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RESEARCH ARTICLE

BAYESIAN ESTIMATION OF THE EXPONENTIATED KUMARASWAMY-EXPONENTIAL DISTRIBUTION

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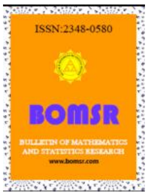

ABSTRACT

In this paper the Bayesian estimation of the parameters of the exponentiatedKumaraswamy-exponential distribution with four parameters, called EK-Exp (α,β,γ,λ), is considered according to informative and non-informative assumptions for the prior distributions. A comparison study for the obtained Bayesian estimates, based on different priors,is carried out through a Monte Carlo simulation study, for different sample sizes.
Keywords: Exponentiated Kumaraswamy-Exponential Distribution, Bayesian Estimation, informative prior distribution, non-informative prior distribution.

1. INTRODUCTION

The Kumaraswamy (K) distribution was proposed by (Kumaraswamy, 1980) to be appropriate for hydrological applications and related areas. Its cumulative distribution function (cdf) and probability density function (pdf) are given respectively by

$$F(x) = 1 - (1 - x^{\alpha})^{\beta}, \qquad 0 < x < 1, \tag{1}$$

and

$$f(x) = \alpha\beta x^{\alpha-1}(1 - x^{\alpha})^{\beta-1}, \qquad 0 < x < 1, \tag{2}$$

where$\alpha > 0$ and $\beta > 0$.

(Reyad and Ahmed, 2016) proposed Bayesian and E-Bayesian estimation for the Kumaraswamy distribution based on type II censoring. (Hussian, 2014) defined the estimated of the K distribution under both simple random sample (SRS) and ranked set sampling (RSS) approaches and made the estimation by maximum likelihood and Bayesian estimation methods. (Mandouh, 2016) introduced a Bayesian analysis for the Kumaraswamy-Weibull distribution.

A generalization of the K distribution called the exponentiated Kumaraswamy (EK) distribution was considered by (Lemonte et al., 2013). The cdf and pdf of the EK distribution are given respectively by

$$F(x) = [1-(1-x^{\alpha})^{\beta}]^{\gamma}, \qquad 0 < x < 1, \tag{3}$$

and

$$f(x) = \qquad \alpha\beta\gamma\, x^{\alpha-1}(1-x^{\alpha})^{\beta-1}[1-(1-x^{\alpha})^{\beta}]^{\gamma-1}, \qquad 0 < x < 1, \tag{4}$$

where$\alpha, \beta$and γ are positive shape parameters. Clearly, when the parameter $\gamma = 1$in the cdf and pdf of the EK distribution, given by (3) and (4), it reduces to the cdf and pdf of the K distribution given by (1) and (2) respectively.

According to (Rodrigues and Silva, 2015), the family of the exponentiated Kumaraswamy-Generalized (EK-G) distributions with cdf and pdf are given, respectively, by

$$F(x) = \{1-[1-\left(G(x)\right)^{\alpha}]^{\beta}\}^{\gamma}, \tag{5}$$

and

$$f(x) = \alpha\beta\gamma\, g(x)(G(x))^{\alpha-1}[1-\left(G(x)\right)^{\alpha}]^{\beta-1}\{1-[1-\left(G(x)\right)^{\alpha}]^{\beta}\}^{\gamma-1}, \tag{6}$$

for an arbitrary cdf, $G(x)$, with pdf, $g(x)$.

Using the previous two equations, (5) and (6), (Huang and Oluyede, 2014) proposed a family of distributions called exponentiated Kumaraswamy-Dagum (EK-D) distribution. They studied the properties of this family. Oluyede and Huang (2015) presented the maximum likelihood estimates of the parameters of the EK-D distribution under type I right censoring and type II double censoring plans. A new distribution called the exponentiated Kumaraswamy inverse Weibull (EK-IW) was proposed by Rodrigues et al., (2016). Eissa (2017) constructer a new five-parameter distribution, named the exponentiated Kumaraswamy-Weibull (EK-W) distribution.

Rodrigues and Silva, (2015) defined the exponentiated Kumaraswamy-exponential (EK-Exp) distribution. The idea of the construction of the EK-Exp distribution applying the EK-G distribution withthe cdf of the exponential distribution. Hence, the cdf of the EK-Exp distribution is givenby

$$F(x) = \{1-[1-\left(1-e^{-\lambda x}\right)^{\alpha}]^{\beta}\}^{\gamma}. \tag{7}$$

The pdf of EK-Exp distribution with parameters α, β, γandλ, denoted by EK-Exp$(\alpha, \beta, \gamma, \lambda)$is

$$f(x) = \alpha\beta\gamma\lambda\, e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}[1-\left(1-e^{-\lambda x}\right)^{\alpha}]^{\beta-1}\{1-[1-\left(1-e^{-\lambda x}\right)^{\alpha}]^{\beta}\}^{\gamma-1}, \quad x > 0, \tag{8}$$

where$\alpha, \beta$and γ are shape parameters and λ is a scale parameter.

