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RESEARCH ARTICLE

THE LOGARITHMICALLY COMPLETE MONOTONICITY AND INEQUALITIES INVOLVING THE k-GAMMA FUNCTION

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ABSTRACT
In this paper, we prove several logarithmically completely monotonic results involving k-gamma function, and deduce some inequalities related to k-multinomial coefficient and k-multinomial beta function. These results can be used to evaluate or estimate some integrals.
Keywords: logarithmically completely monotonic; inequalities; k-multinomial coefficient; k-multivariate beta function

1. Introduction

The k-analogue of the gamma and psi functions are defined as(See [14,15,16,17,18])

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}} dt, \tag{1.1}$$

$$\psi_k(x) = \frac{d}{dx} \ln \Gamma_k(x). \tag{1.2}$$

where $k \in \mathbb{N}$, $x \in (0, \infty)$ and $\Gamma_1(x) = \Gamma(x)$, $\Gamma_k(k) = 1$, $\Gamma_k(x+k) = x\Gamma_k(x)$. The gamma, digamma and polygamma functions and its k-analogues play an important role in the theory of special functions, and have many applications in other many branches, such as statistics, fractional differential equations, mathematical physics and theory of infinite series. The reader may see references [4,5,6,7,8,9,10,30,31,32]. Some of the work about the complete monotonicity, convexity and concavity, and inequalities of these special functions may refer to [21,23,24,25,26,27]. Accordingly, we have

$$\psi_k(x) = \frac{d}{dx} \ln \Gamma_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{i=1}^{\infty} \frac{x}{ik(ik+x)}, \tag{1.3}$$

$$\psi_k^{(m)}(x) = \frac{d^m}{dx^m} \ln \Gamma_k(x) = (-1)^{m+1} m! \sum_{i=0}^{\infty} \frac{1}{(ik+x)^{m+1}}$$

$$= (-1)^{m+1} \int_0^{\infty} \frac{t^m e^{-xt}}{1-e^{-kt}} \mathrm{d}t, \ m \geq 1, \tag{1.4}$$

where$\gamma = \lim_{n\to\infty} \left(\sum_{i=1}^n \frac{1}{i} - \ln n\right) = 0.57721566 \ldots$ is the Euler-Mascheroni's constant for $m \in \mathbb{N}$.

Let the multinomial coefficient and the multivariate beta function be

$$\binom{\sum_{i=1}^n b_i}{b_1, b_2, \cdots, b_n} = \frac{\Gamma(1+\sum_{i=1}^n b_i)}{\prod_{i=1}^n \Gamma(1+b_i)},$$

and

$$\beta(b_1, b_2, \cdots, b_n) = \frac{\prod_{i=1}^n \Gamma(b_i)}{\Gamma(b_1+\cdots+b_n)},$$

respectively. Hence, $Q(x)$ is defined as

$$Q(x) = Q_{b,q;n}(x) = \frac{\Gamma(1+x\sum_{i=1}^n b_i)}{\prod_{i=1}^n \Gamma(1+xb_i)} \prod_{i=1}^n q_i^{xb_i} = \binom{x\sum_{i=1}^n b_i}{xb_1, xb_2, \cdots, xb_n} \prod_{i=1}^n q_i^{xb_i}$$

$$= \frac{\sum_{i=1}^n b_i}{\prod_{i=1}^n b_i} \frac{\prod_{i=1}^n q_i^{xb_i}}{x^{n-1}\beta(xb_1, \cdots, xb_n)}, \ x \in (0, \infty), \ n \in \mathbb{N}, \tag{1.5}$$

where $b = (b_1, b_2, \cdots, b_n)$ with $b_i > 0$, $q = (q_1, q_2, \cdots, q_n)$ with $\sum_{i=1}^n q_i = 1$ and $q_i > 0$ for $1 \leq i \leq n$. In [19] and [28], the authors proved that the function defined in (5) is logarithmically completely monotonic. For $a \in (0, 1)$, the logarithmically complete monotonicity of

$$Q_a(x) = Q_{a;b,q;n}(x) = \frac{\Gamma_a(1+x\sum_{i=1}^n b_i)}{\prod_{i=1}^n \Gamma_a(1+xb_i)} \prod_{i=1}^n q_i^{xb_i}, \tag{1.6}$$

has been proved by Qi in [20]. For more detailed information about the function $Q(x)$, one can refer to [2,3,12,13,19,22,28,33] and their references. Attracted by these work, it is natural to look for an extension involving $\psi_k(x)$and $\psi_{p,k}(x)$. In this paper, we will study the logarithmically complete monotonicity of the function $Q_k(x) = Q_{k;b,q;n}(x)$, $k \in \mathbb{N}$ and establish some inequalities.

