Vol.7.Issue.1.2019 (Jan-Mar.)
©KY PUBLICATIONS

http://www.bomsr.com
Email:editorbomsr@gmail.com

RESEARCH ARTICLE

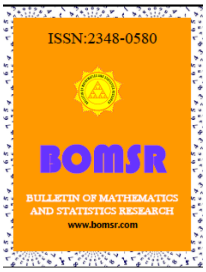
BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

INTERNATIONAL STANDARD SERIAL NUMBER
2348-0580

ALMOST HAUSDORFF SPACES AND NEARLY HAUSDORFF SPACES

MALLIKA MITRA[1], SUBRATA MAJUMDAR[2]
[1]Dept. of EEE, Varendra University, Rajshahi, Bangladesh,
[2] Dept. of Mathematics, Rajshahi University, Rajshahi, Bangladesh
https://doi.org/10.33329/bomsr.71.12



ABSTRACT
Two classes of topological spaces which are very similar to Hausdorff spaces have been defined and studied in this paper. The structures of these spaces have also been determined.

Keywords: Almost Hausdorff space, Nearly Hausdorff space.
AMS Classification (2010): 54D10, 54G12, 54D15

1. Introduction

This paper is a continuation of our programme of generalizing Hausdorff spaces started in [1]. An almost Hausdorff space is a topological space such that at most one pair of distinct points in it cannot be separated by disjoint open sets. A nearly Hausdorff space is a topological space such that there exists at most one point in it with the property that

i) this point cannot be separated from any other point by disjoint open sets,

ii) each pair of distinct points which are different from the above point can be separated from each other by disjoint open sets.

Clearly, the Hausdorff spaces belong to each of the above two classes of spaces.

Here we shall study almost Hausdorff spaces and nearly Hausdorff spaces. We shall prove a number of important properties of these spaces and determine their topological structures. For any non-empty class C of non-empty subsets of a topological space X, <C > will denote the topology generated by C. Generalisations of Regular and normal spaces have been studied in [2], [3] and [4].

2. Almost Hausdorff spaces

Definition 2.1. A topological space X will be called almost Hausdorff space if there exists at most one pair of points {a, b} in X which cannot be separated from each other by two disjoint open sets.

If such a and b exist and X has at least 3 elements, then we shall denote such spaces by $(X;\ \{a,b\})$ and call these spaces properly almost Hausdorff space if X is not Hausdorff.

Example 2.1.1.
Every Hausdorff space is almost Hausdorff space.

Example 2.1.2.
If $X = \{a, b, c, d\}$ & $\Im = \{X, \Phi, \{a, b\}, \{a, b, c\}, \{c, d\}, \{c\}, \{d\}, \{a, b, d\}\}$
Then $(X, \Im)$ is an almost Hausdorff space since only a and b cannot be separated by disjoint open sets. Moreover, X is properly almost Hausdorff.

Example 2.1.3.

Let $X = \Re$, $\Im = <\{X, \Phi, \{1,2\}\} \cup$ Discrete topology on $(\Re - \{1,2\})>$.

Example 2.1.4.
Let $X = \Re \cup \{i, \omega\}$ and $\Im = < U_{\Re} \cup \{i, \omega\}>$, where $U_{\Re}$ is the usual topology on $\Re$. Here i and ω are non-real square and cube root of -1 and 1 respectively. It is clear that i and ω cannot be separated from each other and that for each $x, y \in \Re$, $x \neq y$, x and y can be separated from each other. Also each of i and ω can be separated from every $x \in \Re$. Hence X is properly almost Hausdorff.

Properties

We have proved a few properties almost Hausdorff spaces.

Theorem 2.2.1. *Any subspace of an almost Hausdorff space is also so.*

Proof. Let X be an almost Hausdorff space and A be a subspace of X. Then there exists at most one pair $\{x, y\}$ with open sets G and H in X such that $x \in G$, $y \in H$ and $G \cap H \neq \Phi$. If X is Hausdorff, then so is A. So, suppose X is not Hausdorff. Then there exist exactly one pair of distinct points a, b such that for each pair of open sets G and H with $a \in G$, $b \in H$, $G \cap H \neq \Phi$. If both of a and b belong to A, then $G \cap A$ and $H \cap A$ are open sets in A and $a \in G \cap A$, $b \in H \cap A$ and $(G \cap A) \cap H \cap A \neq \Phi$. If either none of a and b belong to A, or exactly one of a and b belong to A, then A is Hausdorff. Thus in all possible cases A is almost Hausdorff.

