compressibility ($\Delta\beta_{ad}$) and excess molar volume (V^E) of Anisic Aldehyde + Methyl Salicylate at the temperatures 303.15, 308.15, 313.15 and 318.15K

Mole fraction X_1	Cal 303.15K	RK	Cal 308.15K	RK	Cal 313.15K	RK	Cal 318.15K	RK
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1059	-0.1187	-0.1187	-0.1611	-0.1611	-0.2719	-0.2719	-0.3087	-0.3087
0.2104	-0.2258	-0.2133	-0.2881	-0.2695	-0.4197	-0.4145	-0.4810	-0.4478

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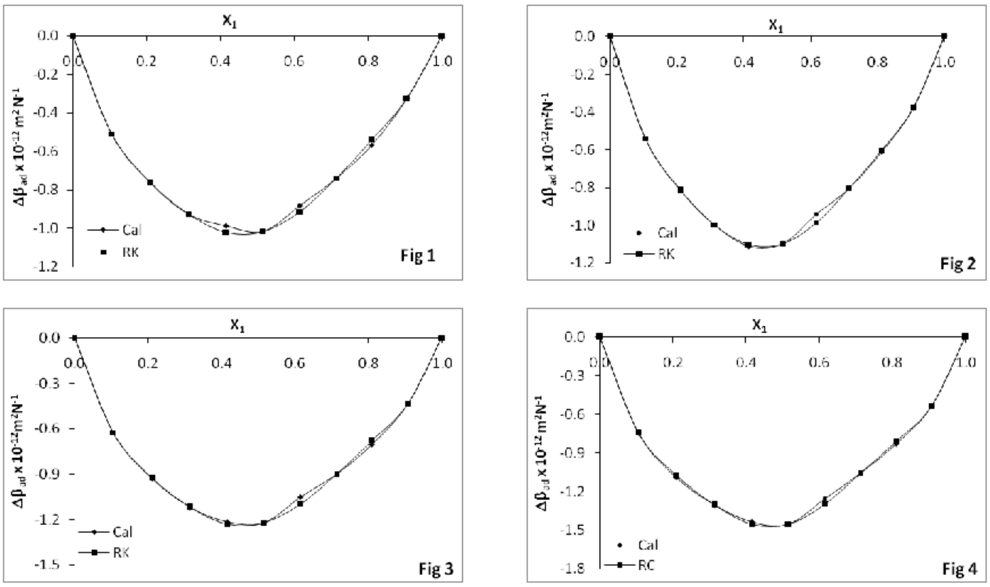
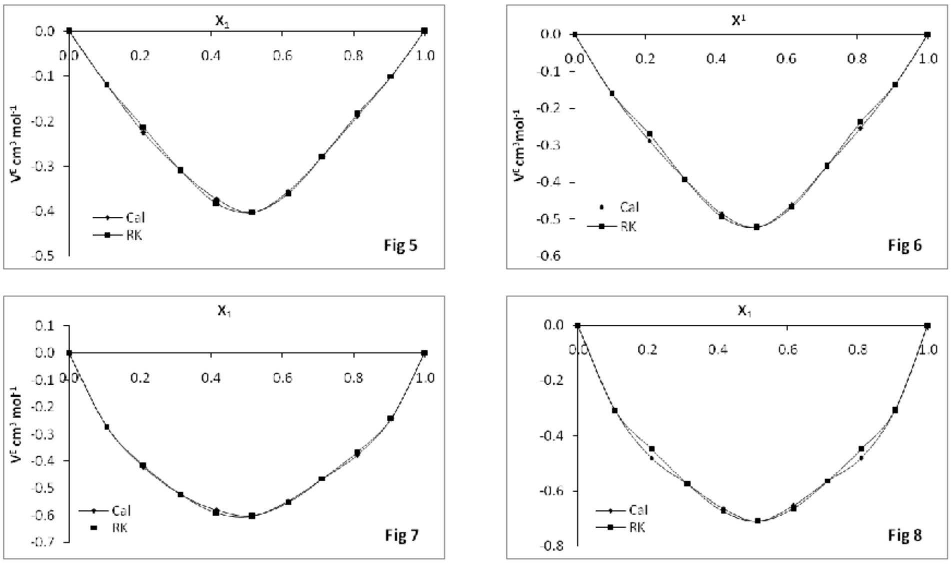
0.3135	-0.3105	-0.3105	-0.3923	-0.3923	-0.5222	-0.5222	-0.5736	-0.5736
0.4153	-0.3731	-0.3835	-0.4851	-0.4927	-0.5794	-0.5920	-0.6648	-0.6758
0.5159	-0.4029	-0.4029	-0.5227	-0.5227	-0.6027	-0.6027	-0.7102	-0.7102
0.6152	-0.3567	-0.3615	-0.4617	-0.4681	-0.5480	-0.5528	-0.6541	-0.6633
0.7132	-0.2780	-0.2780	-0.3570	-0.3570	-0.4650	-0.4650	-0.5632	-0.5632
0.8100	-0.1880	-0.1841	-0.2525	-0.2379	-0.3756	-0.3652	-0.4819	-0.4489
0.9056	-0.0998	-0.0998	-0.1372	-0.1372	-0.2433	-0.2433	-0.3095	-0.3095
1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 5 Values of coefficients and standard deviation (σ)

Temp K	A_0	A_1	A_2	A_3	A_4	σ
Excess adiabatic Compressibility ($\Delta\beta_{ad}$)						
303.15K	-4.0947	-0.7711	1.2045	-0.4383	-3.1313	0.0210
308.15K	-4.4218	-0.8699	1.2692	-0.0213	-3.6166	0.0188
313.15K	-4.9233	-1.0255	1.3687	-0.0277	-4.5157	0.0219
318.15K	-5.8574	-1.1076	2.4792	-0.0108	-6.9569	0.0231
Excess Molar Volume (V^E)						
303.15K	-1.6160	-0.0408	1.5716	-0.0313	-1.4677	0.0071
308.15K	-2.0950	0.0066	2.3526	-0.1409	-2.6032	0.0105
313.15K	-4.9233	-1.0255	1.3687	-0.0277	-4.5157	0.0219
318.15K	-2.8414	0.1156	1.4549	0.0509	-3.7287	0.0199



Figs 1-4: variation of deviation in adiabatic compressibilies with the mole fraction of anisic aldehyde



Figs 1-4: variation of excess molar volume with the mole fraction of anisic aldehyde

Conclusions

Using Redlich – Kister polynomial, we calculated RK coefficients and standard deviation we checked the validity of our data. From the observation it is concluded that our data is accurate.

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SEVARAL SIMPLE REAL-WORLD APPLICATIONS OF LINEAR ALGEBRA

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ABSTRACT

In this paper we discuss about several real world motivated examples illustrating the power of the linear algebra tools as the product of matrices and matrix notation of systems of linear equations. To explain some mathematical terms it is often convenient to illustrate them by suitable examples having applications in our daily life. Such examples are presented which illustrate the use of matrices, the efficiency of their products and an advantage of matrix notation for a system of linear equations.

Key words: demand matrix, price matrix.

Example 1: Three people denoted by P1, P2, P3 intend to buy some rolls, buns, cakes and bread. Each of them needs these commodities in different amounts and can buy them in two different shops S1, S2. Which shop is the best for every person P1, P2, P3 to pay as little as possible? The individual prices and quantities of the commodities are given in the following tables:

Demanded quantity of foodstuff: prices in shops S1, S2:

	roll	Bun	Cake	Bread
P1	**6**	**5**	**3**	**1**
P2	**3**	**6**	**2**	**2**
P3	**3**	**4**	**3**	**1**

	S1	S2
Roll	**1.50**	**1.00**
Bun	**2.00**	**2.50**
Cake	**5.00**	**4.50**
Bread	**16.00**	**17.00**

Prices in shops S1, S2:

For example the amount spend by the person P1 in the shop S1 is :

6(1.50) +5(2) +3(5) +1(16) =50

And in the shop S2:

6(1) +5(2.50) +3(4.50) +1(17) =49, for the other people similarly.

These calculations can be written using a product of two matrices namely

$$P=\begin{pmatrix} 6 & 5 & 3 & 1 \\ 3 & 6 & 2 & 2 \\ 3 & 4 & 3 & 1 \end{pmatrix} \qquad Q=\begin{pmatrix} 1.50 & 1 \\ 2 & 2.50 \\ 5 & 4.50 \\ 16 & 17 \end{pmatrix}$$

The ***price matrix***. for example, the first row of matrix

$$R= PQ = \left(\begin{array}{c|c} 50 & 49 \\ 58.50 & 61 \\ 43.50 & 43.50 \end{array}\right)$$

Expresses the amount spent by the person P1 in the shop S1 (the element ***r*11)** and in the shop S2(the element r12)**.** Hence**,** it is optimal for the person P1 to buy in the shop S2 for the person P2 in S1 and the person P3 will pay the same price in S1 as in S2.

Example 2: To encode a short message a number can be assigned to each letter of the alphabet according to a given table. The text as a sequence of numbers will be organised into a square matrix A; in the case that the number of letters is less than the number of elements of the matrix A, the rest

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of the matrix can be filled with zero elements. Let a non-singular square matrix C be left. Let the following table and the matrix C be given:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
8	7	5	13	9	16	18	22	4	23	11	3	21	1	6	15	12	19	2	14	17	20	25	24	10	26

C = $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

We put the text "BILA KOCKA" (a white cat) into the matrix A:

A = $\begin{pmatrix} 7 & 4 & 3 \\ 8 & 11 & 6 \\ 5 & 11 & 8 \end{pmatrix}$

And encode the text

Z= CA= $\begin{pmatrix} 19 & 19 & 14 \\ 12 & 15 & 11 \\ 8 & 11 & 6 \end{pmatrix}$

To decode the message we have to multiply the matrix Z by the matrix C^{-1} and Z in the opposite order , we obtain

ZC^{-1}= $\begin{pmatrix} 19 & 19 & 14 \\ 12 & 15 & 11 \\ 8 & 11 & 6 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 9 & 19 \\ 1 & 10 & 15 \\ 2 & 4 & 11 \end{pmatrix}$

And it means "CERNY PSIK" (a black dog).

Example3: Let us consider a group of people P1... Pn. We put a_{ij}=1'if the person Pi can send some information to the person Pj, and a_{ij}=0 otherwise (for convenience, we put a_{ii} =0 for all i= 1,.....,n). Organising this elements into a square matrix A, we obtain a so called incidence matrix. Let Pi→Pj denotes the fact that Pi can send information to Pj . Thus, for example the elements of the matrix

A= $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

Tell us that

P1→ $P2$, $P1 \rightarrow P4, P2 \rightarrow P3, P3 \rightarrow P1, P3 \rightarrow P4, P4(\rightarrow P1, P4 \rightarrow P2\ since\ P1 \rightarrow P4 and P4 \rightarrow P1$,it is obvious that P1 and P4 can send information to each other).how may we interpret the matrix.

A^2= $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$?

Denoting the elements of A^2 by (a^2)ij, we obtain for example

(a^2)32=$a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} + a_{34}a_{42}$ =1+ 0+0+1=2

And this result shows that the person P3 can send information to P2 in two stages by two ways P3→$P1$ ^$P1 \rightarrow P2$(because $a_{31}a_{12=1}$)and P3→$P4$^$P4 \rightarrow P2$ (because $a_{34}a_{42}$=1). Similarly, since (a^2)14=0,there is no possibility to send information from P1 to P4 in two stages (but it is possible directly, because a_{14}=1).

Hence, the elements (a^2)ij gives the no. Of ways in which the person Pi

Similarly,(a^3)ij represents the number of ways in which the person Pi can send information Pj in

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three stages

$$A^3 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

And thus for example $a^3)32 = (a^2)31a_{12} + (a^2)32a_{22} + (a^2)33a_{32} + (a^2)34a_{42} =$ 1+0+0+1 =2.

Hence, there are two ways to send information from P3 to P2 in three stages: P3→$p4^{P4\to P1^{P1\to P2}}$ (because $(a^2)31a_{12} = a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31} + a_{34}a_{41})a_{12} = (0+0+0+1).1 = 1)$ and P3→ $P1^{P1\to P4^{P4}\to P2}$(because$(a^2)34a_{42} = (a_{31}a_{14} + a_{32}a_{24} + a_{33}a_{34} + a_{34}a_{44})a_{42}$= (1+0+0+0).1=1).

In general , the no. Of ways in which Pi can send information to Pj in at most k stages is given by the element in the i-th row and j-th coloumn of the matrix A+$A^2 + A^3 + \cdots + A^k$.

Thus in the above example we deduce from the matrix

$$A+A^2 + A^3 = \begin{pmatrix} 2 & 3 & 2 & 3 \\ 2 & 2 & 1 & 1 \\ 3 & 4 & 2 & 3 \\ 3 & 3 & 2 & 2 \end{pmatrix}$$

That for example there are four ways in which P3 can send information to P2 in at most three stages.

Example 4: Three people (denoted by P1, P2, P3) organised in a simple closed society produce three commodities *Z1, Z2, Z3.each* person sells and buys from each other. All their products are consumed by them, no other commodities enter the system (the "closed model ").the proportions of the products consumed by each of P1, P2, P3 are given in the following table:

	Z1	Z2	Z3
P1	0.6	0.2	0.3
P2	0.1	0.7	0.2
P3	0.3	0.1	0.5

For example, the first column lists that 60% of the produced commodity Z1 are consumed by P1, 10% by P2 and 30% by P3 . Thus, it is obvious that the sum of elements in each column is equal to 1.

Let us denote X1,X2,X3 the incomes of the persons P1,P2,P3 .then the amount spent by P1 on Z1 ,Z2 ,Z3 is 0.6 X1+0.2X2+0.3x3=X1,similarly for the other persons .we obtain the system of linear equations

0.6X1+0.2X2+0.3X3 = X1

0.1X1+0.7X2+0.2X3 =X2

0.3X1+0.1X2+0.5X3 = X3

This system can be re written as the equation AX=X, where

A = $\begin{pmatrix} 0.6 & 0.2 & 0.3 \\ 0.1 & 0.7 & 0.2 \\ 0.3 & 0.1 & 0.5 \end{pmatrix}$ and X= (X1,X2,X3)^T

Moreover, we assume the income to be non negative i.e., xi ≥0 for i = 1,2,3(we denote it X ≥0).

We can rewrite this equation into the equivalent form (A-I)X=0

$$\left(\begin{array}{ccc|c} -0.4 & 0.2 & 0.3 & 0 \\ 0.1 & -0.3 & 0.2 & 0 \\ 0.3 & 0.1 & -0.5 & 0 \end{array}\right)$$

An arbitrary solution of the system has the form X= t (13, 11, 10)^t and it is X ≥0 ,for t≥0.

Thus to ensure that this society survives, the persons P1, P2, P3 have to have their incomes in the proportions 13:11:10.

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APPLICATIONS OF MATHEMATICS IN ECONOMICS

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ABSTRACT
Mathematics has been in use in arriving at true economic theory and in solving many economic problems of a state or a country, right from the 19th century. In the process of using mathematics to prove a number of theorems, logic and rationale help very much. Mathematics, when applied to economics, explains (1) the causes of rise in the prices of different commodities, (2) the causes of inflation and (3) the causes of rise in unemployment. Also, mathematics explains them reasonably and convincingly. Economic analysis uses quantitative methods to obtain specific information in an economy. Quantitative methods are, generally, made known in the form of mathematical calculations. With the help of these mathematical calculations, economists compare the current economic situation with the economic situation of previous years. Further, economists use various types of mathematical calculations to support their theories or assumptions or inferences. Economists mainly use calculus (a type of mathematics) in economics. Calculus includes the use of many formulae to measure derivatives, functions and limits. Differential calculus is the specific measuring of a derivative that relates to a specific function. In short, mathematics has been playing a key role in predicting the economic conditions of people at large and in improving their lot. In view of the importance of mathematics, it is useful to the students of economics to take a course in mathematics in order to achieve excellence in economics and contribute their mite to the nation.

Keywords: Applications, mathematics, economics, calculus, problems, uses

> Stop trying to memorise the formulae. You need to understand them and learn how to apply them in a given situation. The best way to practise maths is to solve the questions in a time-bound manner.
>
> -Payal Krishnan, a career expert (*THE HINDU, EDUCATION PLUS,* 13 Jan. 2014)

INTRODUCTION

Mathematics, which is considered the backbone of modern economics, has been in use in the economic life of the public, at large, even from 17th century. The applications of mathematics in economics began in 19th century. Calculus, an important tool of mathematics, was used to explain the economic matters related to the public. The number of applications of mathematics in economics increased in 20th century. Mathematics occupies a significant place in the economic development of a household or a state or a country, in 21st century. Mathematical tools such as differential equations, set theory and graph theory have been employed in economics, just as mathematical models were used in physics. This development comes to be known as an important move from mechanics to axiomatics. In fact, what is mathematics? What is economics? What is mathematical economics? What is Agent-based Computational Economics (ACE)? What is the advantage of the approach to mathematical economics? As a language, is mathematics an easy subject? When will the language of mathematics be understood and appreciated by many people. In the past, what is the role of mathematics? What are the applications of mathematics in economics? What is the conclusion? What is needed today? All these significant and current issues

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are discussed, in detail, hereunder.

What is mathematics?

Mathematics is the study of relationships among quantities, magnitudes and properties. It can also be said that mathematics is "the systematic treatment of magnitude, relationships between figures and forms, and relations between quantities expressed symbolically".

What is Economics?

Economics is a discipline that deals with production, distribution and consumption of goods and services.

What is Mathematical Economics?

Mathematical economics is the application of mathematical methods to present theories and analyse problems in economics. Mathematical economies refers to mathematical methods like, differential calculus , integral calculus, differential equations, algebra, matrix and mathematical programming .

What is Agent- based Computational Economics (ACE) ?

Agent-based Computational Economics is that which studies economic processes as dynamic systems of interaction of agents over time. In corresponding agent-based models, agents are not real people but "computational objects modelled as interacting, according to rules" ... "whose micro-level interactions create emergent patterns" in space and time. The rules are formulated to predict behaviour and social interactions based on incentives and information. "The theoretical assumption of mathematical optimisation by agents markets is replaced by the less restrictive postulate of agents with bounded rationality adopting to market forces". The final scientific objective of the method is described as testing "theoretical findings against real-world data in ways that permit empirically supported theories to cumulate over time, with each researcher's work building appropriately on the work that has gone before".

What is the advantage of the approach to mathematical economics?

The advantage of the approach to mathematical economics is that it makes possible formulation of theoretical relationships with accuracy and simplicity. Further, economists will be able to make meaningful propositions and also propositions that can be tested. Furthermore, economists will be able to present significant decisions about controversial subjects.

As a language, is mathematics an easy subject? When will the language of mathematics be understood and appreciated by many people.

As a language, mathematics is not an easy subject. Only when the language of mathematics is simple, it will be understood and appreciated by many people.

In the past, what is the role of mathematics?

In the past, the role of mathematics is limited to physical sciences such as physics and chemistry.

What are the applications of mathematics in economics?

The following are some of the applications of mathematics in economics.

(1) mathematics is used in planning the economic development of family members in a household.

(2) mathematics is used in performing the business of a firm or a company on sound lines, with reasonable profit.

(3) mathematics is used in the formulation of economic policies of a state or a country.

(4) mathematics in combination with the economic concepts helps to understand the economic theory of a particular organisation, clearly and precisely.

(5) mathematics helps to comprehend macroeconomics, and microeconomics and predict the economic growth and inclusive growth of a nation.

(6) the knowledge of statistics and statistical tools is very helpful in studying and applying economic theories.

(7) differential calculus is applied to original supply and demand models to ascertain how different factors affect supply and demand in the public or private markets.

(8) mathematical and economic models can be used to prove or disprove different economic theories or inferences.

(9) mathematical and economic calculations help persons in business field to know about sales of their products and take appropriate decisions regarding business operations.

(10) large samples of mathematical and economical information help in reducing the number of difficulties that arise because of errors in information.