2. The logarithmically completely monotonic function

In this paper, we denote $Q_k(x) = Q_{k;b,q;n}(x)$ as follows

$$Q_k(x) = Q_{k;b,q;n}(x) = \frac{\Gamma_k(k+x\sum_{i=1}^n b_i)}{\prod_{i=1}^n \Gamma_k(k+xb_i)} \prod_{i=1}^n q_i^{xb_i}, \tag{2.1}$$

where $x \in (0, \infty)$, k, $n \in \mathbb{N}$, $b = (b_1, b_2, \cdots, b_n)$ with $b_i > 0$, $q = (q_1, q_2, \cdots, q_n)$ with $\sum_{i=1}^n q_i = 1$ and $q_i > 0$ for $1 \leq i \leq n$. It is obviously that $Q_1(x) = Q(x)$.

Theorem 2.1. Let $x \in (0, \infty)$, k, $n \in \mathbb{N}$, $b = (b_1, b_2, \cdots, b_n)$with $b_i > 0$, $q = (q_1, q_2, \cdots, q_n)$ with $\sum_{i=1}^n q_i = 1$ and $q_i > 0$ for $1 \leq i \leq n$. Then the function $Q_k(x)$ defined in (7) is logarithmically completely monotonic for $x \in (0, \infty)$.

Proof. By directly calculating, we obtain

$$\ln Q_k(x) = \ln \Gamma_k\left(k + x\sum_{i=1}^n b_i\right) - \sum_{i=1}^n \ln \Gamma_k(k+xb_i) + x\sum_{i=1}^n b_i \ln q_i,$$

$$[\ln Q_k(x)]' = \left(\sum_{i=1}^n b_i\right) \psi_k(k + x\sum_{i=1}^n b_i) - \sum_{i=1}^n b_i \psi_k(k+xb_i) + \sum_{i=1}^n b_i \ln q_i,$$

and

$$[\ln Q_k(x)]'' = \left(\sum_{i=1}^n b_i\right)^2 \psi_k'(k + x\sum_{i=1}^n b_i) - \sum_{i=1}^n b_i^2 \psi_k'(k+xb_i).$$

By using (4), we get

$$\psi_k'(vx+k) = \int_0^\infty \frac{t}{1-e^{-kt}} e^{-(vx+k)t} \mathrm{d}t$$
$$= \int_0^\infty \frac{t}{e^{kt}-1} e^{-vxt} \mathrm{d}t = \int_0^\infty g\left(\frac{\rho}{v}\right) e^{-\rho x} \mathrm{d}\left(\frac{\rho}{v}\right),$$

where $v > 0,\ \rho = vt$, and $g(t) = \frac{t}{e^{kt}-1}$. Therefore, we obtain

$$[\ln Q_k\,(x)]'' = \int_0^\infty G\,(\rho) e^{-\rho x} \mathrm{d}\left(\frac{\rho}{v}\right), \tag{2.2}$$

where

$$G(\rho) = \left(\sum_{i=1}^n b_i\right)^2 g\left(\frac{\rho}{\sum_{i=1}^n b_i}\right) - \sum_{i=1}^n b_i^2\, g\left(\frac{\rho}{b_i}\right). \tag{2.3}$$

In the following, we shall prove $G(\rho) > 0$.
Considering the following identities:

$$\lim_{t\to 0^+} t^2\, g\left(\frac{1}{t}\right) = 0,$$
$$\frac{d}{dt}\left[t^2 g\left(\frac{1}{t}\right)\right] = \frac{d}{dt}\left(\frac{t}{e^{\frac{k}{t}}-1}\right) = \frac{1}{e^{\frac{k}{t}}-1} + \frac{ke^{\frac{k}{t}}}{t(e^{\frac{k}{t}}-1)^2} > 0,$$
$$\frac{d^2}{dt^2}\left[t^2 g\left(\frac{1}{t}\right)\right] = k^2 e^{\frac{k}{t}}\left(\frac{2e^{\frac{k}{t}}}{(e^{\frac{k}{t}}-1)^3 t^3} - \frac{1}{(e^{\frac{k}{t}}-1)^2 t^3}\right) > 0,$$

and based on the results in [11], we obtain $t^2 g(\frac{1}{t})$ is star-shaped and super-additive on $(0,\ \infty)$, that indicates