Corollary 2.2.1. *For any two spaces A and B of a topological space X, $A \cap B$ will be almost Hausdorff space if either A or B is almost Hausdorff.*

Theorem 2.2.2. *The product of two almost Hausdorff spaces is also an almost Hausdorff space.*

Proof. Let X and Y be two almost Hausdorff spaces. If any of X and Y is Hausdorff, then $X \times Y$ is also Hausdorff. Then the proof is obvious. So we assume that neither X nor Y is Hausdorff. Let X and Y be given by $(X; \{x_1, x_2\})$ and $(Y; \{y_1, y_2\})$ respectively.

Let U and V be two open sets in $X \times Y$ such that $(x_1, y_1) \in U$ and $(x_2, y_2) \in V$.

Then $(x_1, y_1) \in G_1 \times H_1$, $(x_2, y_2) \in G_2 \times H_2$, for some open sets G_1, G_2 in X and open sets H_1, H_2 in Y. So, $x_1 \in G_1$, $x_2 \in G_2$, $y_1 \in H_1$, $y_2 \in H_2$. By definition, $G_1 \cap G_2 \neq \Phi$, $H_1 \cap H_2 \neq \Phi$. Hence $(G_1 \times H_1) \cap (G_2 \times H_2) \neq \Phi$. So $U \cap V \neq \Phi$. Thus (x_1, y_1) and (x_2, y_2) cannot be separated by disjoint open sets. However, all other pairs of distinct points can be separated by disjoint open sets.

Let $(x, y) \in X \times Y$ and suppose (x, y) is different from (x_1, y_1) or (x_2, y_2), say, $(x, y) \neq (x_1, y_1)$ i.e., $x \neq x_1$ or $y \neq y_1$. Then there exist open sets G, G_1 in X and H, H_1 in Y such that $x \in G$, $x_1 \in G_1$, $y \in H$, $y_1 \in H_1$, with $G \cap G_1 = \Phi$ or $H \cap H_1 = \Phi$. So, $(G \times H) \cap (G_1 \times H_1) = \Phi$. Thus (x, y) and (x_1, y_1) can be separated by disjoint open sets. Similarly it can be shown that if $(x, y) \neq (x_2, y_2)$, then they can be separated. Also if $\{(x, y), (x', y')\} \neq \{(x_1, y_1), (x_2, y_2)\}$, then it can be similarly shown that they can be separated by disjoint open sets.

Hence $X \times Y$ is almost Hausdorff.

Theorem 2.2.3. *If $X \times Y$ is almost Hausdorff spaces, then at least one of X and Y is almost Hausdorff space.*

Proof. Suppose none of X and Y is almost Hausdorff spaces. Then there exist $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$ such that x_1 cannot be separated from both x_2, x_3; y_1 cannot be separated from y_2 and y_3 . Then there exist open sets G_1 , G_2, G_3 in X and open sets H_1, H_2, H_3 in Y with $x_1 \in G_1$, $x_2 \in G_2$, $x_3 \in G_3$, $y_1 \in H_1$, $y_2 \in H_2$, $y_3 \in H_3$ such that $G_1 \cap G_2 \neq \Phi$, $G_1 \cap G_3 \neq \Phi$, $H_1 \cap H_2 \neq \Phi$, $H_1 \cap H_3 \neq \Phi$. So, $(G_1 \times H_1) \cap (G_2 \times H_2) \neq \Phi$, $(G_1 \times H_1) \cap (G_3 \times H_3) \neq \Phi$. Since $(x_1, y_1) \in (G_1 \times H_1)$,

$(x_2, y_2) \in (G_2 \times H_2)$, $(x_3, y_3) \in (G_3 \times H_3)$, (x_1, y_1) cannot be separated from (x_2, y_2) and (x_3, y_3). This contradicts the fact that $X \times Y$ is almost Hausdorff space.

Structure of almost Hausdorff spaces.