Conclusion

In this age of globalisation, economics has become more mathematical as a discipline. Today, mathematical models are helping in matters of (1) forecasting the economic situation of a country, (2) decision-making related to people's welfare and (3) solving problems regarding allocation of funds to various departments in central government and state governments. **What is needed today is:** (1) to be able to derive benefits from applications of mathematics in economics and improve the conditions of people at large (2) to remember the words of Rajiv Gandhi, former Prime Minister of India about mathematics: Mathematics is the underpinning of the physical sciences and technology. The teaching of mathematics requires no great capital investment as in the other sciences. Its present neglect is a matter of great concern to me. and (3) to give due importance to the words of Dr. S.B. Rao, Director of C.R. Rao Advanced Institute of Mathematics, Statistics and Computers (AIMSCS) about learning mathematics along with other subjects: For those, who want to make it big in mathematics, I would say that like in any subject, they need to work really hard. Learning mathematics with combination of other subjects is the way forward. (M. Sai Gopal. *THE HINDU EDUCATION PLUS* 2 Jan. 2012).

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ANALYSIS OF A REPAIRABLE SYSTEM SUBJECTED TO TWO MODES OF FAILURE

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ABSTRACT
Systems deteriorate over time and hence require maintenance. This paper discusses the operational performance of a repairable system subjected to two modes of failure. Unlike many models in the literature , where after each failure the system is perfectly repaired (replaced) , it is assumed here that the system performance is declined after each failure , i.e., the performed repairs are imperfect. This study presents a model for a system that is subjected to failure. At each failure it is imperfectly repaired resulting in a step wise increasing failure rate of the process. Thus imperfect repairs can be taken in order to approximate a continuous degradation of the process combined with a step wise improvement of the process after each repair.It presents a closed form expression for the long term probability that the system operates in the operational state.
KEYWORDS: Operational probability, two modes of failure, a repairable system, imperfect repair

INTRODUCTION

Systems are subjected to failure if not periodically and properly maintained. If failed they have to be repaired. The type of repair conducted defines the performance of the system. Perfect repair brings the system to an as good as new state while , minimal repair makes the system as good as it was immediately before failure. In imperfect repair which is the topic of this work , the system becomes inferior after ech repair. In this paper, a system with two-failure mode is examined.

Imperfect repair is investigated in the literature , but not as extensively as perfect repair. Sethi[13] used a Markov decision approach to solve a two component discrete time problem with arbitrary failure rates, while Berg [1] solved the continuous time version of the problem. Ross [12] addressed a problem where the failure rates of components depend on the components still working. Jack [4] considered a parallel system of two non-identical components, where failure of one operating component changed the failure rate of the other operating component. Moustafa [8] considered f failure modes and multiple repair facility while presenting a Markov model for the transient analysis of reliability of k-out-of-N: general repair systems with and without repair; M identical and independent components in the system were assumed, when components had either working or failed states. The reliability and the time between failures of the system were derived. Nakagawa [10] also studied a system with two types of failures. A minimal repair corrected Type 1 failure, while Type 2 failure necessitated a system replacement. The system was either replaced at the kth Type 1 failure , or at the first Type 2 failure, whichever occurred first. Love and Guo [7] analyzed three types of repairs namely, perfect, imperfect and minimal using the exponential power distribution in order to develop a bathtub failure and repair model for the systems. They considered imperfect repairs to be more probable and analyzed the data of a hydroelectric turbine using the model. It was discovered that an exponential power distribution gave a bathtub failure model when used with perfect repair.

Ohashi and Nishida [11] presented a two component series system and allowed probabilistic dependencies between the transition states of the two components. A numerical example was presented.. Billinton and Pan [13] developed system of equations to evaluate the failure rate and frequency of failure of a parallel system with two identical components, and found the optimum maintenance intervals that minimized the total failure frequency of the system. Levitin [6] presented an algorithm for finding optimal series-parallel topology for a system with two failure modes. The proposed algorithm is based on a universal moment generating function. Yokota et.al.[15] studied a system with several failure modes. Genetic algorithm was used to find the optimal reliability of the system. Tolio, et. Al.[14] analyzed a two machine system , where each machine could fail in multiple failure modes. The objective is to evaluate the performance of a production line with unreliable machines with finite buffer. Times to failure were assumed to be geometrically distributed. Levantesi et. Al., [5] investigated the performance evaluation of production lines with deterministic processing times, multiple failure modes, and finite buffer capacity.Mohammed Hajeeh [9] analyzed the analysis of a repairable system subjected to multiple modes of failure.

IMPERFECT REPAIR: This study presents a model for a system that is subjected to failure. At each failure it is imperfectly repaired resulting in a step wise increasing failure rate of the process. Thus imperfect repairs can be taken in order to approximate a continuous degradation of the process combined with a step wise improvement of the process after each repair. With imperfect repairs, the failure rate increases after each repair. Similarly, the repair rate decreases over time, i.e., the repair times get on an average longer with the number of repairs performed. After a specified number of repairs the system is replaced by a new one, which starts the next cycle.

All the models in this work are based on the assumptions that the time between failures and repair times are random and exponentially distributed with failure rates λ_i and repair rates μ_i, respectively. Moreover, it is assumed that the travel times from/to repair facility are negligible, one repair man (a single repair channel), and the system becomes as good as new after each replacement. The long term (steady state) behavior of the system is studied, and the probability of being at each state is derived.

1.MULTIPLE MODES OF FAILURE: There are many examples of systems that may be in one of several modes. Electrical systems undergoing open circuit or short circuit failures are an example of two modes of failure model. The possible ways through which a piping system could fail include pipe rupture, pipe clogging, and pipe leakage which is an example of several modes of failure for a system. Here it is assumed that the system can fail in one of M different modes of failure, where a failure is due to mode m with probability p_m for m=1to M. The probabilities p_m sum up to 1.

$$\sum_{m=1}^{M} p_m = 1 \qquad (1)$$

The operation time of the system is divided into cycles, i.e., the times between successive replacements (renewals).There are two types of states, operational (not failed) and failed states. The system starts new at state 1, after sometimes of operation, it fails with failure rate λ_1 and enters with probability p_m state (2,m), m=1toM.The system is repaired with repair rate $\mu_{1,m}$ and enters the operational state 3. After n failures have occurred, the system enters the failure state (2n+2,m) and is subsequently replaced by a new state with replacement rate μ_{n+1}which ends the cycle and the system starts again in the operational state1.

The following notations are used:

M = Number of failure modes.

λ_i: Actual failure rate after exactly i-1 failures without replacement.

$\mu_{i,m}$: Repair rate of ith repair and effective failure mode m with i = 1. . . . n and m=1. .M.

μ_{n+1}:replacement rate

p_m:Probability of failure mode m on condition of failure, m=1 . . M.

π_j : The long run probability of the system being in the operational state j , j= 2v+1 with v=0,1, ... n

$\pi_{j,m}$: The long run probability of the system being in the failed state j due to failure mode m, j=2v with v=1 . . n+1, m= 1 . . M.

k = Number of times the system has been imperfectly repaired, where k=0 means the system is new, k=1 means the system has been repaired once, and so on.

n = Number of repairs carried out on the system before a replacement by a new one.

The system states and transitions are illustrated in Figure 1. The operational states arerepresented by rectangles and have the uneven numbers 2j+1, j=0, 1, . . . n. The ellipses represent the failed states, they are numbered by the even numbers and the mode of failure, i.e., (2j+2,m) , j= 0, 1, . . .n and m= 1 . .M. For example , the failed state (4,1) means the system has failed the second time (j=2) and the failure is due to failure mode m=1.

The total number of states in the system depends on both the number of repairs carried out and the number of failure modes considered. It is equal to (n+1)(M+1). Figure 1 provides a graphical representation of the system states and the possible transitions.

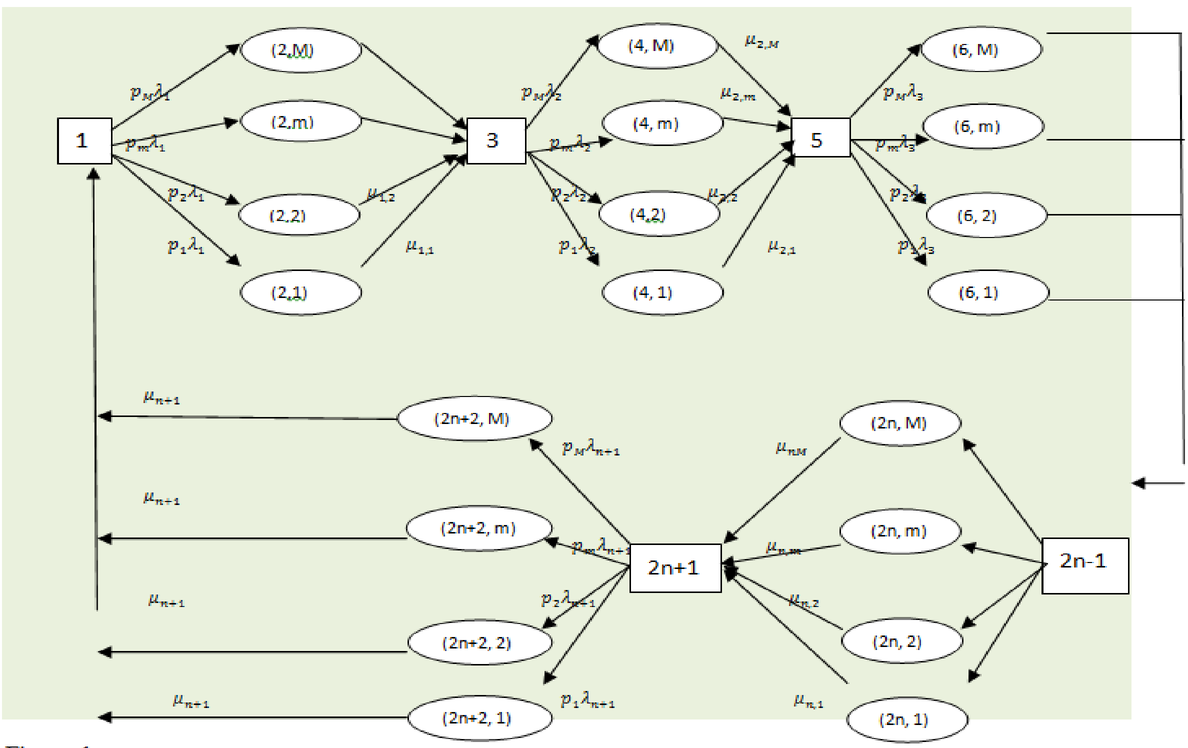

Figure 1

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ANALYSIS OF THE SYSTEM: The steady state equations for the system are as follows

State (1) : $\mu_{n+1}\sum_{m=1}^{M}\pi_{2n+2,m} = \lambda_1\pi_1$ (2)

State (2i, m) : $\lambda_i p_m \pi_{2i-1} = \mu_{i,m}\pi_{2i,m}$ for m=1 . .M and i=1 . . n (3)

State (2i+1) : $\sum_{m=1}^{M}\mu_{i,m}\pi_{2i,m} = \lambda_{i+1}\pi_{2i+1}$ for i=1 . . n (4)

State (2n+2, m) : $\lambda_{n+1}p_m\pi_{2n+1} = \mu_{n+1}\pi_{2n+2,m}$ for m=1 . .M (5)

From the above equations , the steady state probabilities are easily obtained:

$$\pi_{2j+1} = \frac{\sum_{m=1}^{M}\mu_{j,m}\pi_{2j,m}}{\lambda_{j+1}} \quad \text{for } j= 1 . . n \tag{6}$$

$$\pi_{2j,m} = \frac{p_m\lambda_j}{\mu_{j,m}}\pi_{2j-1} \quad \text{for m=1 . .M and j= 1 . . n} \tag{7}$$

$$\pi_{2n+2,m} = \frac{p_m\lambda_{n+1}}{\mu_{n+1}}\pi_{2n+1} \quad \text{for m=1 . .M} \tag{8}$$

Similarly , the long run probabilities of the different system states can be expressed in terms of the probability of being in state 1.

$$\pi_{2j+1} = \frac{\lambda_1}{\lambda_{j+1}}\pi_1 \quad \text{for } j= 1 . . n \tag{9}$$

$$\pi_{2j,m} = \frac{p_m\lambda_1}{\mu_{j,m}}\pi_1 \quad \text{for m=1 . .M and j= 1 . . n} \tag{10}$$

$$\pi_{2n+2,m} = \frac{p_m\lambda_1}{\mu_{n+1}}\pi_1 \quad \text{for m=1 . .M} \tag{11}$$

In order to find the long-run operational probability, the steady state probability π_1 has be derived based on the overall sum of steady state probabilities.

$$\pi_1 + \sum_{j=1}^{n}\sum_{m=1}^{M}\pi_{2j,m} + \sum_{j=1}^{n}\pi_{2j+1} + \sum_{m=1}^{M}\pi_{2n+2,m} = 1 \tag{12}$$

Substituting the expressions (9), (10), (11) into (12), the following equation for π_1 is obtained.

$$\pi_1 + \sum_{j=1}^{n}\sum_{m=1}^{M}\frac{p_m\lambda_1}{\mu_{j,m}}\pi_1 + \sum_{j=1}^{n}\frac{\lambda_1}{\lambda_{j+1}}\pi_1 + \sum_{m=1}^{M}\frac{p_m\lambda_1}{\mu_{n+1}}\pi_1 = 1 \tag{13}$$

From which we obtain the following expression for π_1.

$$\pi_1 = \frac{1}{\lambda_1\left(\sum_{m=1}^{M}p_m\sum_{j=1}^{n}\frac{1}{\mu_{j,m}} + \sum_{j=1}^{n+1}\frac{1}{\lambda_j} + \frac{1}{\mu_{n+1}}\right)} \tag{14}$$

The long run probability of being operational is given by the sum of the corresponding probabilities:

$$\sum_{j=0}^{n}\pi_{2j+1} = \pi_1\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \cdots \frac{\lambda_1}{\lambda_{n+1}}\right) = \pi_1\lambda_1\sum_{j=1}^{n+1}\frac{1}{\lambda_j} \tag{15}$$

Substituting (14) into (15) yields the long run operational probability for the system.

$$P_S\ (\text{operational}) = \frac{\sum_{j=1}^{n+1}\frac{1}{\lambda_j}}{\sum_{m=1}^{M}p_m\sum_{j=1}^{n}\frac{1}{\mu_{j,m}} + \sum_{j=1}^{n+1}\frac{1}{\lambda_j} + \frac{1}{\mu_{n+1}}} \tag{16}$$

SPECIAL CASE

2. TWO MODES OF FAILURE: There are many examples of systems that may be in one of several modes. Electrical systems undergoing open circuit or short circuit failures are an example of two modes of failure model.). Figure 2 provides a graphical representation of the system states and the possible transitions.

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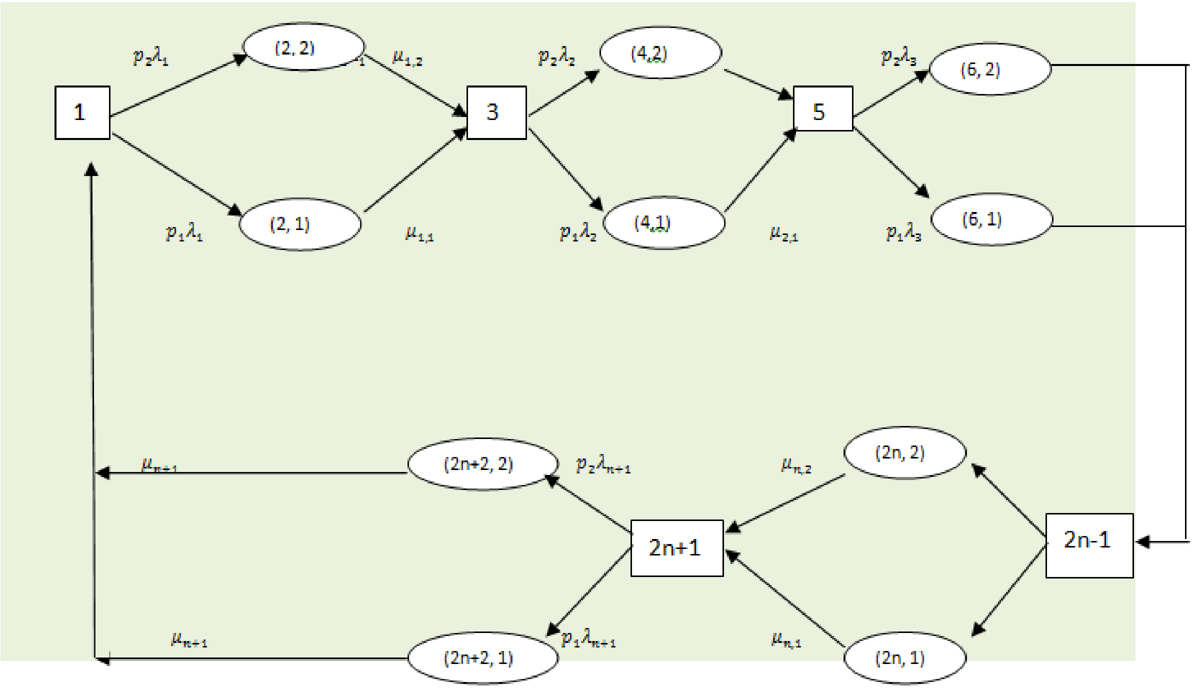

Figure 2

ANALYSIS OF THE SYSTEM: The steady state equations for the system are as follows

State (1) : $\mu_{n+1}\sum_{m=1}^{2}\pi_{2n+2,m} = \lambda_1\pi_1$ (1)

State (2i, m) : $\lambda_i p_m \pi_{2i-1} = \mu_{i,m}\pi_{2i,m}$ for m=1 . .2 and i=1 . . n (2)

State (2i+1) : $\sum_{m=1}^{2}\mu_{i,m}\pi_{2i,m} = \lambda_{i+1}\pi_{2i+1}$ for i=1 . . n (3)

State (2n+2, m) : $\lambda_{n+1}p_m\pi_{2n+1} = \mu_{n+1}\pi_{2n+2,m}$ for m=1 . .2 (4)

From the above equations , the steady state probabilities are easily obtained:

$$\pi_{2j+1} = \frac{\sum_{m=1}^{2}\mu_{j,m}\pi_{2j,m}}{\lambda_{j+1}} \quad \text{for } j= 1 . . n \tag{5}$$

$$\pi_{2j,m} = \frac{p_m\lambda_j}{\mu_{j,m}}\pi_{2j-1} \quad \text{for } m=1 . .2 \text{ and } j= 1 . . n \tag{6}$$

$$\pi_{2n+2,m} = \frac{p_m\lambda_{n+1}}{\mu_{n+1}}\pi_{2n+1} \quad \text{for } m=1 . .2 \tag{7}$$

Similarly , the long run probabilities of the different system states can be expressed in terms of the probability of being in state 1.

$$\pi_{2j+1} = \frac{\lambda_1}{\lambda_{j+1}}\pi_1 \quad \text{for } j= 1 . . n \tag{8}$$

$$\pi_{2j,m} = \frac{p_m\lambda_1}{\mu_{j,m}}\pi_1 \quad \text{for } m=1 . .2 \text{ and } j= 1 . . n \tag{9}$$

$$\pi_{2n+2,m} = \frac{p_m\lambda_1}{\mu_{n+1}}\pi_1 \quad \text{for } m=1 . .2 \tag{10}$$

In order to find the long-run operational probability, the steady state probability π_1 has be derived based on the overall sum of steady state probabilities.

$$\pi_1 + \sum_{j=1}^{n}\sum_{m=1}^{2}\pi_{2j,m} + \sum_{j=1}^{n}\pi_{2j+1} + \sum_{m=1}^{2}\pi_{2n+2,m} = 1 \tag{11}$$

Substituting the expressions (8), (9), (10) into (11), the following equation for π_1 is obtained.

$$\pi_1 + \sum_{j=1}^{n}\sum_{m=1}^{2}\frac{p_m\lambda_1}{\mu_{j,m}}\pi_1 + \sum_{j=1}^{n}\frac{\lambda_1}{\lambda_{j+1}}\pi_1 + \sum_{m=1}^{2}\frac{p_m\lambda_1}{\mu_{n+1}}\pi_1 = 1 \tag{12}$$

From which we obtain the following expression for π_1.

$$\pi_1 = \frac{1}{\lambda_1\left(\sum_{m=1}^{2} p_m \sum_{j=1}^{n} \frac{1}{\mu_{j,m}} + \sum_{j=1}^{n+1} \frac{1}{\lambda_j} + \frac{1}{\mu_{n+1}}\right)} \tag{13}$$

The long run probability of being operational is given by the sum of the corresponding probabilities:

$$\sum_{j=0}^{n} \pi_{2j+1} = \pi_1\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \cdots \frac{\lambda_1}{\lambda_{n+1}}\right) = \pi_1 \lambda_1 \sum_{j=1}^{n+1} \frac{1}{\lambda_j} \tag{14}$$

Substituting (13) into (14) yields the long run operational probability for the system.

$$P_S \text{ (operational)} = \frac{\sum_{j=1}^{n+1} \frac{1}{\lambda_j}}{\sum_{m=1}^{2} p_m \sum_{j=1}^{n} \frac{1}{\mu_{j,m}} + \sum_{j=1}^{n+1} \frac{1}{\lambda_j} + \frac{1}{\mu_{n+1}}} \tag{15}$$

CONCLUSION: Most maintenance problems investigated in literature are based on maintenance using perfect repair. This work attempts to study imperfect repair phenomena which are closer to real life situations. The expression which is derived by (15) is general and includes the expressions for perfect repair , one and two modes of failure and for imperfect repair.