$$\left(\sum_{i=1}^n \frac{b_i}{\rho}\right)^2 g\left(\frac{1}{\sum_{i=1}^n \frac{b_i}{\rho}}\right) \geq \sum_{i=1}^n \left(\frac{b_i}{\rho}\right)^2 g\left(\frac{\rho}{\frac{b_i}{\rho}}\right),$$

which can be simplified as

$$\left(\sum_{i=1}^n b_i\right)^2 g\left(\frac{\rho}{\sum_{i=1}^n b_i}\right) \geq \sum_{i=1}^n b_i^2\, g\left(\frac{\rho}{b_i}\right). \tag{2.4}$$

So, we prove that $G(\rho) > 0$.
Based on Bernstein Theory in [29], we can deduce that $[\ln Q_k\,(x)]''$ is completely monotonic, which also indicates $[\ln Q_k\,(x)]'$ is strictly increasing. Moreover, by applying (3), we can prove $[\ln Q_k\,(x)]' < 0$. Hence, the function $Q_k(x)$ is logarithmically completely monotonic for $x > 0,\ k \in \mathbb{N}$.

3. Inequalities for the k-multinomial coefficient

In this section, we give some inequalities related to the k-multinomial coefficient. The k-multinomial coefficient is defined as

$$\binom{\sum_{i=1}^n b_i}{b_1, b_2, \cdots, b_n}_k = \frac{\Gamma_k(k+\sum_{i=1}^n b_i)}{\prod_{i=1}^n \Gamma_k\,(k+b_i)}, \tag{3.1}$$

where $k,\ n \in \mathbb{N}$, and $b = (b_1, b_2, \cdots, b_n)$ with $b_i > 0$ for $1 \leq i \leq n$.

Theorem 3.1. The inequality

$$\binom{\sum_{j=1}^l \mu_j\, x_j \sum_{i=1}^n b_i}{b_1 \sum_{j=1}^l \mu_j\, x_j, b_2 \sum_{j=1}^l \mu_j\, x_j, \cdots, b_n \sum_{j=1}^l \mu_j\, x_j}_k$$

$$\leq \prod_{j=1}^{l} \binom{x_j \sum_{i=1}^{n} b_i}{b_1 x_j, b_2 x_j, \cdots, b_n x_j}_k^{\mu_j} \tag{3.2}$$

issatisfied and the equality in (3.2) is satisfied if and only if $x_1 = x_2 = \cdots = x_l$, where k, l, $n \in \mathbb{N}$, $b = (b_1, b_2, \cdots, b_n)$ with $b_i > 0$ for $1 \leq i \leq n$, and $\mu_j \in (0,1)$ with $\sum_{j=1}^{l} \mu_j = 1$, $x_j > 0$ for $1 \leq j \leq l$.

Proof. By Theorem 2.1 we obtain that the function $Q_k(x)$ is logarithmically convex on $(0, \infty)$. That is

$$Q_k\left(\sum_{j=1}^{l} \mu_j x_j\right) \leq \prod_{j=1}^{l} Q_k^{\mu_j}(x_j). \tag{3.3}$$

By the definition of $Q_k(x)$, (3.3) can be written as

$$\binom{\sum_{j=1}^{l} \mu_j x_j \sum_{i=1}^{n} b_i}{b_1 \sum_{j=1}^{l} \mu_j x_j, b_2 \sum_{j=1}^{l} \mu_j x_j, \cdots, b_n \sum_{j=1}^{l} \mu_j x_j}_k \prod_{i=1}^{n} q_i^{b_i \sum_{j=1}^{l} \mu_j x_j}$$

$$\leq \prod_{j=1}^{l} \left[\binom{x_j \sum_{i=1}^{n} b_i}{b_1 x_j, b_2 x_j, \cdots, b_n x_j}_k \prod_{i=1}^{n} q_i^{b_i x_j}\right]^{\mu_j}. \tag{3.4}$$

Eliminating the public factor in (3.4), we complete the *proof.*

Theorem 3.2. The inequality

$$\prod_{j=1}^{l} \binom{x_j \sum_{i=1}^{n} b_i}{b_1 x_j, b_2 x_j, \cdots, b_n x_j}_k$$

$$< \binom{\sum_{j=1}^{l} x_j \sum_{i=1}^{n} b_i}{b_1 \sum_{j=1}^{l} x_j, b_2 \sum_{j=1}^{l} x_j, \cdots, b_n \sum_{j=1}^{l} x_j}_k \tag{3.5}$$

is satisfied, where k, l, $n \in \mathbb{N}$, $b = (b_1, b_2, \cdots, b_n)$ with $b_i > 0$ for $1 \leq i \leq n$, and $x_j > 0$ for $1 \leq j \leq l$.