Structure of the topology of an almost Hausdorff space (X; {a, b}) with at least three elements:

Theorem 2.3.1. Every properly almost Hausdorff space $((X; \{a, b\}), \Im)$ with at least three elements is given by

(i) $X = Y \cup \{a, b\}$, for some a, b in X, $(a \neq b)$, with $Y = X - \{a, b\} \neq \Phi$,

(ii) $\Im = \langle \{X, \Phi\} \cup H_Y \cup \{a, b\} \rangle$, where H_Y is a Hausdorff topology on Y.

Proof: It is obvious that $((X; \{a, b\}), \Im)$ given by (i) and (ii) is a properly almost Hausdorff space with at least three elements.

We shall prove the converse. So, let $((X; \{a, b\}), \Im)$ be a properly almost Hausdorff space with at least three elements. By definition, the topology H_Y, induced on $Y = X - \{a, b\}$ is Hausdorff. Then the following cases may arise:

Case-1: $\{a, b\}$ is in $\Im$.

Case-2: Both $\{a\}$ and $\{b\}$ is in $\Im$.

Case-3: $\{a\} \cup G \in \Im$, for some non-empty $G \in H_Y$, or

$\{b\} \cup H \in \Im$, for some non-empty $H \in H_Y$, or

$\{a, b\} \cup K \in \Im$, for some non-empty $K \in H_Y$,

Case-4: One of $\{a\},\{b\} \in \Im$.

If case-1 holds, then $\Im$ is given by (ii).

If case-2 holds, then a and b can be separated from each other by disjoint open sets {a}, {b} in X which is a contradiction to the definition.

If case-3 holds, then in the first situation, there exists $y \in G$ such that a and y cannot be separated from each other by disjoint open sets, in the second situation, then b cannot be separated from some $y \in H$, and in the third situation, both a and b cannot be separated from some $y \in K$. Thus in all situations there is a contradiction to the definition.

If none of the cases (1), (2) and (3) holds, but (4) holds, then if $\{a\} \in \Im$, then b cannot be separated from the elements of Y, and if $\{b\} \in \Im$, then a cannot be separated from the elements of Y. Always there is a contradiction to the definition. The proof is thus complete. We know that for a Hausdorff space every compact subspace is closed ([5], [6]). We now prove the following theorem:

Theorem 2.4.1. *Let (X; {a, b}) be a properly almost Hausdorff space and K be a compact subspace of X. Then K is closed if and only if either both of a, b belong to K or none of a, b belong to K.*

Proof.

Case-i. Both a, b are in K.

Let $x \in K^c$ and $y \in K$. Then $x \neq y$ and $\{x, y\} \neq \{a, b\}$. Hence by the condition on (X; {a, b}) there exist disjoint open sets G_x and H_y such that $x \in G_x$, $y \in H_y$ and $G_x \cap H_y = \Phi$. Then $\{H_y \mid y \in K\}$ is an open cover of K. Since K is compact, there exists $H_{y_1}, H_{y_2}, ..., H_{y_n}$ for some n such that $H_{y_1} \cup H_{y_2} \cup ... \cup H_{y_n} \supseteq K$, and $(H_{y_1} \cup H_{y_2} \cup ... \cup H_{y_n}) \cap (G_{y_1} \cup G_{y_2} \cup ... \cup G_{y_n}) = \Phi$. Hence $(G_{y_1} \cup G_{y_2} \cup ... \cup G_{y_n}) \cap K = \Phi$. i.e., $G_{y_1} \cup G_{y_2} \cup ... \cup G_{y_n} \subseteq K^c$.

Since $x \in G \cup G \cup ... \cup G$ and the latter is open, X is an interior point of K^c.

Hence K^c is open and so K is closed.

Case-ii: None of a, b are in K i.e., $a, b \in K^c$.

Similar argument will show that K is closed.

Case-iii: One of a, b, say $a \in K$.

Then $b \notin K$ i.e., $b \in K^c$.

Let $b \in G$ with G open in X. Then G must contain a by the structure theorem (Theorem 2.3.1). So $G \cap K \neq \Phi$. So $G \not\subset K^c$. Thus b is not an interior point of K^c. Thus K^c is not open, i.e., K is not closed.

The concept of properly almost Hausdorff spaces (X; {a, b}) can be extended to a on- Hausdorff space (X; A), which is less close to a Hausdorff space, where A is a subset of X with more than two elements. The definition of (X; A) is as follows:

If there exists a non-empty subset $A \subset X$ such that $X\text{-}A \neq \Phi$ and

i) no pair of distinct points in A can be separated from each other by disjoint open sets in X,

ii) each pair of distinct points x, y in X such that at least one of x and y does not belong to A, can be separated by disjoint open sets in X.