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DATA SECURITY USING MODULAR OPERATION: A NUMBER THEORETIC TRANSFORM

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ABSTRACT

Because of the specialized advancement in the field of Electronics, the ideas of cryptography and cryptology are these days utilized all the more comprehensively; the objectives of cryptography now cover all parts of security in preparing, transmission and utilization of data within the sight of a rival. Along these lines, cryptographic techniques have entered numerous different zones. One can utilize them to guarantee confidentiality in any sort of electronic correspondence. They are utilized for verification while opening an auto or discharging an immobilizer, pulling back cash with a bank card or distinguishing oneself at an boarder with a visa, for instance. Archives are these days frequently marked carefully with cryptographic strategies, for instance by a public accountant; like this the non-disavowal of understandings can be ensured. With advanced marks one can likewise ensure the respectability of electronic information, that will be, that the information has not been altered. The most imperative regular operation for each vital operations incorporate a polynomial modulo multiplication for large integers, which can be gotten by performing integer transforms, otherwise called number theoretic transform. In this paper, we embrace and alter a calculation for figuring enormous integer multiplication.

Key words- Cryptography, security, data transmission, number theoretic transform.

1. INTRODUCTION

Cryptography or cryptology "shrouded mystery" is the practice and investigation of methods for secure correspondence within the sight of outsiders called foes. For the most part, cryptography is about developing and breaking down conventions that avert outsiders or people in general from perusing private messages. Different perspectives in data security, for example, information classification, information respectability, confirmation, and non-revocation are integral to current cryptography. Uses of cryptography incorporate ATM cards, PC passwords and so on.

In cryptography, encryption is the way toward encoding messages or data in a manner that lone approved gatherings can read it. Encryption does not avoid capture attempt, but rather denies the message substance to the interceptor. In an encryption plan, the expected correspondence data or message, alluded to as plaintext, is encoded utilizing an encryption calculation, producing ciphertext that must be perused if decoded. An approved beneficiary can without much of a stretch decode the message with the key gave by the originator to beneficiaries, yet not to unapproved interceptors.

The motivation behind encryption is to guarantee that exclusive some person who is approved to get to information (e.g. an instant message or a record), will have the capacity to peruse it, utilizing the decryption key, some individual who is not approved can be prohibited.

2. **Utilization of encryption and decryption :**Encryption is the way toward interpreting plain content information (plaintext) into something that gives off an impression of being irregular and inane (ciphertext). Decoding is the way toward changing over ciphertext back to plaintext

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2.1 **Modular Arithmatic:** Measured number-crunching is fundamentally doing expansion (and different operations) - the qualities "wrap around", continually remaining not exactly a settled number called the modulus.

To discover, for instance, 39 modulo 7, you basically compute 39/7 (= 5 4/7) and take the rest of. For this situation, 7 partitions into 39 with a rest of 4. Subsequently, 39 modulo 7 = 4. Take note of that the rest of (partitioning by 7) is constantly under 7. In this way, the qualities "wrap around," as should be obvious beneath:

0 mod 7=0	6 mod 7=6
1 mod 7=1	7 mod 7=0
2 mod 7=2	8 mod 7=1
3 mod 7=3	9 mod 7=2
4 mod 7=4	10 mod 7=3
5 mod 7=5	etc.

To do modular addition, you first add the two numbers normally, then divide by the modulus and take the remainder. Thus (17+20) mod 7 = (37) mod 7 = 2.

Modular arithmetic is not unfamiliar to you;for example, when you would have to get up in the morning if you want to get a certain number of hours of sleep. Say you're planning to go to bed at 10 PM and want to get 8 hours of sleep. To figure out when to set your alarm for, you count, starting at 10, the hours until midnight (in this case, two). At midnight (12), you reset to zero (you "wrap around" to 0) and keep counting until your total is 8. The result is 6 AM. What you just did is to solve (10+8) mod 12. As long as you don't want to sleep for more than 12 hours, you'll get the right answer using this technique. What happens if you slept more than 12 hours?

3. Definition of Key: The following variables are determined from the given safety parameter μ.
δ :bit length of the secret key, ε : bit length of the error , κ : bit lengths of public keys $\mathbf{x_i}$ that include errors , $\boldsymbol{\tau}$: number of public keys $\mathbf{x_i}$; γ' : bit length of the secondary error used in encryption.

Select an odd number ε of an arbitrary bit length δ.

Determine integers x_i that will be used as public keys for $\mathbf{0} \leqq \mathbf{i} \leqq \boldsymbol{\tau}$ as below , where $\mathbf{q_i}$ is arbitrary integer smaller than $\mathbf{s}^{\kappa-\delta}$, $\mathbf{r_i}$ is error values that satisfy $\mathbf{s}^{-\varepsilon} \leqq \mathbf{i} \leqq \mathbf{2}^{\varepsilon}$

$\mathbf{x_i = q_i * p + r_i}$

Rearrange the selected x_i in order of size so that $\mathbf{x_0}$ becomes the largest value, where x_0 is assumed to be an odd number that satisfies

$\mathbf{x_0 (mod\ p) = even}$.

The final public keys are $(x_0, x_1, \cdots, x_\tau)$ and the secret key is p.

4. **Encryption stage:** Plaintext m is assumed to be a value of 0 or 1. This is, $m \in \{0,1\}$.

Arbitrarily select j that satisfies $1 \leqq j \leqq \tau$ and arbitrarily select j pieces of elements from among public keys $(x_1,\cdots,x_\tau)$. Assume the selected elements to be $(x_1',\cdots,x_j')$.

In addition, select an arbitrary random number r with bit length .

Finally calculate the cryptogram for m as shown below

$C = (m+2r+2\Sigma_{0 \leqq i \leqq j}\ x_i') \bmod x_0$.

5. **Decryption stage :** Decrypt the given cryptogram c as shown by equation 4 below using secret key p.

$$m = (c \bmod p) \bmod 2$$

In the encryption organize, plain content m is covered up by the total of arbitrary number r and self-assertively chose open keys. Since open keys xi have been chosen as xi = qi∗p + ri individually, the qi∗p parts gets to be distinctly terminated by the estimation of (c mod p), which is the primary phase of the procedure of decryption and just the mistakes parts ri remains . On audit of the encryption organize, it can be seen that all these blunder parts have been increased by steady 2. Hence every one of them is wiped out by the mod 2 operation, which is the second phase of decryption.

In addition, operations of cryptograms $c_1=m_1 + 2r_1 + 2\Sigma x_i \,(\text{mod } x_0)$ and $c_2 = m_2 + 2r_2 + 2\Sigma x_i \,(\text{mod } x_0)$ for plain text m_1, m_2 can be performed.

$C_1 + C_2 = m_1 + 2r_1 + 2 \sum x_i \,(\text{mod}x_0) + m_2 + 2r_2 + 2\sum x_i \,(\text{mod}x_0)$

$= (m_1+m_2) + 2\,(r_1+r_2) + 2\sum x_i \,(\text{mod}x_0)$

$C_1 * C_2 = (m_1+2r_1 + 2\sum x_i \,(\text{mod}x_0))* (m_2+2r_2 + 2 \sum x_i \,(\text{mod}x_0))$

$= (m_1*m_2) + 2\,(m_2r_1 + m_1r_1 + m_1r_2 + \sum a_ix_i)\,(\text{mod}x_0)$

6. RESULT AND DISCUSSION

Everyday life contains a scope of circumstances where the utilization of cryptography encourages the arrangement of a protected administration: money withdrawal from an ATM, Pay TV, e - mail and document stockpiling utilizing Pretty Good Privacy (PGP) freeware, secure web perusing, and utilization of a GSM cell phone. PGP utilizes both symmetric and deviated cryptography and a two-level key chain of command in which symmetric session keys are utilized to secure information, and hilter kilter keys are utilized for both mark and the assurance of the symmetric session keys.

7. Conclusion

The most vital operation in executing encryption frameworks is secluded augmentation of substantial numbers surpassing one million bits. Duplication for huge whole numbers can be gotten by over and over performing Fourier change and whole number Fourier change, which is secluded lessening. In the present study, number Fourier change was enhanced to utilize the base limit of memory with the goal that it can be connected to different executions.

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A REVIEW ON BIOMATHEMATICS AND ITS APPLICATIONS

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Abstract
Mathematical models are vital instruments in essential logical research in numerous territories of science, including physiology, environment, advancement, toxicology, immunology, common asset administration, and preservation science. Biomathematics is the utilization of Mathematical models to comprehend wonders in science. Present day test science is great at dismantling natural frameworks (at all levels of association, from genome to worldwide supplement cycling), into parts sufficiently basic that their structure and capacity can be concentrated on in seclusion. Dynamic models are an approach to assemble the pieces back, with conditions that speak to the framework's parts, forms, and the structure of their associations. Models are likewise broadly used to integrate accessible data and give quantitative responses to handy inquiries. What measures can be utilized to invert the decrease in ocean turtle populaces, and how soon would we be able to tell on the off chance that they are working? In what capacity can research center trials on substance cancer-causing nature be scaled up to set safe presentation restrains on people? For inquiries like these, where it is alluring to foresee the result precisely before move is made, quantitative demonstrating is basic. This paper clarifies the significance of biomathematics and its noteworthiness significance.
Key words- Bio Maths, Dynamical systems,Cell Biology,Epidemiology,Bio technology,Ecology

1. INTRODUCTION

science, including science. Not at all like material science and science, science is not as a rule a science connected with mathematics. But since there are quantifiable parts of life science, mathematics assumes a basic part in better comprehension the common world. Numerical science is a field of research that looks at scientific representations of organic frameworks. Biomathematics is the usage of Mathematical models to grasp ponders in science. Here we will examine the diverse utilizations of mathematics in science and investigate a few cases. This paper elucidates the essentialness of biomathematics,role of science in science and some scientific applications to the field of science.

1.1 Importance

Numerical and hypothetical science is an interdisciplinary logical research field with a scope of utilizations in science, biotechnology, and medication. The field is likewise called numerical science or biomathematics. scientific science to stretch the numerical side, or hypothetical science to push the natural side. Numerical science goes for the scientific representation, treatment and displaying of natural processess, utilizing an assortment of connected numerical strategies and apparatuses. Biomathematics has both hypothetical and functional applications in organic, biomedical and biotechnology examine. For instance, in cell science, protein connections are regularly spoken to as "toon" models, which, albeit simple to envision, don't precisely portray the frameworks studied[1].

Numerical science may utilize math, likelihood hypothesis, insights, straight variable based math, unique algebra,graph hypothesis, logarithmic geometry, topology, dynamical frameworks, differential conditions and coding hypothesis. scientific regions, for example, certain strategies in insights, were produced as devices amid the lead of research into numerical science [2,3].

What is biomathematics

1. Biomathematics is the utilization of scientific models to comprehend wonders in science.
2. Biomathematics is the science that arrangements with the utilization of numerical techniques to the structure and elements of living creatures.
3. the utilization of mathematics to the investigation of natural frameworks and procedures.

1.2 Use Of Mathematics In The Field Of Biology

1) **Measuring Life**

Unlike physics and chemistry, biology is not usually a science associated with mathematics. But because there are quantifiable aspects of life science, mathematics plays a critical role in better understanding the natural world. Mathematical biology is a field of research that examines mathematical representations of biological systems. Imagine that you are a biologist studying butterfly migrations. You go into the field and count a sample population in a confined region and then multiply your sample numbers by the total geographical range to get a population estimate. You then go back to your lab and review other researcher's reports of butterflies over the span of their migration pattern and use vector calculations to predict their future path. Finally, you examine previous years' data on the butterfly numbers and location to establish a probable error margin for your prediction. At every step of this process, you depend upon mathematics to measure, predict, and understand natural phenomena!

2) In Epidemiology : A particular type of extremely simplified ecosystem is that composed by the interaction between viruses and the immune system. Epidemiology has been another important field in biology where mathematics has allowed a rich conceptual development. Let's consider, for example

Example1: The spread of smallpox within a human population. Epidemiologic models describe the temporal dynamics in the number of hosts infected and make predictions, such as the existence of a critical population size beyond which the diseases fade away. Similarly, epidemiologic models predict a critical fraction of hosts to be vaccinated for the eradication of the diseases. These thresholds are essentially identical to the extinction thresholds described above in the context of habitat loss. Thus, mathematics can guide the design of efficient vaccination programs.

Example 2: A notorious example of virus is the HIV, the infectious agent causing AIDS. These RNA-based viruses mutate at an incredible rate to escape the immune system. This is the reason of their high success. The immune system evolves to recognize the exact nature of an invader and to counterbalance it, but if the virus keeps changing, it escapes this defensive system.

2. Use of mathematical techniques in Biology: Some of the mathematical technique we use in different field of biology to analyse the consequence of the physical situations are mentioned below.

2.1 Chain Rule Applications[1].

i. Effect of rainfall on predators and prey: The population of lions, L, in Africa greatly depends on the population of their prey, P, which consists mainly of large mammals with a preference for zebras, impalas, wildebeest, buffalo, and warthogs.

The extension of vegetation, V , affects the population of prey, which feed on the vegetation The vegetation, and the vegetation if effected by the amount of rainfall, r. These dependencies can be as

$L = f(p), P = g(v) and\ V = h(r)$

$L = f(p) = \frac{1}{2}P^2$A change in the abundance of rainfall will affect the amount of vegetation. This change in vegetation will affect the population of prey which in turn will affect the population of lions on the reserve. Therefore, the amount of rainfall affects the lion population.

ii. Tumor growth: Tumors can differ significantly in shape. For example, an adenocarcinoma (a cancerous tumor originating in glandular tissue) tends to be more irregularly shaped while an adenoma (a benign tumor originating in glandular tissue) tends to be more regularly shaped (for example close to spherical).

Assume we're studying a tumor that is approximately spherical. A tumor grows such that its radius expands at a constant rate k. Determine the rate of growth of the volume of tumor when the radius is 5 millimeters.

iii. Carbon monoxide level: An environmental study of a certain community suggests that the average daily level of carbon monoxide in the air may be modeled by the formula

$$C(p) = \sqrt{0.5p^2 + 17}$$

2.2 Derivative Applications

Drug Sensitivity: It is extremely important for doctors to understand the characteristics of the drugs they prescribe to patients. The strength of the drug is given by R(M) where M measures the dosage, i.e. the amount of medicine absorbed in the blood, and the sensitivity of the patient's body to the drug is the derivative of R with respect to M. For a certain drug, the drug strength is described by

$R(M) = 2M\sqrt{10 + 0.5M}$

Growth of Bacteria: Suppose a droplet of bacterial suspension is introduced into a flask containing nutrients for the bacteria. The bacteria undergo cell divisions and the bacterial density is observed at intervals of time during a short period. The data is then fit to a model describing the bacterial density, N(t), observed at time t. Assume for three different types of bacteria, the growth rates are described by the following differential equations,

$$\frac{dN_1}{dt} = \frac{2N(t)}{t}, N(t) = t^2$$

$$\frac{dN_2}{dt} = 2N(t), N(t) = e^2 t$$

$$\frac{dN_3}{dt} = 0.2[1 + \cos(0.5t)]N(t), N(t) = e^{0.2t + 0.4\sin(0.5t)}$$

2.3 Optimization Applications: Blood Pressure: Blood pressure is the pressure exerted by circulating blood on walls of the arteries. the pressure varies periodically according to the formula

$p(t) = 90 + 15\sin(2.5\pi t)$

number of seconds since the beginning of a cardiac cycle. Fisheries Management: The fishing industry is interested in obtaining the largest yield while maintaining a permanent supply of fish. A differential equation describing the change in the fish population is given by $\frac{df}{dt} = G(x) - H(x)$, where G(x) describes growth of the fish population (the inflow) and H(x) is the harvesting function (the outflow). Assume logistic growth for the fish and assume a linear harvesting term. Therefore, $\frac{df}{dt} = rf\left(1 - \frac{f}{k}\right) - hf$ where r is the growth rate of the fish at low density, K is the carrying capacity, and h is the harvesting effort of the fishermen. We say the system has an equilibrium point whenever there is no change, hence, whenever the derivative is zero.

2.4 Integration Applications

Global Warming: The "greenhouse effect" is the rise in temperature that the Earth experiences because certain gases prevent heat from escaping the atmosphere. According to one

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study, the temperature is rising at the rate of 0.014t 0.4 degrees Fahrenheit per year, where t is the number of years since 2000. Given that the average surface temperature of the earth was 57.8 degrees Fahrenheit in 2000, predict the temperature in 2200.

Neurons: Neurons (nerve cells) provide most of the important functions of the nervous system, such as controlling muscle activity, sensing, thinking, remembering, and regulating glandular secretions. They consist of three basic parts: the dendrites, the cell body, and the axon. Neurons are responsible for transmitting information throughout the body. Electrical signals and chemical messengers are used in order for neurons to transmit and receive information. The dendrites receive information which is then passed down to the cell body and then travels down the axon. These pulse-like waves of voltage that travel along the axons are known as action potentials. Assume that the change in voltage v of a neuron with respect to time follows the differential equation$\frac{dv}{dt} = 1.0 + \frac{1}{1+0.2t} - e^{0.01t}$ over the course of 100 milliseconds, where t is measured in milliseconds and v in millivolts.

2.5 Partial Derivative Applications

Spruce Budworm: A parasitoid is an organism that attaches to or within a host during part of their development. Unlike parasites, parasitoids ultimately kill their hosts. The Nicholson-Bailey model is a frequently used model to describe the population dynamics of the host-parasitoid system, in which it is assumed that the number of parasitized hosts, denoted by Na, is given by $N_a = N[1 - e^{-bp}]$where N is the host density, P is the parasitoid density, and b is the searching efficiency of the parasitoid.

Conclusion

The exponentially expanding measures of natural information at all sizes of biological association, alongside practically identical advances in registering power, make the potential for researchers to build quantitative, prescient models of biological frameworks. Expansive achievement would change fundamental science, drug, agribusiness, and natural science. The fundamental push in science amid the coming decades will be toward an undeniably quantitative comprehension of natural capacity; the rate at which advance happens will rely on upon a more profound, powerful execution of quantitative strategies and a quantitative point of view inside the biological sciences.

The achievement of this change will depend to some degree on the creation and nurturance of a hearty interface amongst science and mathematics, which ought to wind up distinctly a top need of science approach. The strategy difficulties will be generous and multifaceted. The interface amongst science and mathematics is an interdisciplinary boondocks sprawling over a limitless breadth of scholarly landscape that is exceptionally various, vaguely stamped, and developing.