Proof. By the results in [1] and the logarithmically convexity of the function $Q_k(x)$,we can obtain

$$\prod_{j=1}^{l} \binom{x_j \sum_{i=1}^{n} b_i}{b_1 x_j, b_2 x_j, \cdots, b_n x_j}_k \prod_{i=1}^{n} q_i^{b_i x_j}$$

$$< \binom{\sum_{j=1}^{l} x_j \sum_{i=1}^{n} b_i}{b_1 \sum_{j=1}^{l} x_j, b_2 \sum_{j=1}^{l} x_j, \cdots, b_n \sum_{j=1}^{l} x_j}_k \prod_{i=1}^{n} q_i^{b_i \sum_{j=1}^{l} x_j} \tag{3.6}$$

So, we easily obtain the inequality (3.5).

Theorem 3.3. If $d > 0 \; and \; 0 < c \leq d$, the inequality

$$\binom{(c+x) \sum_{i=1}^{n} b_i}{(c+x)b_1, (c+x)b_2 \ldots \ldots (c+x)b_n}_k \binom{d \sum_{i=1}^{n} b_i}{db_1, db_2 \ldots \ldots db_n}_k$$

$$\leq \binom{c \sum_{i=1}^{n} b_i}{cb_1, cb_2, \cdots, cb_n}_k \binom{(d+x) \sum_{i=1}^{n} b_i}{(d+x)b_1, (d+x)b_2 \ldots \ldots (d+x)b_n}_k \tag{3.7}$$

is satisfied and the equality in (3.7) is satisfied if and only if $c = d$, where k , $n \in \mathbb{N} b = b_1, b_2, \ldots \ldots, b_n$ with $b_i > 0$ for $1 \leq i \leq n$ and $x > 0$.

Proof. Let

$$h(x) = lnQ_k(c + x) - lnQ_k(d + x) - lnQ_k(c) + lnQ_k(d)$$

with $0 < c < d$. Since the function $Q_k(x)$ is a logarithmically completely monotonicfunction, the function $\frac{Q'k(x)}{Qk(x)}$ is strictly increasing. This implies

$$h'(x) = \frac{Q'k(c + x)}{Qk(c + x)} - \frac{Q'k(d + x)}{Qk(d + x)} < 0;$$

and $h(x) < h(0) = 0$. That is

$$h(x) = ln\left(Q_k(c + x)Q_k(d)\right) - ln\left(Q_k(d + x)Q_k(c)\right) \leq 0:$$

So we get

$$\ln\left[\binom{(c+x)\sum_{i=1}^{n} b_i}{(c+x)b_1,(c+x)b_2 \ldots\ldots (c+x)b_n}_k \binom{d\sum_{i=1}^{n} b_i}{db_1, db_2 \ldots\ldots db_n}_k \prod_{i=1}^{n} q_i^{b_i(c+x)b_1+b_i d}\right]$$

$$\leq \ln\binom{c\sum_{i=1}^{n} b_i}{cb_1, cb_2 \ldots\ldots cb_n}_k \binom{(d+x)\sum_{i=1}^{n} b_i}{(d+x)b_1,(d+x)b_2 \ldots\ldots (d+x)b_n}_k \prod_{i=1}^{n} q_i^{b_i(d+x)b_1+b_i c}$$

The proof is complete.

4. Inequalities for the k -multivariate beta function

In this section, the k -multinomial beta function is defined as

$$\beta k(b1;\ b2; \ldots.;\ bn) = \frac{\prod_{i=1}^{n} \Gamma_k(b_i)}{\Gamma_k\left(\sum_{i=1}^{n} b_i\right)} \tag{4.1}$$

$where\ k;\ n \in \mathbb{N}, and\ b = (b_1;\ b_2; \ldots.; b_n)\ with\ b_i > 0\ for\ 1 \leq i \leq n.$