The properties of (X; A) will be similar to that of (X; {a, b}).

Example 2.5.1.

If X = {letters of English alphabets} &

$\mathfrak{I} = <\{X, \Phi\} \cup \{a, b, d, e\} \cup$ Discrete topology on $\{X\text{-}\{a, b, d, e\}\}>$. Here A= {a, b, d, e}.

Example 2.5.2.

If $X = \{a, b, c, d\} \cup \Re$

Here A= {a, b, c}., and $\mathfrak{I} = \{\{X, \Phi, \{a, b, c\}, \{d\}\} \cup$ the usual topology on $\Re$ }.

Hausdorffification of (X; {a, b}, $\mathfrak{I}$)

To make (X; {a, b}, $\mathfrak{I}$) Hausdorff, we extend the topology H_Y of Y to the topology $\mathfrak{I}_H$ on X

where $\mathfrak{I}_H = < H_Y \cup \{\{a\}, \{b\}\}>$. Clearly $\mathfrak{I}_H$ is a Hausdorff topology which contains $\mathfrak{I}$.

2. Nearly Hausdorff Space

Definition 3.1. A topological space X will be called nearly Hausdorff if either X is Hausdorff or

i) there exists only one point $a \in X$ which cannot be separated from any other points of X,

ii) every other pair of distinct points of X can be separated from each other. We denote such spaces by (X; a).

Example 3.1.1.

Let X be a non-empty set with at least two elements and let $a \in X$. Let Y be X-{a}, and H_Y a Hausdorff topology on Y. Let $\mathfrak{I} = H_Y \cup \{X\}$. Then ((X; a), $\mathfrak{I}$) is a nearly Hausdorff topological space.

The following are particular cases of the above:

Example 3.1.2.

Let $X= \Re \cup \{i\}$, where $i=\sqrt{-1}$, then (X, $\mathfrak{I}$) is a nearly Hausdorff topological space where

$\mathfrak{I}=U_{\Re} \cup \{X\}$, $U_{\Re}$ being the usual topology on $\Re$.

Example 3.1.3.
Let X= {a, b, c, d, e, f}, Y= {b, c, d, e, f} and let D_Y be the discrete topology on Y. Then ((X: a), $\mathfrak{I}$) is a nearly Hausdorff topological space where $\mathfrak{I} = \{X\} \cup D_Y$

Properties
We have proved a few properties of nearly Hausdorff spaces.

Theorem 3.2.1. *Every nearly Hausdorff space is connected if it is not Hausdorff.*

Proof. Let X be a nearly Hausdorff space which is not Hausdorff. Then there exists a unique element a of X such that for all $x \in X$, $x \neq a$, and for every pair of open sets G and H, with $x \in G$, $x \in H$, $G \cap H \neq \Phi$.

If possible, suppose X is not connected. Then there exists non-empty open sets V, W in X such that $V \cap W = \Phi$ and $X=V \cup W$. Now a is in one of V and W say V. Since $W \neq \Phi$, there exists $x \in W$. Since $V \cap W = \Phi$, we have a contradiction. Thus X is connected.

Theorem 3.2.2. *Every subspace of a nearly Hausdorff space is either nearly Hausdorff or Hausdorff.*

Proof. Let (X; a) be a nearly Hausdorff space and $Y \subseteq X$.

If $a \in Y$, then (Y; a) is obviously nearly Hausdorff.

If $a \notin Y$, then each point of Y can be separated from each other by disjoint open sets. Thus Y is Hausdorff.

Corollary3.2.1. *For any two subspaces C and D of a topological space X, $C \cap D$ will be nearly Hausdorff space if either C or D is nearly Hausdorff.*

Theorem 3.2.3. *The product of two nearly Hausdorff spaces is also a nearly Hausdorff space.*

Proof. Let (X, a) and (Y, b) two nearly Hausdorff spaces.

Then we shall prove that (X×Y, (a, b)) is also a nearly Hausdorff space.

Let(x_1,y_1), $(x_2,y_2) \in X\times Y-\{(a, b)\}$. Let$(x_1,y_1) \neq (x_2,y_2)$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$.