Let $\underline{\theta} = (\alpha, \beta, \gamma, \lambda)$be a vector of all parameters of the EK-Exp distribution, the likelihood function is given by

$$L(\underline{X}|\underline{\theta}) = (\alpha\beta\gamma\lambda)^{n} e^{-\lambda\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left[\left(1-e^{-\lambda x_i}\right)^{\alpha-1}\left[1-\left(1-e^{-\lambda x_i}\right)^{\alpha}\right]^{\beta-1}\left\{1-\left[1-\left(1-e^{-\lambda x_i}\right)^{\alpha}\right]^{\beta}\right\}^{\gamma-1}\right]. \tag{9}$$

Moreover, (Rodrigues and Silva, 2015) discussed the estimation of the model parameters by the method of maximum likelihood. An application on a real data set was studied using maximum likelihood method to estimate the parameters of EK-Exp distribution and comparison these results with exponential (Exp), inverse exponential (IExp) and Weibull (W) distribution. They observed that the EK-Exp distribution provides better fit than Exp, IExp and W distribution because it has the lowest Akaike Information Criterion value compared with other distributions.

In this paper the Bayesian estimation of the parameters of the EK-Exp distribution is considered. The Bayesian estimation is performed under the assumption of the squared error loss function. Both informative and non-informative prior distributions are applied. The Bayes estimates for the parameters of the EK-Expdistribution are obtained via a Monte Carlo simulation study.

The rest of the paper is organized as follows. Section 2 is concerned with the Bayesian estimation for all the four parameters of the EK-Exp$(\alpha, \beta, \gamma, \lambda)$ distribution, assuming both informative and non-informative priors, where numerical solutions are appliedto compute the estimates of all the parameters. A Monte Carlo simulation study and numerical results are held in Section 3. Finally, Section 4 concludes the results.

2. BAYESIAN ESTIMATION

In this section, Bayesian estimation of the unknown vector of parameters $\underline{\theta}$of the EK-Expdistribution is considered under the squared error loss function, say $\hat{\underline{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\lambda})$. Assuming that the unknown parameters are independent. The Bayesian estimation for $\underline{\theta}$ is obtainedassuming the standard exponential distribution as an informative prior for each parameter, in Case 1. While Case 2 assumes a gamma prior for each one of the parameters. However, case 3 deals with the non-informative prior distribution for the parameters.

Case 1 (called B1): suppose that the prior distribution of each element of the vector of parameters $\underline{\theta} = (\alpha, \beta, \gamma, \lambda)$ is a standard exponential distribution, the joint prior density function of parameters $\underline{\theta}$is given by

$$\pi_1(\underline{\theta}) = e^{-\alpha} e^{-\beta} e^{-\gamma} e^{-\lambda}, \tag{10}$$

whereα, β, γ and $\lambda$are positive. Then, the joint posterior density function of $\underline{\theta}$ given the data $\underline{x} = (x_1, \ldots, x_n)$can be obtained from equation (9) and (10) as follows

$$\pi_1(\underline{\theta}|\underline{x}) \propto L(\underline{\theta})\pi_1(\underline{\theta}). \tag{11}$$

Case 2 (called B2): suppose that the prior distribution of each element of the vector of parameters $\underline{\theta} = (\alpha, \beta, \gamma, \lambda)$ is a gamma$(\delta_i, 1)$ distribution; $i = 1, 2, 3, 4$, the joint prior density function of parameters $\underline{\theta}$is given by

$$\pi_2(\underline{\theta}) = \frac{1}{\Gamma\delta_1} \alpha^{\delta_1 - 1} e^{-\alpha} \frac{1}{\Gamma\delta_2} \beta^{\delta_2 - 1} e^{-\beta} \frac{1}{\Gamma\delta_3} \gamma^{\delta_3 - 1} e^{-\gamma} \frac{1}{\Gamma\delta_4} \lambda^{\delta_4 - 1} e^{-\lambda}. \tag{12}$$

Hence, the joint posterior density function of $\underline{\theta}$ can be obtained from equation (9) and (12) as follows

$$\pi_2(\underline{\theta}|\underline{x}) \propto L(\underline{\theta})\pi_2(\underline{\theta}). \tag{13}$$

Case 3 (called B3): suppose a non-informative distribution for each parameter. Hence, the joint prior density function of the vector of the parameters $\underline{\theta}$is given by

$$\pi_3(\underline{\theta}) \propto \frac{1}{\alpha} \frac{1}{\beta} \frac{1}{\gamma} \frac{1}{\lambda}. \tag{14}$$

The joint posterior density function of $\underline{\theta}$ can be obtained from equation (9) and (14) as follows

$$\pi_3(\underline{\theta}|\underline{x}) \propto L(\underline{\theta})\pi_3(\underline{\theta}). \tag{15}$$

For the above three cases (B1, B2 and B3), the Bayesian estimation is performed under the squared error loss function. That is, the Bayesian estimators of the vector of the parameters, $\hat{\underline{\theta}}$, is the mean of the joint posterior distribution $\pi_j(\underline{\theta}|\underline{X})$, $j = 1, 2, 3$, given by equations (11), (13) and (15).

