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SERIES REPRESENTATIONS FOR GENERALIZED COMPLETE ELLIPTIC INTEGRALS WITH THREE PARAMETERS

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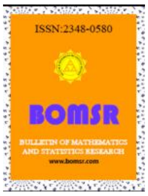

ABSTRACT
In this note, we show series representations for the generalizedcomplete elliptic integrals with three parameters, and generalized the known results.
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1. Introduction

The main object of this note is to present infinite series representations for generalized complete elliptic integrals with three parameters. As we all know, it plays an important role in various of mathematics as well as in physics and engineering. During the past years, many mathematicians have published some series representations for many special functions and important constants. In [3,4,5],series representations of some important mathematical constants were obtained. In particular, Alzer and Richard give a collection of single- parameter series representations for special functions such as complete elliptic integrals,gamma and beta functions, trigonometric and inverse trigonometric functions, logarithm function,and so on, by using the $łambda$ −methods.

This note shows infinite series representations for generalized complete elliptic integrals by generalizing the λ −methods of Alzer and Richard. First of all, we simply introduce definitions about generalized complete elliptic integrals defined by Takeuchi.

Let $k \in [0,1)$. the following classical complete elliptic integrals of the first kind and of the second kind are defined by

$$K(k) = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt$$

and

$$E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt.$$

In 2016, Takeuchi [16] defined a new form of the generalized complete elliptic integrals via generalized trigonometric functions with single parameter. We repeat the definition of complete p −elliptic integrals of the first kind $K_p(k)$ and of the second kind $E_p(k)$: for $k \in (0,1)$

$$K_p(k) = \int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}(1-k^pt^p)^{1-\frac{1}{p}}}, \tag{1.1}$$

$$E_p(k) = \int_0^1 \left(\frac{1-k^pt^p}{1-t^p}\right)^{\frac{1}{p}} dt. \tag{1.2}$$

In this note, we consider generalized elliptic integrals with three parameters[17] defined by

$$K_{p,q,r}(k) := \int_0^1 \frac{1}{(1-t^q)^{1/p}(1-k^qt^q)^{1/r}}\,dt, \tag{1.3}$$

and

$$E_{p,q,r}(k) := \int_0^1 \frac{(1-k^qt^q)^{1/r^*}}{(1-t^q)^{1/p}}\,dt \tag{1.4}$$

where $\frac{1}{r}+\frac{1}{r^*}=1$. In case $p=q=2$, $K_{p,q,r}(k)$ and $E_{p,q,r}(k)$ are reduced to the classical $K(k)$ and $E(k)$, respectively.

The beta function is defined by

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, t \in (0,1].$$

Given complex numbers a, b and c with $c \neq 0,-1,-2,\cdots$, the Gauss hypergeometric function is the analytic continuation to the slit place $\boldsymbol{C}\backslash[1,\infty)$ of the series

$$F(a,b;c;x) = \ {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}, |x|<1$$

For more details, we refer the reader to [7,8,9,10,11,15] and the references therein.

2. Main results

Lemma 2.1. [3] Suppose that the function G given by $G(x) = \frac{g(x)}{(1-\alpha x^\eta)^\xi}$ satisfies $g, G \in L^1[0,1]$ where $0 < \alpha \le 1, \eta > 0, \lambda < \frac{1}{2}, \xi \in \boldsymbol{R}$, and

$$b_j = b_j(\alpha,\eta) = \alpha^j \int_0^1 t^{j\eta}\, g(t)dt,$$

it follows that

$$\int_0^1 \frac{g(x)}{(1-\alpha x^\eta)^\xi}\,dx = \sum_{n=0}^{\infty} \frac{(\xi)_n}{n!\,(1-\lambda)^{n+\xi}} \sum_{j=0}^{n} \binom{n}{j}(-\lambda)^{n-j}\, b_j(\alpha,\eta,x). \tag{2.1}$$

Theorem 2.1. For $p,q,r > 1$ and $k \in [0,1), \lambda < \frac{1}{2}$, we have

$$K_{p,q,r}(k) = \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{r}\right)_n}{n!\,(1-\lambda)^{n+\frac{1}{r}}} \sum_{j=0}^{n} \binom{n}{j} \begin{pmatrix} -\frac{1}{p} \\ j \end{pmatrix} \begin{pmatrix} \frac{1}{p}-\frac{1}{q}-1 \\ j \end{pmatrix}^{-1} k^{jq}. \tag{2.2}$$