Then either x_1, x_2 can be separated from each other by disjoint open sets G_1 , G_2 in X or y_1, y_2 can be separated from each other by disjoint open sets H_1, H_2 in Y. Since either $G_1 \cap G_2 = \Phi$, or Here $H_1 \cap H_2 = \Phi$, $(G_1 \times H_1) \cap (G_2 \times H_2) = \Phi$. Here (x_1,y_1) and (x_2, y_2) can be separated in X×Y.

Now let us consider the points (a,y_1) and (a, y_2) of X×Y , with $y_1, y_2 \in Y$, $y_1 \neq y_2$. Then there exist open sets H_1, H_2 in Y with $y_1 \in H_1, y_2 \in H_2$ and $H_1 \cap H_2 = \Phi$. Hence $(a,y_1) \in X\times H_1$and $(a, y_2) \in X\times H_2$. Thus (a,y_1) and (a, y_2) can be separated in X×Y. Similarly (x_1,b), (x_2,b),with $x_1 \neq x_2$ can be separated in X×Y.

However, x, y where $x \in X$, $y \in Y$ and a, b cannot be separated from each other by disjoint open sets, since the only open sets containing (a, b) is X×Y. Thus (X×Y, (a, b)) is nearly Hausdorff.

Theorem 3.2.4. *Let X and Y be two topological spaces and let $a \in X$, $b \in Y$ such that $(X\times Y,(a, b))$ is nearly Hausdorff. Then both of (X, a) and (Y, b) are nearly Hausdorff spaces.*

Proof. We first prove that (X, a) is nearly Hausdorff. Let x_1, $x_2 \in X$-{a} with $x_1 \neq x_2$, Then (x_1,b) and (x_2, b) can be separated in X×Y. So, there exist open setsG_1, G_2 in X such that $(x_1,b) \in G_1\times Y$, $(x_2, b) \in G_2\times Y$. Then $(G_1\times Y) \cap (G_2\times Y) = \Phi$. Then $G_1 \cap G_2 = \Phi$.

Let $x \in X$-{a}. Since the only open sets in X×Y containing (a, b) is X×Y, the only open set in X which contains a is X. Hence x and a cannot be separated from each other by disjoint open sets. Thus (X, a) is nearly Hausdorff.

Similarly it can be shown that (Y, b) is nearly Hausdorff space.

Structure of nearly Hausdorff spaces.

It is clear that if a topological space X with at least two elements is not Hausdorff, then ((X; a), $\Im$) is nearly Hausdorff and is given by

i) $X=Y\cup \{a\}$, where $a \notin Y$,

ii) Y is a Hausdorff space with a Hausdorff topology H_Y on Y,

iii) $\Im=\{H_Y\cup \{X\}\}$.

The concept of a **nearly Hausdorff space (X; a)** can be extended to a non Hausdorff space (X; A) which is less close to a Hausdorff space. Here A is a subset of X with more than one elements and $X-A \neq \Phi$.

For a topological space (X; A), either X is Hausdorff, or for each pair of elements x, y in X with $x \neq y$, x and y can be separated from each other by disjoint open sets in X if and only if neither x nor y belongs to A.

The properties of (X; A) will be similar to that of (X; a).

References:

[1]. M. Mitra, N. Akhter, and S. Majumdar, Locally Hausdorff spaces, Bulletin of Mathematics and Statistics Research, Vol. 5, Issue. 4 (2017).

[2]. S.K. Biswas, N. Akhter and S. Majumdar, Pseudo Regular and Pseudo Normal Topological Spaces, International Journal of Trend in Research and Development, Vol.5. Issue.1 (2018), 426- 430.

[3]. S.K. Biswas, S. Majumdar and N. Akhter, Strongly Pseudo Regular and Strongly Pseudo Normal Topological Spaces, International Journal of Trend in Research and Development, Vol.5. Issue.3 (2018), 459-464.

[4]. S.K. Biswas, N. Akhter and S. Majumdar, Strictly Pseudo Regular and Strictly Pseudo Normal Topological Spaces, International Journal of Trend in Research and Development, Vol.5. Issue. (2018), 130-132.

[5]. S. Majumdar and N. Akhter , Topology, Somoy Publisher, Dhaka, Bangladesh, January 2009.

[6]. J. R. Munkres, Topology, Prentice-Hall of India Private Limited, New Delhi-110001, 2008.