Hence,

$$\hat{\varphi}_i = E(\varphi_i|\underline{x}) = \int_{\underline{\theta}} \varphi_i \, \pi_j(\underline{\theta}|\underline{X}) \, d\underline{\theta}. \tag{16}$$

whereφ_i is a function of $\underline{\theta}$, $i = 1, 2, 3, 4$, and the integral is taken over four dimensional space. Though, to compute the integral in equation (16), we consider the Markov Chain Monte Carlo (MCMC) methods.

3. NUMERICAL COMPUTATIONS

In this section, the computations regarding the comparisons between the considered three cases (B1, B2 and B3) are performed assuming different sample sizes.

For a given vector $\underline{\theta} = (\alpha, \beta, \gamma, \lambda)$ generate random samples of different sizes$n = 20, 50, 100$from the EK-Exp distribution with selected parameters values$\alpha = 2, \beta = 3, \gamma = 1, \lambda = 2$.

A program code has been designed using R statistical package to solve the integral in equation (16) in order to obtain the estimates $\underline{\hat{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\lambda})$ of $\underline{\theta} = (\alpha, \beta, \gamma, \lambda)$ in all three previous cases, assuming the number of samples $m = 1000$.

Table 1 shows the estimates, Bias and mean squared error (MSE), for the three considered cases (B1, B2 and B3). However, for B2, the shape parameters of the gamma prior distributions, gamma$(\delta_i, 1)$; $\mathrm{i} = 1, 2, 3,4$, take values $\delta_1 = 2.2, \delta_2 = 4, \delta_3 = 3, \delta_4 = 2.8$.

Table 1: Bayesian estimates, Bias and MSE according to different prior assumptions.

n	case	$\hat{\theta}$				Bias				MSE			
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$
20	B1	1.88	2.44	1.57	2.37	-0.12	-0.56	0.57	0.37	0.60	0.87	0.47	0.50
	B2	1.67	3.38	2.19	2.17	-0.33	0.38	1.19	0.17	0.40	0.63	1.62	0.22
	B3	0.11	0.052	0.013	0.101	-1.88	-2.95	-0.98	-1.89	29.65	17.57	3.53	259.8
50	B1	1.92	2.55	1.57	2.43	-0.08	-0.45	0.57	0.43	0.51	0.77	0.45	0.50
	B2	1.62	3.35	2.02	2.06	-0.38	0.35	1.02	0.06	0.40	0.91	1.25	0.11
	B3	0.08	0.03	0.01	0.02	-1.92	-2.96	-0.99	-1.98	32.06	12.73	1.33	6.75
100	B1	1.88	2.55	1.55	2.49	-0.12	-0.45	0.55	0.49	0.33	0.59	0.43	0.44
	B2	1.62	3.31	1.89	2.02	-0.38	0.31	0.89	0.02	0.38	0.65	1.02	0.08
	B3	0.07	0.07	0.01	0.04	-1.93	-2.93	-0.98	-1.95	14.94	36.05	2.01	68.65

Table 1 shows that the MSE of all parameters are decreasing when the sample size nis increasing, only for cases B1 and B2. However, the MSE of all parameters isvery large, when considering Bayesian estimation based on the non-informative prior (B3). That is, in general, the Bayesian estimation based on the informative priors provides smaller MSE than the Bayesian estimation based on the non-informative prior. For all sample sizes, it is clear that the Bayesian estimation according to the standard exponential prior distributions (B1) provides the best estimates for the parameters, since their corresponding MSE is the smallest.

4. CONCLUSION

The Bayesian estimation of the parameters under squared error loss function was considered for the EK-Exp $(\alpha, \beta, \gamma, \lambda)$ distribution. The joint posterior distribution was introduced by using both informative and non-informative prior distribution. Based on Monte Carlo simulation study, it has been observed that the Bayesian estimates of $\underline{\theta}$under the assumption of the standard exponential prior distributions have the smallest MSE, when compared to the other cases. Also, Bayesian estimation based on informativeprior distributions were better than Bayesian estimation based on non-informative prior distributions.

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