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EXISTENCE OF EVEN NUMBER POSITIVE SOLUTIONS FOR A COUPLED SYSTEM OF FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS WITH (*p, q*)-LAPLACIAN

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Abstract
In this paper we establish sufficient conditions for the existence of even number of multiple positive solutions for a coupled system of (*p, q*)– Laplacian fractional order boundary value problems

$$
\begin{gathered}
D_{0+}^{\beta_1}\left(\phi_p(D_{0+}^{\alpha_1}x(t))\right) = f_1\big(t, x(t), y(t)\big), \quad t \in (0,1),\\
D_{0+}^{\beta_2}\left(\phi_q(D_{0+}^{\alpha_2}y(t))\right) = f_2\big(t, x(t), y(t)\big), \quad t \in (0,1),\\
x^{(i)}(0) = 0, \quad i = 0, 1, \cdots, n-2, \quad x^{(n-2)}(1) = 0,\\
\phi_p\left(D_{0+}^{\alpha_1}x(0)\right) = 0, \quad \phi_p\left(D_{0+}^{\alpha_1}x(1)\right) = 0,\\
y^{(i)}(0) = 0, i = 0, 1, \cdots, n-2, \quad y^{(n-2)}(1) = 0,\\
\phi_q\left(D_{0+}^{\alpha_2}y(0)\right) = 0, \quad \phi_q\left(D_{0+}^{\alpha_2}y(1)\right) = 0,
\end{gathered}
$$

by an application of Avery–Henderson fixed point theorem

AMS Subject Classification: 34A08, 34B15, 34B18 , 34B27, 35J05.
Key words: Fractional derivative, boundary value problem, system, Green's function, positive solution.

1. INTRODUCTION

Differential equations deals with change, with flux, with flow and in par- ticular, with the rate at which the change takes place. Every living thing changes. The tides fluctuate over the course of a day, countries increase and diminish their stockpiles of weapons, the price of oil rises and falls. Nearly all the fundamental principles that govern physical processes of engineering interest are described by differential equations. Hence it is fair to say that the ability to analyze, solve and understand differential equations is fundamentally important for engineers and scientists.

Fractional calculus is an extension of classical calculus and deals with the generalization of integration and differentiation to an arbitrary real order. The history, definitions, theory and applications of fractional calculus are well laid out in the books by Miller and Ross [?], Oldham and Spanier [?], Podlubny [?], and Samko, Kilbas and Marichev [?].

A class of differential equations governed by nonlinear differential opera- tors appears frequently and generated by great deal of interest in studying special types of problems. In this theory, the most applicable operator is the classical p–Laplacian operator. These types of problems arise in math- ematical modeling of viscoelastic flows, turbulent filtration in porous media, biophysics, plasma physics, control theory, image processing, electromagnetic theory and chemical reaction design. For a detailed description on application of p–Laplacian operator, we refer [?] and for p–Laplacian fractional order boundary value problems, one can see [?, ?, ?, ?, ?, ?, ?, ?].

In [?], Chen, Liu and Hu studied the existence of solutions by coincidence degree for the p–Laplacian Caputo type fractional order differential equations

$$D_{0+}^{\beta}\Big(\phi_p\big(D_{0+}^{\alpha}x(t)\big)\Big) = f\big(t, x(t), D_{0+}^{\alpha}x(t)\big), \quad t \in (0,1),$$
$$D_{0+}^{\beta}x(0) = D_{0+}^{\alpha}x(1) = 0,$$

where $0 < \alpha, \beta < 1, 1 < \alpha + \beta \leq 2, \phi_p(s) = |s|^{p-2}s,\ p > 1, f$ is continuous, D_{0+}^{β} and D_{0+}^{α} are the standard Caputo type fractional order derivatives.

Chai [?] investigated the existence and multiplicity of positive solutions for a p–Laplacian fractional order boundary value problems

$$D_{0+}^{\beta}\Big(\phi_p\big(D_{0+}^{\alpha}x(t)\big)\Big) + f(t, x(t)) = 0, \quad t \in (0,1),$$
$$x(0) = 0, x(1) + \sigma D_{0+}^{\gamma}x(1) = 0, D_{0+}^{\alpha}x(0) = 0,$$

by means of a fixed point theorem in a cones.

Prasad and Krushna [?, ?, ?] derived sufficient conditions for the existence of positive solutions for a coupled system of p–Laplacian fractional order boundary value problems. Prasad, Krushna and Wesen [?] established the existence of at least three positive solutions for a system of (p,q)–Laplacian fractional order two-point boundary value problems.

Inspired by the aforementioned work, we establish the existence of even number of multiple positive solutions to a coupled system of (p,q)–Laplacian fractional higher order two-point boundary value problems

$$D_{0+}^{\beta_1}\Big(\phi_p\big(D_{0+}^{\alpha_1}x(t)\big)\Big) = f_1\big(t, x(t), y(t)\big), \quad t \in (0,1), \tag{1.1}$$

$$D_{0+}^{\beta_2}\Big(\phi_q\big(D_{0+}^{\alpha_2}y(t)\big)\Big) = f_2\big(t, x(t), y(t)\big), \quad t \in (0,1) \tag{1.2}$$

satisfying the boundary conditions

$$\left.\begin{array}{l} x^{(i)}(0) = 0, \quad i = 0, 1, \cdots, n-2, \quad x^{(n-2)}(1) = 0, \\ \phi_p\Big(D_{0+}^{\alpha_1}x(0)\Big) = 0, \quad \phi_p\Big(D_{0+}^{\alpha_1}x(1)\Big) = 0, \end{array}\right\} \tag{1.3}$$

$$\left.\begin{array}{l} y^{(i)}(0) = 0, i = 0, 1, \cdots, n-2, \quad y^{(n-2)}(1) = 0, \\ \phi_q\Big(D_{0+}^{\alpha_2}y(0)\Big) = 0, \quad \phi_q\Big(D_{0+}^{\alpha_2}y(1)\Big) = 0, \end{array}\right\} \tag{1.4}$$

where $\phi_p(s) = |s|^{p-2}s, p > 1, \phi_p^{-1} = \phi_q, \frac{1}{p} + \frac{1}{q} = 1$ and $\alpha_i \in (n-1, n), n \geq 3, \beta_i \in (1,2), f_i : (0,1) \times \mathbb{R}^2 \to \mathbb{R}^+$ is continuous, $D_{0+}^{\alpha_i}$ and $D_{0+}^{\beta_i}$ for $i = 1, 2$ are the standard Riemann–Liouville fractional derivatives by an application of Avery–Henderson fixed point theorem [?].

where $\phi_p(s) = |s|^{p-2}s, p > 1, \phi_p^{-1} = \phi_q, \frac{1}{p} + \frac{1}{q} = 1$ and $\alpha_i \in (n-1, n), n \geq 3, \beta_i \in (1,2), f_i : (0,1) \times \mathbb{R}^2 \to \mathbb{R}^+$ is continuous, $D_{0^+}^{\alpha_i}$ and $D_{0^+}^{\beta_i}$ for $i = 1, 2$ are the standard Riemann–Liouville fractional derivatives by an application of Avery–Henderson fixed point theorem [?].

The rest of the paper is organized as follows. In Section 2, we construct the Green functions for the associated linear boundary value problems (??), (??) and (??), (??) and estimate the bounds for these Green functions. In Section 3, we establish criteria for the existence of at least two positive solutions for the boundary value problem (??)–(??) by using an Avery–Henderson functional fixed point theorem [?]. And then we establish the existence of at least $2n$ positive solutions to the boundary value problem (??)–(??) for an arbitrary positive integer n. Finally as an application, we give an example to illustrate our results.

2 Green Functions and Bounds

In this sections we express the boundary value problems (??), (??) and (??), (??) in terms of the equivalent integral equations involving Green functions and in the meantime, we estimate the bounds of these Green functions.

Lemma 2.1 *Let $h_1(t) \in C[0,1]$. Then the fractional order differential equation,*

$$D_{0^+}^{\alpha_1} x(t) + h_1(t) = 0, \quad t \in (0,1) \tag{2.1}$$

satisfying

$$x^{(i)}(0) = 0, i = 0, 1, \cdots, n-2, \quad x^{(n-2)}(1) = 0, \tag{2.2}$$

has a unique solutions,

$$x(t) = \int_0^1 G_1(t,s) h_1(s) ds,$$

where $G_1(t,s)$ is the Green's function for the problem (??)–(??) and is given by

$$G_1(t,s) = \begin{cases} G_{11}(t,s), & 0 \leq t \leq s \leq 1, \\ G_{12}(t,s), & 0 \leq s \leq t \leq 1, \end{cases} \tag{2.3}$$

$$G_{11}(t,s) = \frac{t^{\alpha_1 - 1}(1-s)^{\alpha_1 - n + 1}}{\Gamma(\alpha_1)},$$

$$G_{12}(t,s) = \frac{t^{\alpha_1 - 1}(1-s)^{\alpha_1 - n + 1} - (t-s)^{\alpha_1 - 1}}{\Gamma(\alpha_1)}.$$

Proof: The proof of this result is based on the equivalence of (??) to the integral equation of the form:

$$x(t) = \frac{1}{\Gamma(\alpha_1)} \int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha_1}} ds + \sum_{j=1}^{j=n} \frac{b_j}{\Gamma(j-1)} t^{\alpha_1 - j}$$

From the boundary condition (??) we obtain $b_j = 0, j = 2, 3, \ldots, n$. Then

$$x(t) = \frac{1}{\Gamma(\alpha_1)} \int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha_1}} ds + b_1 t^{\alpha_1 - 1}$$

From $x^{(n-2)}(1) = 0$, we get $b_1 = \frac{1}{\Gamma(\alpha_1)} \int_0^1 \frac{h_1(s)}{(1-s)^{n-\alpha_1-1}} ds$.
Thus, the unique solutions of (??)-(??) is

$$\begin{aligned}
x(t) &= -\frac{1}{\Gamma(\alpha_1)}\int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}}ds + \frac{t^{\alpha-1}}{\Gamma(\alpha_1)}\int_0^1 \frac{h_1(s)}{(1-s)^{n-\alpha_1-1}}ds \\
&= -\frac{1}{\Gamma(\alpha_1)}\int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha_1}}ds + \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_0^t \frac{h_1(s)}{(1-s)^{n-\alpha_1-1}}ds + \frac{t^{\alpha-1}}{\Gamma(\alpha_1)}\int_t^1 \frac{h_1(s)}{(1-s)^{n-\alpha_1-1}}ds \\
&= -\frac{1}{\Gamma(\alpha_1)}\int_0^t \left((t-s)^{\alpha-1} - t^{\alpha_1-1}(1-s)^{\alpha_1-n+1}\right)h_1(s)ds + \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1)}\int_t^1 (1-s)^{\alpha_1-n+1}h_1(s)ds \\
&= \int_0^1 G_1(t,s)h_1(s)ds.
\end{aligned}$$

Lemma 2.2 *[?] Let $h_2(t) \in C[0,1]$. Then the fractional order differential equations,*

$$D_{0+}^{\beta_1}(\phi_p(D_{0+}^{\alpha_1}x(t))) = h_2(t), \quad t \in (0,1) \tag{2.4}$$

satisfying

$$\phi_p(D_{0+}^{\alpha_1}x(0)) = 0, \phi_p(D_{0+}^{\alpha_1}x(1)) = 0 \tag{2.5}$$

has unique solution,

$$x(t) = \int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)h_2(\tau)d\tau\Big)ds \tag{2.6}$$

where

$$H_1(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \le t \le s \le 1, \\ \dfrac{[t(1-s)]^{\beta_1-1} - (t-s)^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \le s \le t \le 1, \end{cases} \tag{2.7}$$

Lemma 2.3 *For $t \in I = [\frac{1}{4}, \frac{3}{4}]$, the Green's function $G_1(t,s)$ given in (??) satisfies the following inequalities*

$$\begin{array}{ll}
(W1) & G_1(t,s) \ge 0, \text{ for all } (t,s) \in [0,1]\times[0,1], \\
(W2) & G_1(t,s) \le G_1(1,s), \text{ for all } (t,s) \in [0,1]\times[0,1], \\
(W3) & G_1(t,s) \ge \left(\dfrac{1}{4}\right)^{\alpha_1-1} G_1(1,s), \text{ for all } (t,s) \in I\times[0,1].
\end{array}$$

For details refer [?].

Lemma 2.4 *For $(t,s) \in [0,1], \xi_1 \in (\frac{1}{4}, \frac{3}{4})$ the Green's functions $H_1(t,s)$ given in(??) satisfies the following inequalities*

$$\begin{array}{ll}
(Q1) & H_1(t,s) \ge 0, \\
(Q2) & H_1(t,s) \le H_1(s,s), \\
(Q3) & \min\limits_{t\in I} H_1(t,s) \ge \vartheta_1^*(s)H_1(s,s), \text{ for } 0 < s < 1,
\end{array}$$

where

$$\vartheta_1^*(s) = \begin{cases} \dfrac{[\frac{3}{4}(1-s)]^{\beta_1-1} - (\frac{3}{4}-s)^{\beta_1-1}}{[s(1-s)]^{\beta_1-1}}, & s \in (0,\xi_1], \\ \dfrac{1}{(4s)^{\beta_1-1}}, & s \in [\xi_1, 1). \end{cases}$$

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For details refer [?].

Lemma 2.5 *Let $g_1(t) \in C[0,1]$. Then the fractional order differential equation,*

$$D_{0+}^{\alpha_2} y(t) + g_1(t) = 0, \ t \in (0,1), \tag{2.8}$$

satisfying

$$y^{(i)}(0) = 0, \quad i = 0,1,...,n-2, \quad y^{(n-2)}(1) = 0, \tag{2.9}$$

has a unique solution

$$y(t) = \int_0^1 G_2(t,s) g_1(s) ds,$$

where $G_2(t,s)$ is the Green's function for the problem (??)-(??) and is given by

$$G_2(t,s) = \begin{cases} G_{21}(t,s), & 0 \le t \le s \le 1, \\ G_{22}(t,s), & 0 \le s \le t \le 1, \end{cases} \tag{2.10}$$

$$G_{21}(t,s) = \frac{t^{\alpha_2 - 1}(1-s)^{\alpha_2 - n + 1}}{\Gamma(\alpha_2)},$$
$$G_{22}(t,s) = \frac{t^{\alpha_2 - 1}(1-s)^{\alpha_2 - n + 1} - (t-s)^{\alpha_2 - 1}}{\Gamma(\alpha_2)}.$$

The proof is in similar way with ??.

Lemma 2.6 *[?] Let $h_2(t) \in C[0,1]$. Then the fractional order differential equations,*

$$D_{0+}^{\beta_2}(\phi_q(D_{0+}^{\alpha_2} y(t))) = g_2(t), \quad t \in (0,1) \tag{2.11}$$

satisfying

$$\phi_q(D_{0+}^{\alpha_2} y(0)) = 0, \quad \phi_q(D_{0+}^{\alpha_2} y(1)) = 0 \tag{2.12}$$

has unique solution,

$$y(t) = \int_0^1 G_2(t,s) \phi_p\Big(\int_0^1 H_2(s,\tau) g_2(\tau) d\tau \Big) ds \tag{2.13}$$

where

$$H_2(t,s) = \begin{cases} \frac{[t(1-s)]^{\beta_2 - 1}}{\Gamma(\beta_2)}, & 0 \le t \le s \le 1, \\ \frac{[t(1-s)]^{\beta_2 - 1} - (t-s)^{\beta_2 - 1}}{\Gamma(\beta_2)}, & 0 \le s \le t \le 1. \end{cases} \tag{2.14}$$

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Lemma 2.7 *For $t \in I = [\frac{1}{4}, \frac{3}{4}]$, the Green's function $G_1(t,s)$ given in (??) satisfies the following inequalities*

$$\begin{array}{ll} (C1) & G_2(t,s) \geq 0, \text{ for all } (t,s) \in [0,1] \times [0,1], \\ (C2) & G_2(t,s) \leq G_2(1,s) \text{ for all } (t,s) \in [0,1] \times [0,1], \\ (C3) & G_2(t,s) \geq (\frac{1}{4})^{\alpha_2 - 1} G_2(1,s), \text{ for all } (t,s) \in I \times [0,1]. \end{array}$$

For details refer [?].

Lemma 2.8 *For $(t,s) \in [0,1], \xi_2 \in (\frac{1}{4}, \frac{3}{4})$ the Green's functions $H_2(t,s)$ given in(??) satisfies the following inequalities*

$$\begin{array}{ll} (H1) & H_2(t,s) \geq 0, \\ (H2) & H_2(t,s) \leq H_2(s,s), \\ (H3) & \min\limits_{t \in I} H_2(t,s) \geq \vartheta_2^*(s) H_2(s,s), \text{ for } 0 < s < 1, \end{array}$$

where

$$\vartheta_2^*(s) = \begin{cases} \frac{[\frac{3}{4}(1-s)]^{\beta_2-1} - (\frac{3}{4}-s)^{\beta_2-1}}{[s(1-s)]^{\beta_2-1}}, & s \in (0, \xi_2], \\ \frac{1}{(4s)^{\beta_2-1}}, & s \in [\xi_2, 1), \end{cases}$$

For details refer [?]. From Lemma ?? and ?? the boundary value problem (??)–(??) is equivalent to the following integral equations.

$$\begin{aligned} x(t) &= \int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau) f_1(\tau, x(\tau), y(\tau)) d\tau\Big) ds, \\ y(t) &= \int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau) f_2(\tau, x(\tau), y(\tau)) d\tau\Big) ds. \end{aligned} \tag{2.15}$$

To establish criteria for existence of $2n, n \in \mathbb{N}$ positive solutions for the coupled system of Boundary Value Problem(BVP) (??)–(??), we will apply the following Avery-Henderson fixed point theorem.

Theorem 2.9 *[?] Let P be a cone in a real Banach space. Let α and γ are increasing nonnegative continuous functionals on P and θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that for some positive constants r and η,*

$$\gamma(u) \leq \theta(u) \leq \alpha(u) \text{ and } \|u\| \leq \eta\gamma(u)$$

for all $u \in \overline{P(\gamma, c)}$. Suppose that there exist positive numbers a and b with $a < b < c$ such that $\theta(\lambda u) \leq \lambda\theta(u)$, for $0 \leq \lambda \leq 1$ and $u \in \partial P(\theta, b)$ and $T : \overline{P(\gamma, c)} \to P$ is a completely continuous operator satisfying

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(i.) $\gamma(Tu) > c$ *for all* $u \in \partial P(\gamma, c)$

(ii.) $\theta(Tu) < b$ *for all* $u \in \partial P(\theta, b)$

(iii.) $P(\alpha, a) \neq \emptyset$ *and* $\alpha(Tu) > a$ *for all* $u \in \partial P(\alpha, a)$,

then T *has at least two fixed points* u_1 *and* u_2 *such that*

$$a < \alpha(u_1) \quad with \quad \theta(u_1) < b \quad and \quad b < \theta(u_2) \ with \ \gamma(u_2) < c.$$

3 Existence of Even Number Positive Solutions

In this section, we prove the existence of at least two positive solutions for the BVP (??)–(??) by implementing Avery-Henderson fixed-point theorem. And then, we establish the existence of at least $2n$ positive solutions to the boundary value problem (??)-(??) for an arbitrary positive integer n.

Consider the Banach Space $B = E \times E$, where $E = \{x : x \in C[0, 1]\}$ equipped with the norm $\|(x, y)\| = \|x\|_0 + \|y\|_0$, for $(x, y) \in B$ and the norm is defined as

$$\|x\|_0 = \max_{t \in [0,1]} |x(t)|.$$

Define a cone $P \subset B$ by

$$P = \{(x, y) \in B : x(t) \geq 0, y(t) \geq 0, t \in [0, 1] \text{ and } \min_{t \in I}[x(t)+y(t)] \geq \eta \|(x, y)\|\}$$

where $\eta = \min\{(\frac{1}{4})^{\alpha_1 - 1}, (\frac{1}{4})^{\alpha_2 - 1}\}$. We define the non–negative, increasing continuous functionals γ, θ and α on P by

$$\gamma(x, y) = \min_{t \in I}\{|x| + |y|\} \tag{3.1}$$

$$\theta(x, y) = \max_{t \in I}\{|x| + |y|\} \tag{3.2}$$

$$\alpha(x, y) = \max_{t \in [0,1]}\{|x| + |y|\} \tag{3.3}$$

Let ψ be nonnegative continuous functional on cone P, of the real Banach space B. Then for real number c, we define the sets

$$P(\psi, c) = \{(x, y) \in P : \psi(x, y) < c\},$$
$$\partial P(\psi, c) = \{(x, y) \in P : \psi(x, y) = c\}.$$
$$P_c = \{(x, y) \in P : \|(x, y)\| < c\},$$
$$\partial P_c = \{(x, y) \in P : \|(x, y)\| = c\}.$$

Let $T_1, T_2 : P \to E$ and $T : P \to B$ be the operators defined as

$$\begin{cases} T_1(x,y)(t) = \int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1(\tau,x(\tau),y(\tau))d\tau\Big)ds, \\ \\ T_2(x,y)(t) = \int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2(\tau,x(\tau),y(\tau))d\tau\Big)ds \end{cases}$$

and

$$T(x,y)(t) = \big(T_1(x,y)(t), T_2(x,y)(t)\big), \text{ for } (x,y) \in B.$$

From (??) we assure that a fixed point of T is the solution of the fractional order boundary value problem (??)-(??). In order to use Theorem ??, we must show that $T : P \to P$ must be completely continuous.