Theorem 4.1. The inequality

$$\frac{\prod_{i=1}^{n}\sum_{j=1}^{l}\mu_j\ x_j\ b_i}{\sum_{j=1}^{l}\mu_j\ x_j\sum_{i=1}^{n} b_i}\beta k\left(b_1\sum_{j=1}^{l}\mu_j\ x_j, b_2\sum_{j=1}^{l}\mu_j\ x_j, \ldots.., b_n\sum_{j=1}^{l}\mu_j\ x_j\right)$$

$$\geq \prod_{j=1}^{l}\left[\frac{\prod_{i=1}^{n} x_j b_i}{x_j\sum_{i=1}^{n} b_i}\beta k\left(b_1 x_j, b_2 x_j, \ldots. b_n x_j\right)\right]^{\mu_j} \tag{4.2}$$

is satisfied and the equality in (4.2) is satisfied if and only if $x_1 = x_2 = \cdots = x_l$.

where $k;\ l;\ n \in \mathbb{N}, b = (b_1;\ b_2;\ \_\_\_;\ b_n)$ with $b_i > 0\ for\ 1 \leq i \leq n$, and $\mu_j \in (0; 1)$ with $\sum_{j=1}^{l}\mu_j = 1$; $x_j > 0$ for $1 \leq j \leq l$.

Proof. Substituting (3.1), (4.1) and $\Gamma\_k\ (x + k) = x\Gamma\_k\ (x)$ into (3.2), we obtain

$$\frac{\sum_{j=1}^{l}\mu_j\ x_j\sum_{i=1}^{n} b_i}{\prod_{i=1}^{n}\sum_{j=1}^{l}\mu_j\ x_j\ b_i}\frac{1}{\beta k\left(b_1\sum_{j=1}^{l}\mu_j\ x_j, b_2\sum_{j=1}^{l}\mu_j\ x_j, \ldots.., b_n\sum_{j=1}^{l}\mu_j\ x_j\right)}$$

$$\leq \prod_{j=1}^{l}\left[\frac{x_j\sum_{i=1}^{n} b_i}{\prod_{i=1}^{n} x_j b_i}\frac{1}{\beta k\left(b_1 x_j, b_2 x_j, \ldots. b_n x_j\right)}\right]^{\mu_j} \tag{4.3}$$

Taking the reciprocal of (4.3), the inequality (4.2) is proved.

Corollary 4.2. Let $k;\ n \in \mathbb{N}$, and $b = (b_1;\ b_2; ..;\ b_n)\ with\ b_i > 0\ for\ 1 \leq i \leq n$.

Then the function

$$\frac{\prod_{i=1}^{n} x b_i}{x\sum_{i=1}^{n} b_i}\beta k\left(b_1 x, b_2 x, \ldots.., b_n x\right) \tag{4.4}$$

is logarithmically concave for $x > 0$. That is

$$\frac{x\sum_{i=1}^{n} b_i}{\prod_{i=1}^{n} x b_i}\frac{1}{\beta k\left(b_1 x, b_2 x, \ldots\ldots, b_n x\right)} \tag{4.5}$$

is logarithmically completely monotonic for $x > 0$.

Proof. By (4.2) we can obtain the proof immediately.

Using similar procedure into Theorem (3.2) and (3.3), we can obtain the followinginequalities:

1. $\prod_{j=1}^{l}\frac{\prod_{i=1}^{n} x b_i}{x\sum_{i=1}^{n} b_i}\beta k\left(b_1 x, b_2 x, \ldots.., b_n x\right)$

$$> \frac{\prod_{i=1}^{n} b_i\sum_{j=1}^{l} x_j}{\sum_{j=1}^{l} x_j\sum_{i=1}^{n} b_i}\beta k\left(b_1\sum_{j=1}^{l}\mu_j\ x_j, b_2\sum_{j=1}^{l}\mu_j\ x_j, \ldots.., b_n\sum_{j=1}^{l}\mu_j\ x_j\right) \tag{4.6}$$

2. $\left(\frac{d+x}{c+x}\right)^{n-1}\frac{\beta k(b_1(d+x), b_2(d+x), \ldots, b_n(d+x))}{\beta k(b_1 d, b_2 d, \ldots, b_n d)}$

$$> \left(\frac{d}{c}\right)^{n-1} \frac{\beta k(b_1 d, b_2 d, \ldots, b_n d)}{\beta k(b_1 c, b_2 c, \ldots, b_n c)} \tag{4.7}$$

where $k;\ l;\ n \in \mathbb{N}, b = (b_1;\ b_2; \ldots;\ b_n)$ *with* $b_i > 0$ *for* $1 \leq i \leq n, x_j > 0$ *for* $1 \leq j \leq l, x > 0;\ 0 < c \leq d,$ *and the equality in* (4.7) *is satisfied if and only if* $c = d$.

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