Proof. Taking $\alpha = k^q < 1, \eta = q > 0, \xi = \frac{1}{r}$ and $g(t) = (1-t^q)^{-\frac{1}{p}}$ in Lemma 2.1, we have

$$\begin{aligned} b_j &= \alpha^j \int_0^1 t^{j\eta}\ g(t)dt \\ &= k^{jq} \int_0^1 t^{jq}\ (1-t^q)^{-\frac{1}{p}} dt \\ &= \frac{k^{jq}}{q} \int_0^1 u^{j+\frac{1}{q}-1}\ (1-u)^{-\frac{1}{p}} du \\ &= \frac{k^{jq}}{q} B\left(j+\frac{1}{q}, 1-\frac{1}{p}\right) \\ &= \frac{k^{jq}}{q} \frac{\Gamma\left(j+\frac{1}{q}\right)\Gamma\left(1-\frac{1}{p}\right)\Gamma\left(\frac{1}{q}\right)}{\Gamma\left(j+\frac{1}{q}-\frac{1}{p}-1\right)\Gamma\left(\frac{1}{q}\right)}. \end{aligned} \tag{2.3}$$

Considering the formula $\pi_{p,q} = \frac{2}{q} B\left(\frac{1}{q}, 1-\frac{1}{p}\right)$ in [8], we have

$$b_j = \frac{\pi_{p,q} k^{jq}}{2} \cdot \frac{\left(\frac{1}{q}\right)_j}{\left(\frac{1}{q}-\frac{1}{p}+1\right)_j}. \tag{2.4}$$

Applying the known formula

$$(a)_j = (-1)^j j! \binom{-a}{j}, \tag{2.5}$$

we have

$$b_j = \frac{\pi_{p,q} k^{jq}}{2} \cdot \frac{\binom{-\frac{1}{p}}{j}}{\binom{\frac{1}{p}-\frac{1}{q}-1}{j}}. \tag{2.6}$$

The proof is complete. W

Remark 2.1. Taking $\lambda = 0$ in Theorem 2.1, the formula (2.2) changes into

$$\begin{aligned} K_{p,q,r} &= \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{r}\right)_n}{n!} \binom{n}{n} \binom{-\frac{1}{q}}{n} \binom{\frac{1}{p}-\frac{1}{q}-1}{n}^{-1} k^{nq} \\ &= \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{r}\right)_n \left(\frac{1}{q}\right)_n}{\left(-\frac{1}{p}+\frac{1}{q}+1\right)_n} \cdot \frac{(k^q)^n}{n!} \\ &= \frac{\pi_{p,q}}{2} F\left(\frac{1}{q}, \frac{1}{r}, -\frac{1}{p}+\frac{1}{q}+1, k^q\right). \end{aligned} \tag{2.7}$$

If $q = p$ and $\frac{1}{r} = 1 - \frac{1}{p}$, we obtain

$$K_p(k) = K_{p,p,1-\frac{1}{p}}(k) = \frac{\pi_{p,q}}{2} F\left(\frac{1}{q}, 1-\frac{1}{p}, 1, k^q\right). \tag{2.8}$$

If $q = p = 2$, we have

$$K(k) = K_{2,2,\frac{1}{2}}(k) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1, k^2\right). \tag{2.9}$$

So the $K_{p,q,r}$ is reduced to the classical $K_p(k)$ and $K(k)$ respectively.

Remark 2.2. Taking $p = q = r = 2$ in Theorem 2.1, the formula (2.2) reduces to (2.15) in [6].

Similar to the Theorem 2.1, we easily obtain following result.

Theorem 2.2. For $p, q, r > 0$ and $k \in [0,1), \lambda < \frac{1}{2}$, we have

$$E_{p,q,r}(k) = \frac{\pi_{p,q}}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{r}-1\right)_n}{n!\,(1-\lambda)^{n+\frac{1}{r}-1}}\sum_{j=0}^{n}\binom{n}{j}\binom{-\frac{1}{p}}{j}\binom{\frac{1}{p}-\frac{1}{q}-1}{j}^{-1}k^{jq} \tag{2.10}$$

Remark 2.3. Taking $\lambda = 0$ in Theorem 2.2, the formula (2.3) changes into

$$E_{p,q,r}(k) = \frac{\pi_{p,q}}{2}F\left(\frac{1}{q},\frac{1}{r}-1,1-\frac{1}{p}+\frac{1}{q},k^q\right). \tag{2.11}$$

Remark 2.4. Taking $p = q = r = 2$ in Theorem 2.2, the formula (2.10) reduces to (3.16) in [6].

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