Lemma 3.1 *$T : P \to P$ is completely continuous.*

Proof: Let $(x,y) \in P$. Clearly, $T_1(x,y)(t) \geq 0$ and $T_2(x,y)(t) \geq 0$ for $t \in [0,1]$. Also for $(x,y) \in P$,

$$\begin{cases} \|T_1(x,y)\|_0 \leq \int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1(\tau,x(\tau),y(\tau))d\tau\Big)ds, \\ \\ \|T_2(x,y)\|_0 \leq \int_0^1 G_2(1,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2(\tau,x(\tau),y(\tau))d\tau\Big)ds \end{cases}$$

and

$$\begin{aligned} \min_{t\in I} T_1(x,y)(t) &= \min_{t\in I}\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1(\tau,x(\tau),y(\tau))d\tau\Big)ds \\ &\geq \eta\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1(\tau,x(\tau),y(\tau))d\tau\Big)ds \\ &\geq \eta\|T_1(x,y)\|_0. \end{aligned}$$

Similarly, $\min_{t\in I} T_2(x,y)(t) \geq \eta\|T_2(x,y)\|_0$. Therefore,

$$\begin{aligned} \min_{t\in I}\{T_1(x,y)(t) + T_1(x,y)(t)\} &\geq \eta\|T_1(x,y)\|_0 + \eta\|T_2(x,y)\|_0 \\ &= \eta\big(\|T_1(x,y)\|_0 + \|T_2(x,y)\|_0\big) \\ &= \eta\|\big(T_1(x,y), T_2(x,y)\big)\| \\ &= \eta\|T(x,y)\|. \end{aligned}$$

Hence $T(x, y) \in P$ and so $T : P \to P$. Moreover the operator T is completely continuous by standard arguments involving Arzela-Ascoli theorem.

Let

$$\vartheta^*(s) = \min\{\vartheta_1^*(s), \vartheta_2^*(s)\} \tag{3.4}$$

Define

$$L = min\Big\{\Big(\int_0^1 G_1(1,s)\phi_q(\int_0^1 H_1(\tau,\tau)d\tau)ds\Big)^{-1}, \\ \Big(\int_0^1 G_2(1,s)\phi_p(\int_0^1 H_2(\tau,\tau)d\tau)ds\Big)^{-1}\Big\}$$

and

$$M = \max\Big\{\Big(\int_{s\in I} \eta G_1(1,s)\phi_q(\int_{\tau\in I} \vartheta^*(\tau)H_1(\tau,\tau)d\tau)ds\Big)^{-1}, \\ \Big(\int_{s\in I} \eta G_2(1,s)\phi_p(\int_{\tau\in I} \vartheta^*(\tau)H_2(\tau,\tau)d\tau)ds\Big)^{-1}\Big\}.$$

In our result, we show the existence of at least two positive solutions of (??)-(??).

Theorem 3.2 *Suppose there exists $0 < a < b < c$ such that f_i satisfies the following condition*

(D1.) $f_1(t, x(t), y(t)) > \phi_p(\frac{aM}{2})$ *and* $f_2(t, x(t), y(t)) > \phi_q(\frac{aM}{2})$ *for* $t \in I$ *and* $x, y \in [\eta a, a]$,

(D2.) $f_1(t, x(t), y(t)) < \phi_p(\frac{bL}{2})$ *and* $f_2(t, x(t), y(t)) < \phi_q(\frac{bL}{2})$ *for* $t \in [0, 1]$ *and* $x, y \in [0, \frac{b}{\eta}]$,

(D3.) $f_1(t, x(t), y(t)) > \phi_p(\frac{cM}{2})$ *and* $f_2(t, x(t), y(t)) > \phi_q(\frac{cM}{2})$ *for* $t \in I$ *and* $x, y \in [c, \frac{c}{\eta}]$.

Then the (p, q)*–Laplacian fractional order boundary value problem* (??)-(??) *has at least two positive solutions* $(x_1, y_1), (x_2, y_2)$ *such that*

$$a < \alpha(x_1, y_1) \ with \ \theta(x_1, y_1) < b,$$

$$b < \theta(x_2, y_2) \ with \ \gamma(x_2, y_2) < c.$$

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Proof: From (??)-(??), for each $(x,y) \in P$ we have

$$\gamma(x,y) \leq \theta(x,y) \leq \alpha(x,y) \tag{3.5}$$

$$\|(x,y)\| \leq \frac{1}{\eta}\min_{t\in I}\{x(t)+y(t)\} = \frac{1}{\eta}\gamma(x,y) \leq \frac{1}{\eta}\theta(x,y) \leq \frac{1}{\eta}\alpha(x,y). \tag{3.6}$$

For any $(x,y) \in P$, inequalities (??) and (??) imply

$$\|(x,y)\| \leq \frac{1}{\eta}\gamma(x,y).$$

Also for all $(x,y) \in P, \lambda \in [0,1]$ we have

$$\theta(\lambda x, \lambda y) = \max_{t\in I}\{\lambda x(t) + \lambda y(t)\} = \lambda \max_{t\in I}\{x(t)+y(t)\} = \lambda\theta(x,y). \tag{3.7}$$

It is clear that $\theta(0,0) = 0$. We know show that the remaining conditions of Theorem ?? are satisfied.

At the beginning we prove that condition (i) of Theorem ?? is satisfied. Since $(x,y) \in \partial P(\gamma, c)$, from (??) we have that $c = \min_{t\in I}\{x(t)+y(t)\}$ and $c \leq \|x\| + \|y\| \leq \frac{c}{\eta}$. Then

$$\begin{aligned}
\gamma(T(x,y)(t)) &= \min_{t\in I}\Bigg[\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1(\tau,x(\tau),y(\tau))d\tau\Big)ds \\
&+ \int_0^1 G_2(t,s)\phi_q\Big(\int_0^1 H_2(s,\tau)f_1(\tau,x(\tau),y(\tau))d\tau\Big)ds,\Bigg] \\
&\geq \eta\Bigg[\int_{s\in I} G_1(1,s)\phi_q(\int_{\tau\in I}\vartheta^*(\tau)H_1(\tau,\tau)\phi_p(\frac{cM}{2})d\tau)ds \\
&+ \int_{s\in I} G_2(1,s)\phi_q(\int_{\tau\in I}\vartheta^*(\tau)H_2(\tau,\tau)\phi_p(\frac{cM}{2})d\tau)ds\Bigg] \\
&\geq \frac{cM}{2}\int_{s\in I}\eta G_1(1,s)\phi_q(\int_{\tau\in I}\vartheta^*(\tau)H_1(\tau,\tau)d\tau)ds \\
&+ \frac{cM}{2}\int_{s\in I}\eta G_2(1,s)\phi_q(\int_{\tau\in I}\vartheta^*(\tau)H_2(\tau,\tau)d\tau)ds \\
&\geq \frac{c}{2} + \frac{c}{2} = c,
\end{aligned}$$

using hypothesis (D1). Therefore we have shown that $\gamma(T(x,y)(t)) > c$ for all $(x,y) \in \partial P(\gamma, c)$. Now we shall show that condition (ii) of Theorem ?? is

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satisfied. Since $(x,y) \in \partial P(\theta,b)$, from (??) we have that $0 \leq x(t)+y(t) \leq \|(x,y)\| \leq \frac{b}{\eta}$, for $t \in [0,1]$. Thus

$$\begin{aligned}
\theta(T(x,y)(t)) &= \max_{t\in I} T(x,y)(t) \\
&= \max_{t\in I}\Bigg[\int_0^1 G_1(t,s)\phi_q\Bigg(\int_0^1 H_1(s,\tau)f_1(\tau,x(\tau),y(\tau))d\tau\Bigg)ds \\
&+ \int_0^1 G_2(t,s)\phi_q\Bigg(\int_0^1 H_2(s,\tau)f_1(\tau,x(\tau),y(\tau))d\tau\Bigg)ds,\Bigg] \\
&\leq \Bigg[\int_0^1 G_1(1,s)\phi_q\Bigg(\int_0^1 H_1(s,\tau)\phi_p(\frac{bL}{2})d\tau\Bigg)ds \\
&+ \int_0^1 G_2(1,s)\phi_p\Bigg(\int_0^1 H_2(s,\tau)\phi_q(\frac{bL}{2})d\tau\Bigg)ds,\Bigg] \\
&< \frac{bL}{2}\int_0^1 G_1(1,s)\phi_q\Bigg(\int_0^1 H_1(\tau,\tau)d\tau\Bigg)ds \\
&+ \frac{bL}{2}\int_0^1 G_2(1,s)\phi_p\Bigg(\int_0^1 H_2(\tau,\tau)d\tau\Bigg)ds \\
&\leq \frac{b}{2}+\frac{b}{2} = b.
\end{aligned}$$

using hypothesis (D2). At the end, using hypothesis (D3), we proceed condition (iii) of Theorem ?? is satisfied. Since $(0,0) \in P$ and $P(\alpha,a) \neq \emptyset$. For $(x,y) \in \partial P(\alpha,a), \eta a \leq \max_{t\in[0,1]}\{|x|+|y|\} \leq \|x\|+\|y\| = a$. Therefore,

$$\begin{aligned}
\alpha(T(x,y)(t)) &= \max_{t\in[0,1]} T(x,y)(t) \\
&= \max_{t\in[0,1]}\Bigg[\int_0^1 G_1(t,s)\phi_q\Bigg(\int_0^1 H_1(s,\tau)f_1(\tau,x(\tau),y(\tau))d\tau\Bigg)ds \\
&+ \int_0^1 G_2(t,s)\phi_q\Bigg(\int_0^1 H_2(s,\tau)f_1(\tau,x(\tau),y(\tau))d\tau\Bigg)ds,\Bigg] \\
&\geq \eta\Bigg[\int_{s\in I} G_1(1,s)\phi_q\Bigg(\int_{\tau\in I} \vartheta^*(s)H_1(\tau,\tau)\phi_p(\frac{aM}{2})d\tau\Bigg)ds \\
&+ \int_{s\in I} G_2(1,s)\phi_p\Bigg(\int_{\tau\in I} \vartheta^*(s)H_2(\tau,\tau)\phi_q(\frac{aM}{2})d\tau\Bigg)ds,\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&> \frac{aM}{2}\int_{s\in I}\eta G_1(1,s)\phi_q\Big(\int_{\tau\in I}\vartheta^*(s)H_1(\tau,\tau)d\tau\Big)ds\\
&+\frac{aM}{2}\int_{s\in I}\eta G_2(1,s)\phi_p\Big(\int_{\tau\in I}\vartheta^*(s)H_2(\tau,\tau)d\tau\Big)ds\\
&\geq \frac{a}{2}+\frac{a}{2}=a.
\end{aligned}
$$

using hypothesis (D1). This ends the proof.

Theorem 3.3 *Let n be an arbitrary positive integer. Assume that there exist numbers $a_r(r=1,2,\cdots,n+1)$ and $b_s(s=1,2,\cdots,n)$ with $0<a_1<b_1<a_2<b_2<\cdots<a_n<b_n<a_{n+1}$ such that*

$$
\begin{gathered}
f_1(t,x(t),y(t))>\phi_p(\frac{Ma_r}{2}),\quad f_2(t,x(t),y(t))>\phi_q(\frac{Ma_r}{2})\\
for\ t\in I\ and\ x,y\in[\eta a_r,a_r],\ r=1,2,\cdots,n+1,
\end{gathered}
\tag{3.8}
$$

$$
\begin{gathered}
f_1(t,x(t),y(t))<\phi_p(\frac{b_sL}{2}),\quad f_2(t,x(t),y(t))<\phi_q(\frac{b_sL}{2})\\
for\ t\in[0,1]\ and\ x,y\in[0,\frac{b_s}{\eta}],\ s=1,2,\cdots,n.
\end{gathered}
\tag{3.9}
$$

Then the boundary value problem (??)-(??) has at least $2n$ positive solutions in $\overline{P}_{a_{n+1}}$.

Proof: We use Mathematical induction on n. For $n=1$, we know from (??) and (??) that $T:\overline{P}_{a_2}\to P_{a_2}$, then, it follows from Avery-Henderson fixed point theorem that the boundary value problem (??)-(??) has at least two positive solutions in $\overline{P}_{a_2}$. Next, we assume that this conclusion holds for $n=l$. In order to prove this conclusion holds for $n=l+1$. We suppose that there exist numbers $a_r(r=1,2,\cdots,l+2)$ and $b_s(s=1,2,\cdots,l+1)$ with $0<a_1<b_1<a_2<b_2<\cdots<a_{l+1}<b_{l+1}<a_{l+2}$ such that

$$
\begin{gathered}
f_1(t,x,y)>\phi_p(\frac{Ma_r}{2}), f_2(t,x,y)>\phi_q(\frac{Ma_r}{2})\text{ for }t\in(0,1)\\
\text{and }x,y\in[\eta a_r,a_r],\ r=1,2,\cdots,l+2,
\end{gathered}
\tag{3.10}
$$

$$
\begin{gathered}
f_1(t,x,y)<\phi_p(\frac{Lb_s}{2}), f_2(t,x,y)<\phi_q(\frac{Lb_s}{2})\text{ for }t\in(0,1)\\
\text{and }x,y\in[0,\frac{b_s}{\eta}],\ s=1,2,\cdots,l+1.
\end{gathered}
\tag{3.11}
$$

By assumption, the BVP (??)-(??) has at least $2l$ positive solutions $(x_i,y_i)(i=1,2,\cdots,2l)$ in $\overline{P}_{a_{l+1}}$. At the same time, it follows from Theorem ??, and the inequalities (??) and (??) that the boundary value problem (??)-(??) has at least two positive solutions $(x_{2l+1}, y_{2l+1}), (x_{2l+2}, y_{2l+2})$ in $\overline{P}_{a_{l+2}}$ such that $a_{l+1} < \alpha(x_{2l+1}, y_{2l+1})$ with $\theta(x_{2l+1}, y_{2l+1}) < b_{2l+1}$ and $b_{2l+1} < \theta(x_{2l+2}, y_{2l+2})$ with $\gamma(x_{2l+2}, y_{2l+2}) < a_{l+2}$. Obviously (x_{2l+1}, y_{2l+1}) and (x_{2l+2}, y_{2l+2}) are different from $(x_i, y_i)(i = 1, 2, \cdots, 2l)$. Therefore, the boundary value problem (??)-(??) has at least $2l + 2$ positive solutions in $\overline{P}_{a_{l+2}}$, which shows that the conclusion holds for $n = l + 1$.

4 Example

In this section, we present examples to illustrate our results.

Example 4.1 *Consider the boundary value problem*

$$\begin{aligned}
D_{0^+}^{1.6}\Big(\phi_p\big(D_{0^+}^{3.7}x(t)\big)\Big) &= f_1\big(t, x(t), y(t)\big), \quad t \in (0,1),\\
D_{0^+}^{1.5}\Big(\phi_q\big(D_{0^+}^{3.8}y(t)\big)\Big) &= f_2\big(t, x(t), y(t)\big), \quad t \in (0,1),\\
x(0) = x'(0) = x''(0) = 0, &\quad x''(1) = 0,\\
\phi_p\Big(D_{0^+}^{3.7}x(0)\Big) = 0, &\quad \phi_p\Big(D_{0^+}^{3.7}x(1)\Big) = 0,\\
y(0) = y'(0) = y''(0) = 0, &\quad y''(1) = 0,\\
\phi_q\Big(D_{0^+}^{3.8}y(0)\Big) = 0, &\quad \phi_q\Big(D_{0^+}^{3.8}y(1)\Big) = 0.
\end{aligned} \tag{4.1}$$

$$f_i(t, x, y) = e^{2t} + \cos(x + y) + \frac{(x + y)^3}{228762} \textit{ for } i = 1, 2$$

Let $p = 2$, by direct calculations, we can get $\eta = 0.0206, L = 35.5923$, and $M = 13240.888$. If we take $a = 0.0001, b = 0.5, c = 13760$, then all the conditions in Theorem ?? are fulfilled. Hence, the BVP (??) has at least two positive solutions by Theorem ??.

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BIANCHI TYPE-*IX* SCALAR FIELD COSMOLOGICAL MODEL IN *f*(*R; T*) THEORY OF GRAVITY

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Abstract
This work is devoted to the investigation of anisotropic Bianchi type-*IX* universe in the context of *f*(R; T) theory of gravity proposed by Harko et al. (Physics review D 84, 024020, 2011) with scalar field (quintessence or phantom). Here R is Ricci scalar and T is the trace of the energy momentum tensor. The field equations are solved by using the fact that the expansion scalar is proportional to the shear scalar of the model, which gives the relation between metric potentials of the space-time. Some physical and kinematical properties of the model are also discussed.
Key words: Bianchi type-*IX* model, *f*(R; T) gravity, Scalar field cosmology, Modified gravity

1. INTRODUCTION

In the light of the recent discovery of the accelerated expansion of the universe (Perlmutter et al. [1]; Riess et al. [2]), modified theories of gravity are attracting more and more attention of cosmologists because of the fact that these theories may serve as the possible candidates for explaining the late time acceleration of the universe. Among the various modified theories of gravity, *f(R)* theory of gravity which provides a natural unification of early-time inflation and late-time acceleration (Capozziello and Francaviglia [3]; Nojiri and Odintsov [4]), *f(G)* gravity (Nojiri and Odintsov [5]) and *f(T)* gravity (Linder [6]), where T is the torsion have been proposed to explain the accelerated expansion of universe.

Recently, Harko et al. [7] proposed another modified theory known as *f(R; T)* gravity, wherein the gravitational Lagrangian contains the Ricci scalar R and trace of the energy-momentum tensor T. It is well known that in the study of early stages of evolution of the universe anisotropic models play a vital role. Reddy and Santhi Kumar [8] have discussed some anisotropic cosmological models in *f(R;T)* theory of gravity. Rao and Neelima [9] have obtained perfect fluid Bianchi type-$V I_0$ universes in *f(R; T)* gravity. Rao et al. [10] have investigated Bianchi type-*II, V III* and *IX* cosmological models in *f(R; T)* theory of gravity. Shri Ram and Chandel [12] have discussed dynamics of magnetized string cosmological model in *f(R; T)* theory of gravity. Rao et al. [13] have obtained anisotropic Bianchi type-V Ih perfect fluid cosmological models in *f(R; T)* theory of gravity. Aditya et al. [14] have studied Bianchi type-II, V III and IX cosmological models in *f(R; T)* theory of gravity with variable Λ. Scalar fields play a crucial role in particle physics and cosmology. Olive [15] has shown that, during inflation, the potential of a scalar field acts as a dynamical vacuum energy. This prominent role of

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scalar fields is also evident in models proposed to explain the late time accelerated expansion of the universe in vacuum energy and in evolving quintessence models ([16]-[18]). Further, it was recently proposed that a scalar field can also be the source of the anomalous acceleration [19]. Sharif and Zubair [20] have investigated the anisotropic universe models in *f(R; T)* gravity in the presence of perfect fluid and scalar field. Singh and Singh [21] have obtained the Friedmann-Robertson-Walker (FRW) models with perfect fluid and scalar field in higher derivative theory. Sharif and Jawad [22] have studied

reconstruction of scalar field dark energy models in Kaluza-Klein universe. Singh and Singh [23] have discussed the behavior of scalar field in modified *f(R; T)* gravity within the framework of a flat FRW cosmological model. Later, Singh et al. [24] have investigated Bianchi type-I universe with scalar field and time varying cosmological constant in *f(R; T)* gravity. Very recently, Santhi et al. [25] have studied Kantowski-Sachs scalar field cosmological models in *f(R; T)* theory of gravity.

Motivated by the above investigations, in this paper, we consider Bianchi type-*IX* space-time in presence of scalar field within the framework of *f(R; T)* theory of gravitation proposed by Harko et al. [7]. The plan of the work as follows: Sect. 2 describes *f(R; T)* gravity formalism in the presence of scaler field. Sect. 3 is devoted to the derivation of field equations and scalar field model. Sect. 4 contains a detailed physical discussion of the model. Summery and conclusions are presented in the last section.

2 *f(R; T)* gravity formalism with scalar field:

The field equations of *f(R; T)* gravity are derived from the Hilbert-Einstein type variation principle. The action for the *f(R; T)* gravity with scalar field is [23]

$$S = \frac{1}{16\pi}\int f(R,T)\sqrt{-g}d^4x + \int L_\phi\sqrt{-g}d^4x, \tag{1}$$

where $f(R,T)$ is an arbitrary function of Ricci scalar R, T is the trace of stress-energy tensor (T_{ij}) of the matter and L_ϕ is the matter Lagrangian of scalr field.

The energy momentum tensor T_{ij} is defined as

$$T_{ij} = -\left(\frac{2}{\sqrt{-g}}\right)\frac{\delta(\sqrt{-g})L_\phi}{\delta g^{ij}}. \tag{2}$$

Here we consider that the dependence of matter Lagrangian is merely on the metric tensor g_{ij} rather than on its derivatives and we obtain

$$T_{ij} = g_{ij}L_\phi - \frac{\partial L_\phi}{\partial g^{ij}}. \tag{3}$$

Now varying the action S with respect to metric tensor g_{ij}, $f(R,T)$ gravity field equations are obtained as

$$f_R(R,T)R_{ij} - \frac{1}{2}f(R,T)g_{ij} + (g_{ij}\Box - \nabla_i\nabla_j)f_R(R,T) = 8\pi T_{ij} - f_T(R,T)T_{ij} - f_T(R,T)\Theta_{ij}, \tag{4}$$

where

$$\Theta_{ij} = -2T_{ij} + g_{ij}L_\phi - 2g^{\alpha\beta}\frac{\partial^2 L_\phi}{\partial g^{ij}\partial g^{\alpha\beta}}. \tag{5}$$

Here $f_R(R,T) = \frac{\partial f(R,T)}{\partial R}$, $f_T(R,T) = \frac{\partial f(R,T)}{\partial T}$ and $\Box = \nabla^\mu \nabla_\mu$, where ∇_μ denotes the covariant derivative.

Here we assume that the universe is filled with scalar field minimally coupled to gravity. Therefore, the energy-momentum tensor of a scalar field ϕ with self-interacting scalar field potential $\mathcal{V}(\phi)$ has the form

$$T_{ij} = \epsilon \phi_{,i}\phi_{,j} - g_{ij}\left(\frac{\epsilon}{2} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \mathcal{V}(\phi)\right) \tag{6}$$

where $\epsilon = \pm 1$ correspond to quintessence and phantom scalar fields respectively. The trace of the energy-momentum tensor $T = g^{ij}T_{ij}$ is given by

$$T = -\epsilon\dot{\phi}^2 + 4\mathcal{V}(\phi) \tag{7}$$

hereafter dot denotes differentiation with respect to time t. The matter Lagrangian of the scalar field is given by

$$L_\phi = -\frac{1}{2}\epsilon\dot{\phi}^2 + \mathcal{V}(\phi). \tag{8}$$

Now from equations (5) and (8), we have

$$\Theta_{ij} = -2T_{ij} - g_{ij}\left(\frac{1}{2}\epsilon\dot{\phi}^2 - \mathcal{V}(\phi)\right). \tag{9}$$

Generally, the field equations also depend, through the tensor Θ_{ij}, on the physical nature of the matter field. Hence in the case of $f(R,T)$ gravity depending on the nature of the matter source, we obtain several theoretical models corresponding to different matter contributions for $f(R,T)$ gravity. However, Harko et al. [7] gave three classes of these models:

$$f(R,T) = \begin{cases} R + 2f(T) \\ f_1(R) + f_2(T) \\ f_1(R) + f_2(R)f_3(T). \end{cases}$$

Here we consider the first case, i.e $f(R,T) = R + f(T)$, where $f(T)$ is an arbitrary function of the trace of stress-energy tensor T_{ij}. Using this relation $f(R,T)$ gravity field equations (4) reduced to

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij} - 2(T_{ij} + \Theta_{ij})f'(T) + f(T)g_{ij}, \tag{10}$$

where a prime denotes differentiation with respect to the argument.

3 Field equations and the scalar field model:

We consider spatially homogeneous Bianchi type-IX space-time in the form

$$ds^2 = dt^2 - A^2dx^2 - B^2dy^2 - (B^2 sin^2 y + A^2 cos^2 y)dz^2 + 2A^2 cosy\, dx\, dz, \tag{11}$$

where A and B are functions of cosmic time t only.

For the particular choice of the function $f(T) = \lambda T$ (Harko et al. [7]), where λ is a constant, the field equations (10) for the metric (11) using (6) and (9) can be written as

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$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} - \frac{3A^2}{4B^4} = \left(\frac{1+2\lambda}{2}\right)\epsilon\dot{\phi}^2 - (4\lambda+1)\mathcal{V}(\phi) \tag{12}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{A^2}{4B^4} = \left(\frac{1+2\lambda}{2}\right)\epsilon\dot{\phi}^2 - (4\lambda+1)\mathcal{V}(\phi) \tag{13}$$

$$\frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} + \frac{1}{B^2} - \frac{A^2}{4B^4} = -\left(\frac{1+2\lambda}{2}\right)\epsilon\dot{\phi}^2 - (4\lambda+1)\mathcal{V}(\phi) \tag{14}$$

Equations (12)-(14) are a set of three independent equations with four unknowns (A, B, ϕ and $\mathcal{V}(\phi)$). Therefore, we need an additional condition to solve the above field equations. Here we use the physical condition that the expansion scalar θ is proportional to the shear scalar σ, which leads to

$$A = B^k \tag{15}$$

where $k \neq 1$ is a positive constant. Collins [26] have studied the physical significance of this condition for a perfect fluid.
Now from equations (12), (13) and (15), we get

$$\frac{\ddot{B}}{B} + (k+1)\frac{\dot{B}^2}{B^2} + \frac{1}{(1-k)B^2} - B^{2k-4} = 0, \tag{16}$$

By substituting $\dot{B} = f(B)$, the above equation becomes

$$\frac{df^2}{dB} + \frac{2(k+1)}{B}f^2 = \frac{2}{(k-1)B} + \frac{2B^{2k-3}}{1-k} \tag{17}$$

which admits a solution for $k = 2$

$$\begin{aligned} A &= \frac{2}{\sqrt{3}} sin\left(\frac{t}{2}\right) \\ B &= \frac{4}{3} sin^2\left(\frac{t}{2}\right) \end{aligned} \tag{18}$$

Now the metric (11) can be written as

$$\begin{aligned} ds^2 = dt^2 - \frac{4}{3}sin^2\left(\frac{t}{2}\right)dx^2 - \frac{16}{9}sin^4\left(\frac{t}{2}\right)dy^2 + \frac{8}{3}sin^2\left(\frac{t}{2}\right)cosy\,dx\,dz \\ - \left(\frac{16}{9}sin^4\left(\frac{t}{2}\right)sin^2y + \frac{4}{3}sin^2\left(\frac{t}{2}\right)cos^2y\right)dz^2 \end{aligned} \tag{19}$$

From equations (12)-(14) and (19), we get scalar field potential as

$$\mathcal{V}(\phi) = \frac{-3cot^2\left(\frac{t}{2}\right)}{2(4\lambda+1)} \tag{20}$$

and the scalar field ϕ as

$$\phi = 2\sqrt{\frac{-1}{\epsilon(1+2\lambda)}}\ log|tan\left(\frac{t}{2}\right)| \tag{21}$$

The metric (19) together with (20)-(21) constitutes Bianchi type-IX scalar field cosmological model in $f(R,T)$ theory of gravity. From Eq. (20), it is observed that the scalar field potential $\mathcal{V}(\phi)$ is positive for $\lambda < \frac{-1}{4}$. From Eq. (21) it can be seen that the scalar field ϕ is real on the assumption that $\lambda < \frac{-1}{2}$ for quintessence ($\epsilon = 1$) scalar field model and $\frac{-1}{4} < \lambda < \frac{-1}{2}$ for phantom ($\epsilon = -1$) scalar field model.

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4 Some properties of the model:

We compute the following dynamical parameters which are significant in the physical discussion of the cosmological model (19):
Volume and average scale factor are

$$V = \sqrt{-g} = \left(\frac{16}{9} sin^4 \frac{t}{2}\right) sin\theta \tag{22}$$

$$a(t) = V^{\frac{1}{3}} = \left\{ \left(\frac{16}{9} sin^4 \frac{t}{2}\right) sin\theta \right\}^{\frac{1}{3}}. \tag{23}$$

The directional Hubble parameters and average Hubble parameter takes the form

$$H_x = 2H_y = cot\left(\frac{t}{2}\right) \tag{24}$$

$$H = \frac{2}{3} cot\left(\frac{t}{2}\right). \tag{25}$$

Expansion scalar (θ) and shear scalar (σ^2) can be obtained as

$$\theta = 3H = 2cot\left(\frac{t}{2}\right) \tag{26}$$

$$\begin{aligned}\sigma^2 &= \frac{1}{3}(H_x - H_y)^2 \\ &= \frac{1}{6} cot^2\left(\frac{t}{2}\right).\end{aligned} \tag{27}$$

Average anisotropic parameter is

$$\begin{aligned}A_h &= \frac{1}{3}\left(\frac{(H_x - H)^2 + 2(H_y - H)^2}{H^2}\right) \\ &= \frac{1}{8}.\end{aligned} \tag{28}$$

Deceleration parameter is given by

$$\begin{aligned}q &= \frac{-a\ddot{a}}{\dot{a}^2} \\ &= -1 + \frac{3}{4} sec^2 \frac{t}{2}.\end{aligned} \tag{29}$$

To differentiate various DE models Sahni et al. [27] introduced two new cosmological parameters (r, s) named as statefinder parameters which are defined as

$$\begin{aligned}r &= \frac{\dddot{a}}{aH^3} \\ &= 1 - \frac{9}{4} sec^2 \frac{t}{2} + \frac{9}{2} cosec\, t\end{aligned} \tag{30}$$

$$s = \frac{r-1}{3(q-\frac{1}{2})} = \frac{2\,cosec\,t - sec^2\frac{t}{2}}{sec^2\frac{t}{2} - 2} \qquad (31)$$

5 Conclusions:

$f(R,T)$ theory of gravity can be treated as a possible candidate for explaining the role of DE in the accelerating universe. A suitable form of Lagrangian which can explain the cosmic evolution in a definite way is still under consideration. In this study we have obtained solution of the $f(R,T)$ field equations with scalar field for Bianchi type-IX space time.

The spatial volume of the model (19) increases with time and also the model have no initial singularity at $t = 0$. The expansion scalar θ, the shear scalar σ and the Hubble parameter H decrease with the increase of time. Anisotropic features can reveal key information on the structure and the nature of the components of the universe, and provide hints on the origin of the high-energy emission. The study of anisotropies connects and unravels fundamental issues in various fields of astrophysics and cosmology. Hence anisotropic space-times are important. Since from Eq. (28) $A_h \neq 0$, our model remains anisotropic throughout the evolution of the universe. Initially, the deceleration parameter q is negative, and then for all values of time the deceleration parameter gives positive values. Hence the model exhibits inflation at initial epoch and decelerate in later epoch. However, the model is decelerates in the standard way it will accelerate in finite time due to cosmic re collapse where the universe in turns inflates " decelerates and then accelerates " [28]. Also, we have found that the solution is compatible with the both quintessence ($\epsilon = 1$) and phantom ($\epsilon = -1$) scalar fields for $\lambda < \frac{-1}{2}$ and $\frac{-1}{4} < \lambda < \frac{-1}{2}$ respectively.

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COMPUTING PRINCIPLES MAPCODE THEORY

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Abstract
We introduce the terminology of mapcode thoery and use it to understand the models of computation invented by Donald Knuth, Edsger Dijkstra, Mani Chandy and Jayadev Misra, Robin Milner, and C.A.R. Hoare. Our objective is to present mapcode theory as a connected account of some of the ways the idea of computation can be modeled mathematically.

1 The Idea of Computation

From our familiarity with mathematics we can see that there are two kinds of computational procedures, those that terminate in finite time and those which run forever. Computing the LCM or HCF of two positive integers is an example of the first kind and computing the decimal expansion of$\sqrt{2}$ is an example of the second kind. In the subject of computer science (more correctly called 'computing science') this distinction is maintained but the idea of computation is given a more general meaning than that of calculating numbers. Examples are: sorting a list of numbers, locating a number in a telephone directory, and finding a spanning tree in a connected graph. There are even more general ideas of computation Instead of saying that the earth goes around the sun obeying the gravitational force field one can say that the earth senses its environment and computes its way around the sun [6]. Similarly one can also say that the body constantly senses the temperature of its environment and computes its own temperature to maintain a stable value. Or that the code in the DNA computes the physiology of the baby in the womb.

Computation need not be done sequentially one step after another by one agent[1]. Three other kinds of computation are currently talked about. In 'parallel' computation a single computation is broken into parts so that the parts can be simultaneously carried out, by more than one agent if necessary, with the idea of speeding up the computation. In 'distributed' computation agents located at different places communicate with one another and carry out a single computation. In 'concurrent' computation each agent computes according to its own agenda but constrained by what other agents are doing. In this context the words 'parallel', 'distributed', and 'concurrent' are not precisely defined and are indeed overlapping. Our task as mathematicians is to make precise definitions and create mathematical models to handle all these very different kinds of computations. In this article we take initial steps towards this goal by gathering together and explaining five different models of computation by well known computer scientists.

2 Computational Mechanisms

For our present purposes we define a computational mechanism as a method of generating

[1] We use the word agent to refer neutrally to a person or a machine.

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sequences of elements from a set X called the state space[2]. Computational mechanisms are designed to meet practical requirements that need to be specified in advance. A complete theory of computation should give a mathematically precise way of stating the specifications in all the cases mentioned above and then proving that the designed mechanism meets the specification requirements. However, this article will not take up this general question, but only consider the problem of specification for computations that are expected to terminate. We hope to return to the larger question at a future date. A specification for a terminating computation is given by a map $f : S \rightarrow T$, where S is called the input space and T is called the output space. What we mean is that when we pass the input $s \in S$ to the computational mechanism we expect it start the computation and then to halt in finite time and give us the output $f(s)$.

In three of the models presented below we show what it means for the computing mechanism to compute the specification map. The other two models are not meant to model terminating computations.

3 Discrete Flows

In mathematics we study many sets with an additional structure. Groups, rings, fields, metric spaces, and topological spaces are some examples. In the study of discrete dynamics a topological space X is endowed with the additional structure of a continuous total map $f : S \rightarrow T$ and the pair $(X; F)$ is called a discrete flow. X is called the state space and F the flow map. Let us endow X with the discrete topology. Then every flow map is continuous. For $x \in X$, a run at x is an infinite sequence of the form $(x = x_0; x_1; x_2; : : : ; x_n; x_{n+1}; : : :)$ where $x_{n+1} = F(x_n)$ for all $n \geq 0$. So we can also write a run as $(x; F_{(x)}; F^2_{(x)}; : : : ; F^n_{(x)}; F^{n+1}_{(x)}; : : :)$: x is said to be a convergent point if the run $(F^n_{(x)})$ starting at x is convergent

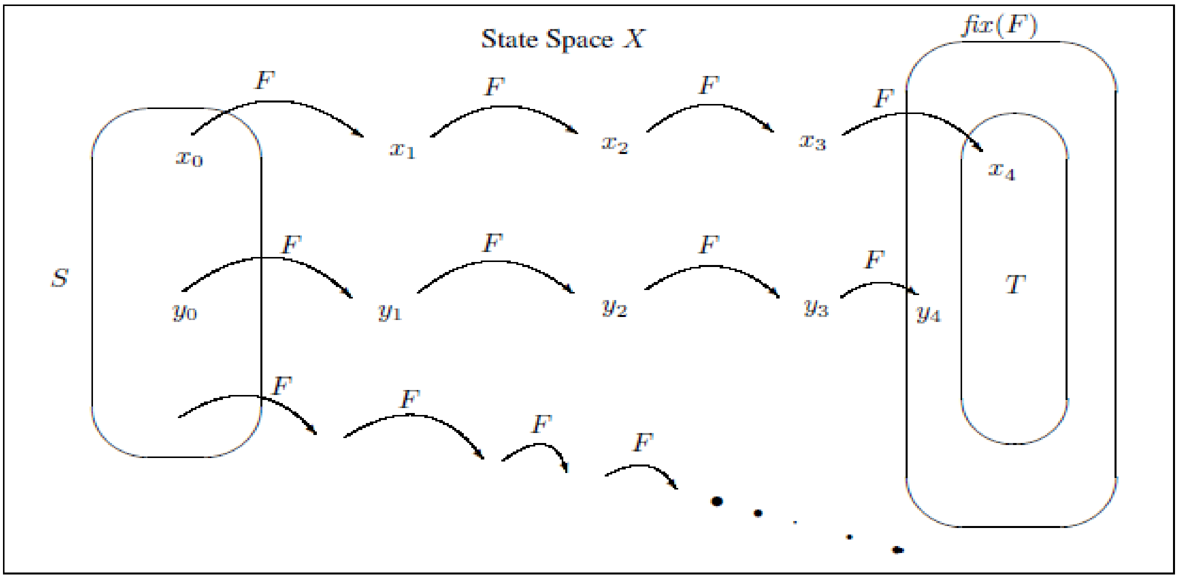

Figure 1: Three Types of Runs in Knuth's Model

[2] There are computational methods that do not involve the concept of a state space. We do not consider them here.

in the discrete topology of X. The set of all convergent points is denoted by $con(F)$. Let $fix(F) = \{x \mid F(x) = x\}$ denote the set of fixed points of F.

Recall that in the discrete topology a sequence is convergent if and only if it is eventually constant. This means that x is convergent if and only if there exists $n > 0$ such that $F^n(x) = F^{n+1}(x) = F^{n+2}(x) = \cdots$. So it is convenient to denote the limit of the run starting at x by $F^\infty(x)$. The map $F^\infty : con(F) \to fix(F)$ is called the limit map [9].

4 Knuth's Model

Suppose we need to compute a specification map $f : S \to T$. In Knuth's model we seek to construct a discrete flow (X, F) such that S and T are subsets of X and further $T \subseteq fix(F)$. Knuth calls the quadruple (X, F, S, T) a computational method and in this context F is called the computational rule. If $x \in S$ a run at x is called a computational sequence. If $x \in S \cap con(F)$ and $F^\infty(x) \in T$, then the computation is said to terminate. The state $F^\infty(x)$ is called the output for the input x. The computational method is called an algorithm if $S \subseteq con(F)$ and $F^\infty(x) \in T$ for all $x \in S$. The map computed by the algorithm is the map $F^\infty : S \to fix(F)$. The computation is correct if $f(x) = F^\infty(x)$ for every $x \in S$.

We refer the reader to Knuth [5] for an example.

5 The Model of Chandy and Misra

Historically, Dijkstra's weakest predicate model came before the UNITY model of Chandy and Misra [1, 3], but because the latter is conceptually simpler and closer to Knuth's model we present it first. 'UNITY' stands for Unbounded Nondeterministic Iterative Transformation theorY.

Knuth's model is a deterministic one. This means that an input always results in the same output every time the computation is invoked. However there are problems in which we can allow the result of the computation to be one of several possibilities. For example, suppose we are given a finite cover for a set E and we are asked to find sets from the cover that constitute a partition of E. Such a partition need not exist, and even if it exists it need not be unique. We call this a nondeterministic problem.

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The specification for a nondeterministic problem is given by a set valued map. If $\mathcal{P}(T)$ denotes the class of all subsets of T we can write down the specification map as $f : S \to \mathcal{P}(T)$. If $f(s)$ is a singleton for all $s \in S$, then we are back in the deterministic case.

UNITY addresses nondeterministic problems also. In UNITY the single flow map F of Knuth's model is replaced by a finite collection $F_1, \cdots, F_N$ of flow maps on the same state space X. Let $\mathbf{F} = \{F_1, \cdots, F_N\}$. An infinite sequence (x_n) in X starting at $x = x_0$ is called a run if $x_{n+1} = F_i(x_n)$ for some $F_i \in \mathbf{F}$. A run is said to be fair if for every i between 1 and N there are infinitely many indices n_i such that $x_{n_i+1} = F_i(x_{n_i})$. Note that if $N \geq 2$ there could be infinitely many distinct fair runs starting at every x. x is said to be convergent if every fair run starting at x is convergent in the discrete topology of X. $con(\mathbf{F})$ denotes the set of all convergent points. If x is convergent and a fair run at x has the limit x^* then $F_1(x^*) = F_2(x^*) = \cdots = F_N(x^*) = x^*$. The set of points satisfying this latter condition is denoted by $fix(\mathbf{F})$. We write $\Lambda(x) = \{x^* \mid x^* \text{ is the limit of some fair run starting at } x\}$. $\Lambda : con(\mathbf{F}) \to \mathcal{P}(fix(\mathbf{F}))$ is called the limit map.

If a nondeterministic specification is given by a map $f : S \to \mathcal{P}(T)$ then a UNITY computation is correct if we can find an X subject to the given conditions containing S and T such that $S \subseteq con(\mathbf{F}), T \subseteq fix(\mathbf{F})$ and $\Lambda(x) \subseteq f(x)$ for all $x \in S$.

6 Nondeterministic Flows

In the UNITY model of Chandy and Misra, the result of the computation at a state x is a new state $x' \in \{F_1(x), F_2(x), \ldots, F_N(x)\}$. While observing the outcome x', which of the F_i is responsible for the outcome is also equally important in determining fair convergence. If we are interested in the outcome only but not which computation rule F_i gave rise to it, then we have a new structure Δ on X defined by $\Delta(x) = \{F_1(x), \ldots, F_N(x)\}$ for all $x \in X$. This motivates us to define a nondeterministic flow as a pair (X, Δ) where $\Delta : X \to \mathcal{P}(X)$. Δ is called a choice map on X. For simplicity we assume here that $\Delta(x) \neq \emptyset$ for all x.

A choice map may also be seen as a relation on X. For any $x, y \in X$ we write $x\Delta y$ if $y \in \Delta(x)$. Thus a relation on X determines a nondeterministic structure on X where the outcomes at a given state x do not have *a priori*

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names.

Let (X, Δ) be a nondeterministic flow. An element $x \in X$ is called a fixed point of Δ if $\Delta(x) = \{x\}$. The set of all fixed points of Δ is denoted by $fix(\Delta)$. For any $A \subseteq X$, $\Delta(A) \doteq \cup\{\Delta(x) \mid x \in A\}$, the union of the sets $\Delta(x)$, $x \in A$. By convention $\Delta(\emptyset) = \emptyset$. At any $x \in X$, we define $\Delta^k(x)$ recursively by $\Delta^0(x) = \{x\}$, and $\Delta^{k+1}(x) = \Delta(\Delta^k(x))$ for $k \geq 0$.

An infinite sequence $(x_0, x_1, x_2, \ldots, x_n, \ldots)$ in X is called a run at $x = x_0$ if $x_{n+1} \in \Delta(x_n)$ for $n \geq 0$. Suppose a run $(x_0, x_1, x_2, \ldots, x_n, \ldots)$ in X is convergent in the discrete topology. Then $x_n \in fix(\Delta)$ for some n. Let n_0 be the least such integer. We call n_0 the runtime. It is clear that $x_n = x_{n_0}$ for $n \geq n_0$. The element x_{n_0} is called the limit of the run. If x^* is the limit of a run at x then x^* is called an output for the input x. The set of all outputs is denoted by $\Delta^\infty(x)$.

We are interested in those x such that not only is every run at x convergent in the discrete topology but also there is an upper bound for the runtimes at x. We call such an x a convergent point of X. The set of all convergent points is denoted by $con(\Delta)$. It may be seen that

$$con(\Delta) = \{x \in X \mid \Delta^n(x) \subseteq fix(\Delta) \text{ for some } n\}.$$

If the specification is given by a map $f : S \to \mathcal{P}(T)$ then the choice map Δ is said to compute f if $S \subseteq con(\Delta)$ and $\Delta^\infty(x) \subseteq f(x)$ for every $x \in X$.

Nondeterministic flows have been investigated extensively in [7].

7 Dijkstra's model

Dijkstra's model is called the *weakest predicate* model. It is equivalent to a nondeterministic flow as described above but Dijkstra approaches it from completely different heuristic point of view.

Suppose $f : S \to \mathcal{P}(T)$ is a nondeterministic specification map and the state space X contains both S and T.

We argue backwards. Let $s \in S$ and $R = f(s)$. At the end of the computation we want an element in R. It does not really matter if we start at s or some other state. So we ask the question, what is the largest set $Q = \mu(R)$ such that starting the computation in Q we are sure to finish in R? If we knew the map $R \to \mu(R)$ for all $R \subseteq X$, we would have solved the problem of computation.

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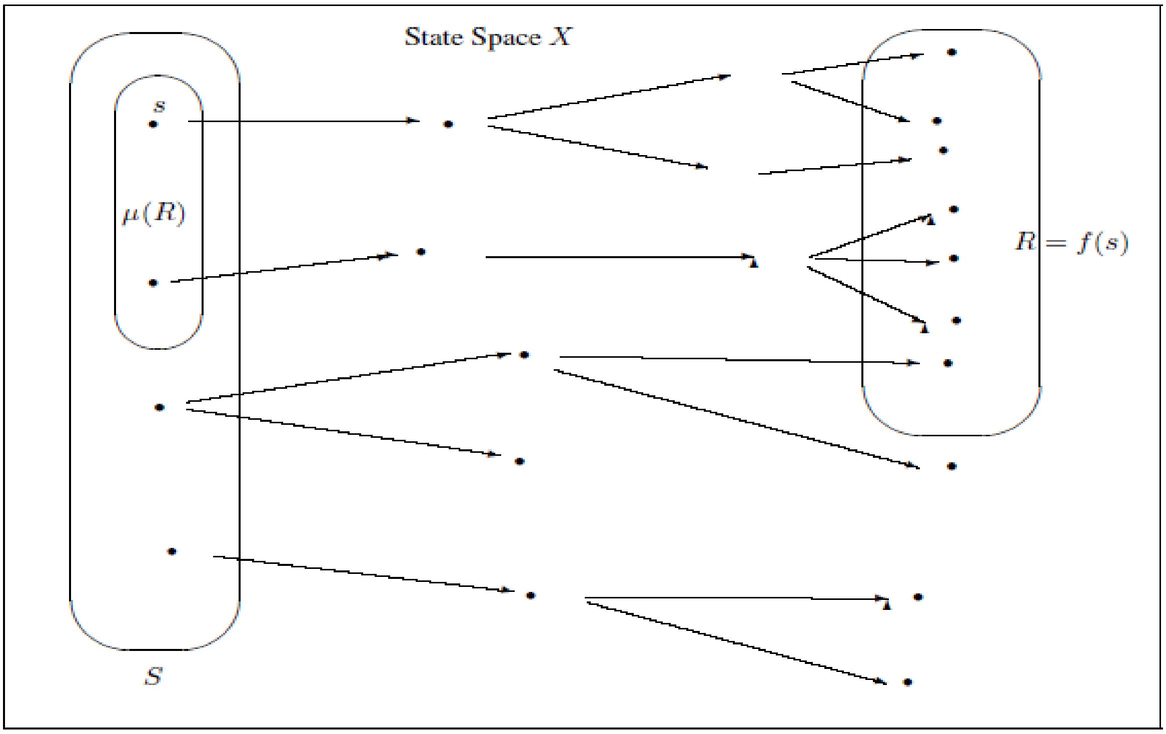

Figure 2: Dijkstra's Model

A typical mathematical way of proceeding from here is to ask: If such a map μ were to exist, what should be its properties?

It is not difficult to argue intuitively [2, 7, 8] and see that the map μ must be such that

1. $\mu(\emptyset) = \emptyset$
2. $\mu(\cap_j A_j) = \cap_j \mu(A_j)$ for an arbitrary family of sets A_j.

For simplicity of discussion here let us add the condition that $\mu(X) = X$. We call μ a multiplicative map.

We now ask the question: What type of computational mechanism could generate a map μ of this kind?

Suppose $\Delta : X \to \mathcal{P}(X)$ is a choice map. Let $\Delta^{-1}(A) = \{x \mid \Delta(x) \subseteq A\}$. Define $\mu_\Delta = \Delta^{-1}$. Then μ_Δ a multiplicative map.

Conversely suppose μ is a multiplicative map. Define $\Delta_\mu(A) = \{x \mid x \in \mu(A)\}$. We then have the following theorem [7].

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Theorem 7.1 *1. If μ is a multiplicative map then there exists a unique choice map $\Delta(=\Delta_\mu)$ on X such that $\mu = \Delta^{-1}$.*

2. If Δ is a choice map on X then there exists unique multiplicative map $\mu(=\mu_\Delta)$ such that $\Delta = \Delta_\mu$.

This shows that Dijkstra's model is equivalent to a nondeterminstic flow.

8 Milner's Model

We have seen that Knuth's model involves a single total map, the UNITY model involves a collection of total maps, and Dijkstra's model involves a single relation. It is therefore natural that there should be a model that involves a collection of relations. Milner's model is exactly that. After understanding the earlier sections it is not difficult to see how to define runs for this model also.

What is very interesting about Milner's studies is that he considers a very different question than that of computing for a specification. He asks the question: What is a symmetry of a computational model?

Suppose that there is only one relation and that relation is a total map F on X, so that (X, F) is a discrete flow. Let $h : X \to X$ be a bijection that commutes with F: $F(h(x)) = h(F(x))$ for all $x \in X$. Then x is a fixed point of F if and only if $h(x)$ is a fixed point F. x is a periodic point of F of period k if and only if $h(x)$ is a periodic point of F of period k. In general the dynamic behaviour of x is identical to that of $h(x)$. It is natural to think of h as a symmetry of the discrete flow (X, F).

Suppose (X, Δ) is a nondeterminstic flow. What can we mean for another choice map Γ on X to be a symmetry of the flow? At first sight we could think of requiring that $\Gamma(\Delta(x)) = \Delta(\Gamma(x))$ for all $x \in X$. It turns out that this is not quite the right condition. Milner asks for the following much weaker condition (yet one that will yield the above definition in the deterministic case):

1. $\forall y \in \Delta(x)\ \forall x' \in \Gamma(x)\ \exists y' \in \Gamma(y) \ni y' \in \Delta(x')$ and
2. $\forall y \in \Delta(x)\ \forall y' \in \Gamma(y)\ \exists x' \in \Gamma(x) \ni y' \in \Delta(x')$.

He calls Γ a bisimulation of Δ. If Milner's model consists of a set X together with a collection of choice maps Δ_a then Γ is a bisimulation of the model if Γ is a bisimulation of every Δ_a. With this definition Milner constructs a beautiful theory of bisimulations. The reader is referred to [?] for details.

9 Hoare's model

We have considered four possibilities. A single map, a single relation, a family of maps and a family of relations. What else can there be? Hoare [4] comes up with a completely different model by asking, not what the next state can be, but what the next state should not be. Let us look at the details.

Let Σ be a finite set and let the state space $X = \Sigma^*$ be the set of all strings of elements from Σ. For x and y in X let $x \cdot y$ denote the concatenation of x with y. Consider a typical state $x = a_0a_1a_2 \ldots a_{n-1}$ of length n. In Hoare's model this string can only change to a string of length $n+1$ got by concatenating x with an element a_n to get the next state $x' = a_0a_1a_2 \ldots a_{n-1}a_n$.

Hoare departs from the method of reasoning of our earlier models by asking, not how a_n should be chosen, but how a_n should *not* be chosen. Let us call a finite set $A \subseteq \Sigma$ *disallowed* at x if a_n can not be chosen from A. There could be several such disallowed sets at x. We can list some properties that we intuitively feel disallowed sets should satisfy.

1. If A is a disallowed set at x and $B \subseteq A$, then B is also a disallowed set at x.
2. If A is a disallowed set and y is not a possible value for a_n at x then $A \cup \{y\}$ is also a disallowed set at x.
3. If we write $(x, \emptyset)$ to mean that $x = a_0a_1a_2 \ldots a_{n-1}$ is a possible state, then $(a_0a_1a_2 \ldots a_{n-1}a_n, \emptyset)$ would imply $(a_0a_1a_2 \ldots a_{n-1}, \emptyset)$ is also valid.

Let us call (x, A) a failure set if A is disallowed at x. We can now define failure sets precisely.

Definition 9.1 A subset $F \subseteq \Sigma^* \times \mathbb{P}(\Sigma)$ is called a failure set if it has the following properties:

1. $(x, A) \in F$ implies A is finite.
2. $(\varepsilon, \emptyset) \in F$.
3. $(x \cdot y, \emptyset) \in F$ implies $(x, \emptyset) \in F$.
4. $(x, A) \in F$ and $B \subseteq A$ implies $(x, B) \in F$.

5. $a \in \Sigma$, $(x, A) \in F$ and $(x \cdot a, \emptyset) \notin F$ implies $(x, X \cup \{a\}) \in F$.

We are not in a position to explore this model further here. The question for us mathematicians is: Is there a more intuitive way of characterizing Hoare's theory?

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BIANCHI TYPE-*II* MODIFIED HOLOGRAPHIC RICCI DARK ENERGY MODEL IN SAEZ-BALLESTER THEORY OF GRAVITATION

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Abstract

We investigate anisotropic Bianchi type-II universe with modified holographic Ricci dark energy in a scalar-tensor theory of gravitation proposed by Saez and Ballester (J. Phys. Lett. A 113, 467, 1986). To get a deterministic solution we assume time varying deceleration parameter, which generates a smooth transition of the universe from the early decelerated phase to the recent accelerating phase. The physical and kinematical aspects of the model are discussed and also found to be they are in good agreement with recent astrophysical observations under suitable conditions.

Key words: Bianchi type-II, Ricci dark energy, Holographic dark energy, Seaz-Ballester theory, Deceleration parameter.

1 Introduction

The accelerated expansion of the universe evidenced by a number of cosmological data (Perlmutter 1999; Fedeli et al. 2009; Caldwell and Doran 2004; Koivoirto and Mota 2006) is well established nowadays. In order to explain this late-time acceleration of the universe, a dark-energy (DE) component which represents the $2/3$ of entire energy content of the universe has been proposed. This component is described by negative pressure (p) leading to a negative equation of state (EoS) parameter, $\omega = \frac{p}{\rho}$, with ρ the energy density of the universe. Extensive reviews on DE are carried out (Sahni 2005; Copeland et al. 2006; Peebles and Ratra 2003; Bamba et al. 2012) in order to speed the late-time acceleration. There are two major approaches to address the problem of cosmic acceleration: either introducing a DE component in the universe and study its dynamics (Sahni 2004) or interpreting it as a failure of general relativity (GR) and consider modifying GR theory, termed as the modified gravity approach (Nojiri and Odintsov 2011).

There are different dynamical DE models out of which holographic dark energy (HDE) has attained much attraction because of its direct relationship with space time. This HDE model is developed using the holographic principle (Susskind 1995). According to this principle, the vacuum energy density can be bounded as $\rho_{vac}L^2 \leq M_p^2 L$, where ρ_{vac} is the vacuum energy density and M_p^2 is the reduced plank mass. The HDE models with Hubble or particle horizons as the IR cut-off, cannot follow the current accelerated expansion (Hsu 2004) of the universe. When the event horizon is taken as the length scale, the model suffers from some disadvantage. Future event horizon is a global concept of space-time but on the other hand, density of DE is a local quantity. So the relation between them will enact challenges to the causality concept. These lead to the introduction of a new HDE, where the length scale is given by the average radius of the Ricci scalar curvature, $R^{-1/2}$. The holographic Ricci dark energy model introduced by Granda and Oliveros (2009) based on the space-time scalar curvature, is fairly good in

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fitting with the observational data. In this model the fine tuning and causality problems can be avoided. The coincidence problem can also be solved effectively in this model. Recently, a modified form of Ricci dark energy was studied (Chen and Jing 2009) by assuming the density of DE contains the Hubble parameter H, the first order and the second order derivatives (i.e., $\dot{H}$ and $\ddot{H}$). The expression of the energy density of this modified holographic Ricci dark energy (MHRDE) is given by

$$\rho_\Lambda = 3M_p^2(\beta_1 H^2 + \beta_2 \dot{H} + \beta_3 \ddot{H} H^{-1}), \tag{1}$$

here H is the Hubble parameter, M_p^2 is the reduced Planck mass, β_1, β_2 and β_3 are constants. Recently, Das and Sultana (2015) have studied anisotropic MHRDE cosmological model with hybrid expansion law. Santhi et al. (2016a, b) have investigated anisotropic Bianchi type-VI_0 MHRDE models in scalar tensor theories of gravitation.

Brans-Dicke (1961) gravity is one of the earlier modifications to Einstein's theory, in which besides a gravitational part, a dynamical scalar field was introduced to account for a variable gravitational constant (Brans and Dicke 1961). This modification was introduced due to lack of compatibility of Einsteins theory with the Mach principle. Later Saez & Ballester (1986) introduced a scalar-tensor theory of gravity in which metric is coupled to a scalar field. Here the strength of the coupling between gravity and the field was governed by a parameter w. With this modification, they were able to solve a 'missing-mass problem'. Reddy et al. (2006), Adhav et al. (2007), Tripathy et al. (2008), Rao et al. (2011), Jamil et al. (2012), Rao et al. (2012), Pradhan et al. (2013), Rao et al. (2015) and Rao et al. (2016) are some of the authors who have investigated several aspects of the cosmological models in Saez-Ballester (1986) theory.

Study of spatially homogeneous and anisotropic models are very important to understand the early stages of evolution of the universe. Bianchi type universes are spatially homogeneous and anisotropic. So, in recent times, there has been a lot of interest in Bianchi type cosmological models. Recently, Sarkar and Mahanta (2013), Sarkar (2014), Adhav et al. (2015) have investigated holographic models in general relativity, while Kiran et al. (2014, 2015), Rao and Suryanarayana (2016), Reddy et al. (2016a) and Rao & Prasanthi (2016) have studied holographic dark energy models in various scalar-tensor theories of gravitation with anisotropic background. Very recently, Santhi et al. (2016a) have discussed Bianchi type-III, V and VI_0 generalized ghost pilgrim dark energy models.

Motivated by above investigations in this paper, we consider spatially homogeneous and anisotropic Bianchi type-II space times filled with matter and MHRDE in Saez-Ballester (1986) scalar-tensor theory of gravitation. This paper is organized as follows: In section 2, the field equations and energy momentum tensor are described. In section 3 the solution of the field equations have been obtained using time varying deceleration parameter proposed by Mishra et al. (2013). Some physical and geometrical properties are discussed in section 4. We conclude our results in section 5.

2 Field equations and energy momentum tensor

We consider the spatially homogeneous and anisotropic Bianchi type-II space-time described by the line element

$$ds^2 = dt^2 - A^2[dx^2 + dz^2] - B^2[dy + xdz]^2, \tag{2}$$

where A and B are the functions of cosmic time t only.

Saez-Ballester field equations for combined scalar and tensor fields are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} - w\phi^l \left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,r}\phi^{,r}\right) = -(T_{ij} + \overline{T}_{ij}) \tag{3}$$

and the scalar field ϕ satisfies the equation

$$2\phi^l\phi^i_{;j} + n\phi^{l-1}\phi_{,r}\phi^{,r} = 0 \tag{4}$$

where T_{ij} is energy-momentum tensor of matter, $\overline{T}_{ij}$ is the energy-momentum tensor of MHRDE fluid and w is a dimensionless constant. Here we choose $8\pi G = c = 1$ in relativistic units.
Also, the energy conservation equation is given by

$$(T_{ij} + \overline{T}_{ij})_{;j} = 0. \tag{5}$$

Here we consider a more general form of energy-momentum tensors for matter and the anisotropic MHRDE fluid in the following form

$$\begin{aligned} T_{ij} &= diag[\rho_m, 0, 0, 0] \\ \overline{T}_{ij} &= diag[\rho_\Lambda, -(\omega_\Lambda + \delta)\rho_\Lambda, -\omega_\Lambda\rho_\Lambda, -(\omega_\Lambda + \delta)\rho_\Lambda] \end{aligned} \tag{6}$$

where $\omega_\Lambda = \frac{p_\Lambda}{\rho_\Lambda}$ is equation of state (EoS) parameter of DE, p_Λ and ρ_Λ are pressure and energy density of dark energy respectively. ρ_m is the energy density of the matter and δ is deviation (skewness parameters) from ω_Λ on both x and z directions.

Now the Saez-Ballester field equations (3) and (4) for the metric (2) with the help of (6) can be written as

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{B^2}{4A^4} - \frac{w}{2}\phi^l\dot{\phi}^2 = -(\omega_\Lambda + \delta)\rho_\Lambda \tag{7}$$

$$2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} - \frac{3B^2}{4A^4} - \frac{w}{2}\phi^l\dot{\phi}^2 = -\omega_\Lambda\rho_\Lambda \tag{8}$$

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}^2}{A^2} - \frac{B^2}{4A^4} + \frac{w}{2}\phi^l\dot{\phi}^2 = \rho_m + \rho_\Lambda \tag{9}$$

$$\ddot{\phi} + \dot{\phi}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \frac{l}{2}\frac{\dot{\phi}^2}{\phi} = 0 \tag{10}$$

and the energy conservation equation (5) as

$$\dot{\rho_m} + \left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)(\rho_m + (1 + \omega_\Lambda)\rho_\Lambda) + \dot{\rho_\Lambda} + \delta\frac{\dot{B}}{B}\rho_\Lambda = 0, \tag{11}$$

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3 Solution of the field equations

The set of field equations (7)-(10) is a system of four independent equations with seven unknowns A, B, ϕ, ω_Λ, ρ_Λ, ρ_m and δ. In order to solve the above system completely we need three more equations relating these parameters, so we consider the following physically plausible conditions:

(i) The shear scalar σ is proportional to expansion scalar θ, which leads to the relationship between the metric potentials (Collins et al. 1980)

$$A = B^n \tag{12}$$

(ii) The varying deceleration parameter proposed by Mishra et al. (2013) is given by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = b(t) \tag{13}$$

where $a(t)$ is the average scale factor of the universe and $b(t)$ is an arbitrary function of time. By solving the above equation (13) using some suitable assumptions the average scale factor obtained as (Mishra et al. 2013)

$$a(t) = [sinh(\alpha t)]^{\frac{1}{k}} \tag{14}$$

where α is an arbitrary constant and k is a positive constant. This average scale factor yields a deceleration parameter (q) varying from early decelerated phase to current accelerated phase. Pradhan (2014), Reddy et al. (2016b) and Mishra et al. (2016) have investigated some cosmological models using this average scale factor. So, in this paper we consider average scale factor given in equation (14).

(iii) We consider MHRDE as the DE candidate given by Chen and Jing (2009). Since $M_p^2 = 1/8\pi G = 1$, we can take the energy density of MHRDE in Saez-Ballester theory as

$$\rho_\Lambda = 3(\beta_1 H^2 + \beta_2 \dot{H} + \beta_3 \ddot{H} H^{-1}), \tag{15}$$

here $H = \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)$ is the Hubble parameter, β_1, β_2 and β_3 are constants.

Now using equations (12), (10) and (14), we obtain the expressions for metric potentials

$$\begin{aligned} A &= [sinh(\alpha t)]^{\frac{3n}{k(2n+1)}} \\ B &= [sinh(\alpha t)]^{\frac{3}{k(2n+1)}}. \end{aligned} \tag{16}$$

and the scalar field ϕ as

$$\phi = \left[\frac{l+2}{2}\left(\frac{k\phi_1 tanh(\alpha t)}{\alpha(k-3)[sinh(\alpha t)]^{\frac{3}{k}}} + \psi_1\right)\right]^{\frac{2}{l+2}} \tag{17}$$

where ϕ_1 and ψ_1 are integrating constants.

Now the metric (2) can be written as

$$ds^2 = dt^2 - [sinh(\alpha t)]^{\frac{6n}{k(2n+1)}}[dx^2 + dz^2] - [sinh(\alpha t)]^{\frac{6}{k(2n+1)}}[dy + xdz]^2 \tag{18}$$

From equations (15) and (16), we get the energy density of MHRDE as

$$\rho_\Lambda = 3\alpha^2\left[\frac{\beta_1}{k^2}coth^2\alpha t + \left(2\beta_3 - \frac{\beta_2}{k}\right)csch^2\alpha t\right]. \tag{19}$$

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Now from the field equations (7)-(10), (15)-(17) and (19), we get
the EoS parameter of MHRDE as

$$\omega_\Lambda = -\left\{\frac{\frac{27n^2\alpha^2}{k^2(2n+1)^2}coth^2\alpha t - \frac{6n\alpha^2}{k(2n+1)}csch^2\alpha t - \frac{(3+2w)}{4}(sinh\alpha t)^{\frac{6(1-2n)}{k(2n+1)}}}{3\alpha^2\left[\frac{\beta_1}{k^2}coth^2\alpha t + \left(2\beta_3 - \frac{\beta_2}{k}\right)csch^2\alpha t\right]}\right\}, \tag{20}$$

the skewness parameter as

$$\delta = \frac{\frac{9\alpha^2(1+\mu_2-2\mu_2^2)}{k^2(2n+1)^2}coth^2\alpha t - \frac{3\alpha^2(1-\mu_2)}{k(2\mu_2+1)}csch^2\alpha t - (sinh\alpha t)^{\frac{-6(\mu_2)}{k(2\mu_2+1)}}}{3\alpha^2\left[\frac{\beta_1}{\mu_2^2}coth^2\alpha t + \left(2\beta_3 - \frac{\beta_2}{\mu_2}\right)csch^2\alpha t\right]}, \tag{21}$$

and the energy density of matter can be written as

$$\begin{aligned}\rho_m &= \frac{9\alpha^2 n(n+2)}{k^2(2n+1)^2}coth^2\alpha t - \frac{(1-2w\phi_0^2)}{4}(sinh\alpha t)^{\frac{6(1-2n)}{k(2n+1)}} \\ &\quad -3\alpha^2\left[\frac{\beta_1}{k^2}coth^2\alpha t + \left(2\beta_3 - \frac{\beta_2}{k}\right)csch^2\alpha t\right].\end{aligned} \tag{22}$$

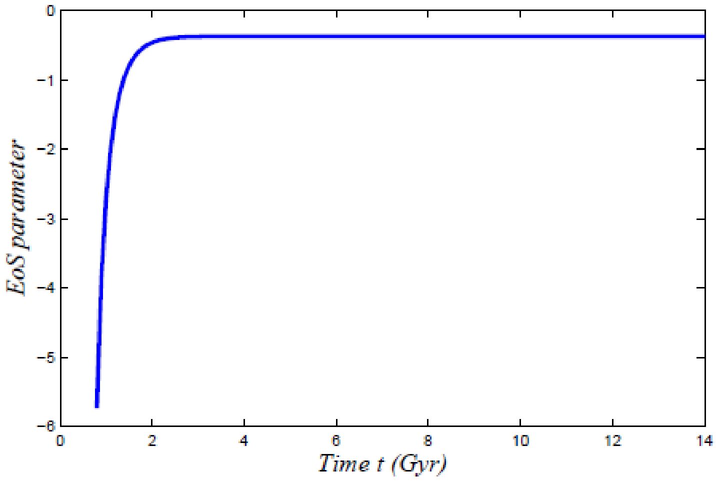

Figure 1: Plot of EoS parameter of MHRDE ω_Λ versus time t for $n = 1.5$, $k = 1.8$, $\alpha = 0.8$, $\beta_1 = 1.5$, $\beta_2 = 0.2$, $\beta_3 = 0.8$ and $w = 2$.

4 Some properties of the model

Spatial volume of the model as

$$V = \sqrt{-g} = (sinh\alpha t)^{3/k} \tag{23}$$

Hubble's parameter

$$H = \frac{\dot{a}}{a} = \frac{\alpha}{k}coth\alpha t \tag{24}$$

Expansion scalar is

$$\theta = 3H = \frac{3\alpha}{k}coth\alpha t \tag{25}$$

Shear scalar is

$$\sigma^2 = \frac{1}{2}\sum_{i=1}^{3} H_i^2 - \frac{1}{6}\theta^2 = \frac{3\alpha^2(n-1)^2}{k^2(2n+1)^2}coth^2\alpha t \tag{26}$$

Anisotropic parameter is

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$$A_h = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 = \frac{2(n-1)^2}{(2n+1)^2} \tag{27}$$

The deceleration parameter is

$$\begin{aligned} q &= \frac{-a\ddot{a}}{\dot{a}^2} \\ &= -1 + k(sech^2 \alpha t) \end{aligned} \tag{28}$$

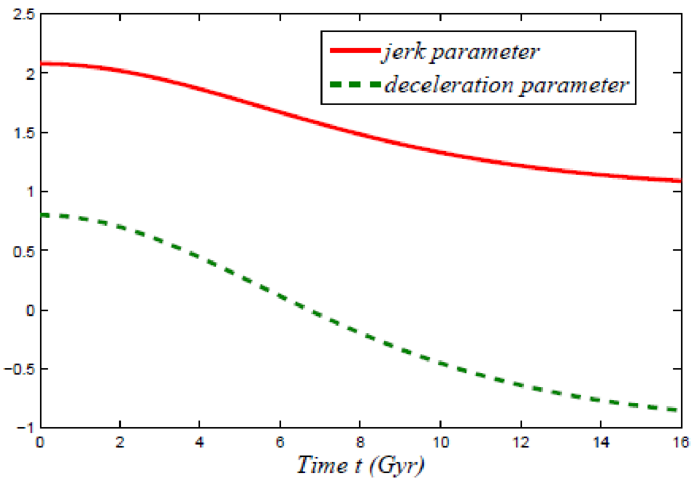

Figure 2: Plot of jerk parameter and deceleration parameter versus time t for $n = 1.5$, $k = 1.8$, $\alpha = 0.15$.

Jerk parameter is given by

$$j = 1 + k(2k-3)sech^2 \alpha t. \tag{29}$$

It is believed that in cosmology the cosmic jerk parameter can explain the transition of the universe from the decelerating to accelerating phase. This transition of the universe occurs for different models with positive value of the jerk parameter and the negative value of the deceleration parameter (Chiba and Nakamura 1998; Visser 2004). For example, ΛCDM models have a constant jerk parameter and equal to unity.

Figure 2 describes variation of deceleration parameter (q) and jerk parameter (j) versus cosmic time. It shows that the model is evolving from early decelerating phase ($q > 0$) to present accelerating phase ($q < 0$). Recent SNe Ia observations, explication that the present universe is accelerating and value of deceleration parameter lies within the range $-1 \leq q < 0$. It follows that in our model q is consistent with the recent observations. Also, we observe that jerk parameter is positive throughout the evolution and finally tends to one and q is tends to -1.

5 Conclusions

In this paper, we have presented anisotropic and homogeneous Bianchi type-II model filled with matter and anisotropic MHRDE in a scalar tensor theory of gravitation proposed by Saez-Ballester (1986). To obtain a viable solution of the field equations we have considered the average scale factor $a(t) = [sinh(\alpha t)]^{\frac{1}{k}}$ proposed by Mishra et al. (2013). We have outlined our results as follows:

Since the spatial volume of the models increase with cosmic time, the universe is expanding spatially. The scalar field (ϕ), matter and MHRDE energy densities (ρ_m and ρ_Λ) approach to zero at late times and they all diverse at initial epoch. The Hubble's parameter (H) and expansion scalar (θ) become constant at late times showing a uniform expansion of the universe. In the evolution of the universes there is a smooth transition from early decelerated phase to present accelerating phase (fig. 2), which is in good agreement with recent accelerated expansion of the universe conformed by many experiments. Since, $j \to 1$ and $q \to -1$, the model finally approaches to ΛCDM model. The behavior of EoS parameters of MHRDE show that the model starts their evolution from phantom region and crosses PDL then ultimately reaches a constant value in quintessence region. Thus our models are in good agreement with modern cosmological observations.

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APPLICATIONS OF DIFFERENTIAL EQUATIONS IN ENGINEERING

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Abstract

Differential equations have wide applications in various engineering and science disciplines. In general modeling of the variation of a physical quantity such as temperature, pressure, displacement, velocity, stress, strain, current, voltage or concentration of a pollutant with the change of time or location or both would result in differential equations. Similarly studying the variation of some physical quantities on other physical quantities would also lead to differential equations. In fact many engineering subjects such as mechanical vibration or structural dynamics, heat transfer, or theory of electric circuits are founded on the theory of differential equations. It is practically important for engineers to be able to model physical problems using mathematical equations and then solve these equations so that the behavior of the systems concerned can be studied.

Key Words: Electromagnetism, Harmonic functions.

Introduction

A differential equation is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change and the equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.

The study of differential equations is a wide field in pure and applied mathematics, physics and engineering. All of these disciplines are concerned with the properties of differential equations of various types. Pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions. Differential equations play an important role in modeling virtually every physical, technical or biological process, from celestial motion, to bridge design, to interactions between neurons. Differential equations such as those used to solve real-life problems may not necessarily be directly solvable, i.e. do not have closed form solutions. Instead, solutions can be approximated using numerical methods. Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to model the behavior of complex systems. The mathematical theory of differential equations first developed together with the sciences where the equations had originated and where the results found application. However, diverse problems, sometimes originating in quite distinct scientific fields, may give rise to identical differential equations. Whenever this happens, mathematical theory behind the equations can be viewed as a unifying principle behind diverse phenomena.

The purpose of learning the theory of differential equations is to be able to solve practical problems where differential equations are used. For engineering students, it is more important to know the applications and techniques for solving application problems than to delve into the nuances of mathematical concepts and theorems. Knowing the appropriate applications can motivate them to study the mathematical concepts and techniques. Engineers are more interested in mathematical modeling of a practical problem and actually solving the equations to find the solutions using the easiest possible method. Hence, a detailed step-by-step approach, especially applied to practical engineering problems, helps students to develop problem solving skills.

Applications in Engineering:

In classical mechanics the motion of a body is described by its position and velocity as the time value varies. Laws allow to express these variables dynamically as a differential equation for the unknown position of the body as a function of time. The force acting on a particle is known, Newton's second law is sufficient to describe the motion of a particle. Once independent relations for each force acting on a particle are available, they can be substituted into Newton's second law to obtain an ordinary differential equation, which is called the *equation of motion*.

In Electrodynamics Maxwell's equations are a set of partial differential equations that, together with the Lorentz force law, form the foundation of classical electrodynamics, classical optics, and electric circuits. These fields in turn underlie modern electrical and communications technologies. Maxwell's equations describe how electric and magnetic fields are generated and altered by each other and by charges and currents.

In General relativity the Einstein field equations are a set of ten partial differential equations in Albert Einstein's general theory of relativity which describe the fundamental interaction of gravitation as a result of space-time being curved by matter and energy. a tensor equation, the EFE equate local space-time curvature with the local energy and momentum within that space-time.

In Quantum mechanics the analogue of Newton's law is Schrödinger's equation for a quantum system. It is not a simple algebraic equation but in general a linear partial differential equation describing the time-evolution of the system's wave function.

Hamiltonian mechanics is a theory developed as a reformulation of mechanics and predicts the same outcomes as non-Hamiltonian classical mechanics. It uses a different mathematical formalism, providing a more abstract understanding of the theory. Historically, it was an important reformulation of classical mechanics, which later contributed to the formulation of statistical mechanics and quantum mechanics.

The wave equation is an important second-order linear partial differential equation for the description of waves as they occur in physics such as sound waves, light waves and water waves. It arises in fields like acoustics, electromagnetic and fluid dynamics. The problem of a vibrating string such as that of a musical instrument was studied by Jean le Rond d'Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-Louis Lagrange. d'Alembert discovered the one-dimensional wave equation, and within ten years Euler discovered the three-dimensional wave equation.

The heat equation is a parabolic partial differential equation that describes the distribution of heat in a given region over time. This is used in many engineering applications.

In mathematics Laplace's equation is a second-order equation. The general theory of solutions to Laplace's equation is known as potential theory. The solutions of Laplace's equation are the harmonic functions which are important in many fields of science, notably the fields of electromagnetism, astronomy, and fluid dynamics, because they can be used to accurately describe the behavior of electric, gravitational, and fluid potentials. In the study of heat conduction, the Laplace equation is the steady-state heat equation.

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In mathematics Poisson's equation is a partial differential equation of elliptic type with broad utility in mechanical engineering and theoretical physics. It arises, for instance to describe the potential field caused by a given charge or mass density distribution with the potential field known, one can calculate the associated gravitational or electrostatic field. It is a generalization of Laplace's equation which is also frequently seen in physics.

The Diffusion equation is a partial differential equation. In physics, it describes the behavior of the collective motion of micro-particles in a material resulting from the random movement of each micro-particle. In mathematics, it is applicable in common to a subject relevant to the Markov process as well as in various other fields, such as the material sciences, information science, life science, social science, and so on. These subjects described by the diffusion equation are generally called Brown problems.

The convection–diffusion equation is a combination of the diffusion and convection equations and describes physical phenomena where particles, energy or other physical quantities are transferred inside a physical system due to two processes: diffusion and convection. Depending on context, the same equation can be called the advection–diffusion equation, drift–diffusion equation or scalar transport equation.

The Lorenz system is a system of ordinary differential equations. It is notable for having chaotic solutions for certain parameter values and initial conditions. In particular, the Lorenz attractor is a set of chaotic solutions of the Lorenz system.

Radioactive decay is the process by which the nucleus of an unstable atom loses energy by emitting radiation, including alpha particles, beta particles, gamma rays, and conversion electrons. A material that spontaneously emits such radiation is considered radioactive. Radioactive decay is a stochastic process at the level of single atoms, in that, according to quantum theory, it is impossible to predict when a particular atom will decay, regardless of how long the atom has existed. For a collection of atoms however, the collection's decay rate can be calculated from their measured decay in Newton's law of cooling which states that *the rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.* As such, it is equivalent to a statement that the heat transfer coefficient, which mediates between heat losses and temperature differences, is a constant.

In Modeling a real world problem using differential equations is the determination of the velocity of a ball falling through the air, considering only gravity and air resistance. The ball's acceleration towards the ground is the acceleration due to gravity minus the acceleration due to air resistance. Gravity is considered constant, and air resistance may be modeled as proportional to the ball's velocity. This means that the ball's acceleration, which is a derivative of its velocity, depends on the velocity. Finding the velocity as a function of time involves solving a differential equation and verifying its validity.

In Engineering a series LCK network is chosen as the fundamental circuit; the voltage equation of this circuit is solved for a number of different forcing functions including a sinusoid, an amplitude modulated (AM) wave, a frequency modulated (KM) wave, and some exponentials. Some well-known formulas such as the AM-PM conversion mechanism and the derivation of the quasi-stationary approximation in time-invariant LCR networks are discussed.

The circuit containing a periodically varying parameter, i.e. the capacitance of the circuit is linearly time-varying. An introduction to the Mathieu equation is presented in general terms and examples have boon worked out for a number of electrical analogues and in this process the transformation of the Mathieu equation into Hill's equation is also discussed, the mechanism of generation of sub-harmonics is discussed by solving the Mathieu equation in non-linear form. Finally, the solutions of Mathieu equations are discussed briefly in general terms.

The applications of differential equations are an important tool to solve engineering problems in all branches.

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