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DYNAMIC BUCKLING OF A VISCOUSLY DAMPED MODIFIED QUADRATIC MODEL ELASTIC STRUCTURE STRUCK BY AN IMPULSE

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ABSTRACT

In this work, asymptotic expansions were carried out on the perturbed system of nonlinear coupled nonhomogeneous ordinary differential equations originally derived by Danielson [1]. Impulse loading was thereafter imposed on the viscously damped system with light damping coefficients ε. Q-basic codes were finally used in the analysis and the result revealed an increasing dynamic buckling load I_D with increase in the damping coefficients.

Key words: Modified, viscously damped, quadratic, model structure, perturbation, elastic, impulse

1. Introduction:

The concept of buckling, though as old as creation, was neglected and overlooked until Koiter [2] overcame the frontiers of this inaction. This elicited various interests in the area. Prominent amongst them include Budiansky and Hutchinson [3] and Hutchinson and Budiansky [4]. The duo investigated buckling in line with imperfections in the structures. In their work, they used a quadratic model structure to study the dynamic stability of elastic structures laden with imperfection. Danielson [1] modified the Budiansky / Hutchinson's structure by incorporating an additional mass M_0 and a spring with spring constant K_0 (see fig. 1), thus enhancing a pre – buckling motion. Ette [5] extended the work of Danielson by the inclusion of an arbitrary explicitly time dependent slowly varying load $\bar{F}(T)$. Other researchers in buckling include Gladden et al. [6] who investigated the dynamic buckling and fragmentation of slender rods axially impacted by a projectile, Amit [7] investigated nonlinear response of shallow arches under dynamic and static loadings while Wei et al. [8] investigated dynamic buckling of thin cylindrical shells under axial impact. Others include Lu and Wang [9], Osuji et al. [10], Udo – Akpan and Ette [11], Chukwuchekwa and Ette [12] amongst others. This work is an extension of a similar one by Ette [5] vis – a – vis Danielson's [1] earlier investigation. Here, we have imposedan impulse loading on the system and incorporated light viscous damping of coefficient ε and imperfection coefficient $\bar{\xi}$ .We recall that Etteand Osuji [13]had earlier considered the same structure for the case of periodic loadingwhile Lillermae et al. [14], Paulo et al. [15] and Ronning et al. [16] had,in the same token, made insightful contributions on the subject matter.

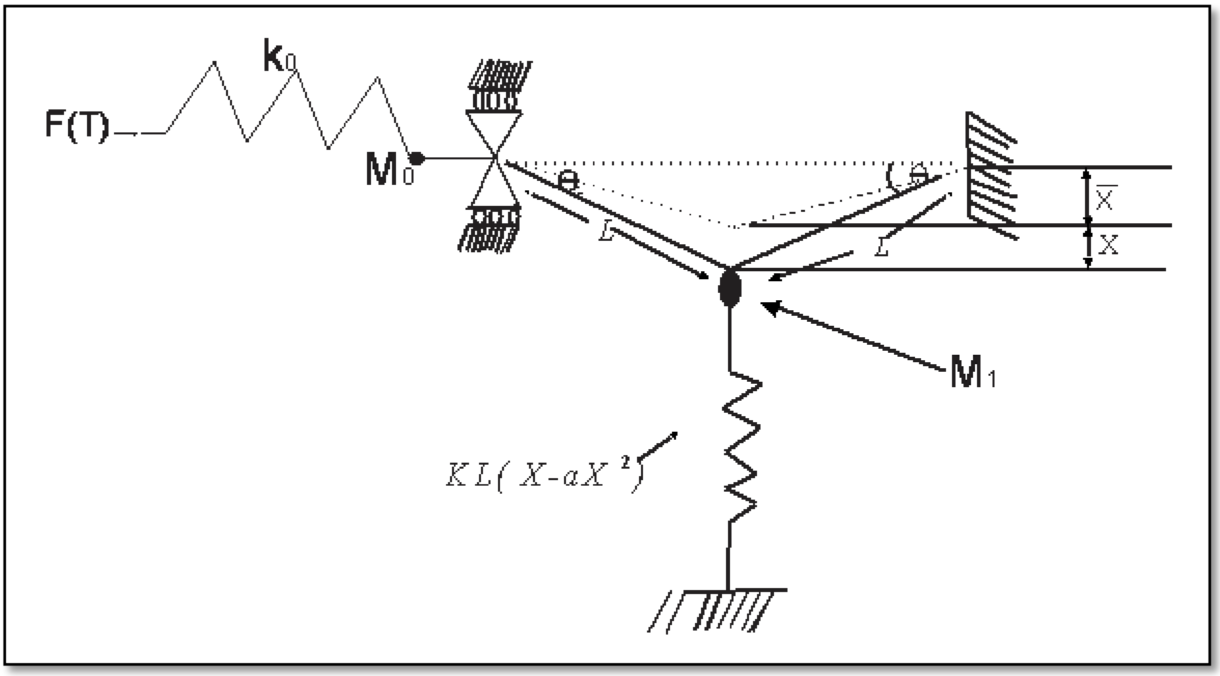

Figure 1: A modified simple quadratic model structure (Danielson's model)

2. Formulation of the problem

The original equations given by Danielson [1] are

$$\frac{1}{\omega_0^{\,2}}\frac{d^2\xi_0}{dT^2}+\xi_0-\frac{K_0}{\lambda_c}\xi_1\left(\xi_1+2\bar{\xi}\right)=\lambda f\left(T\right) \tag{1}$$

$$\frac{1}{\omega_1^{\,2}}\frac{d^2\xi_1}{dT^2}+\xi_1\left(\xi_1-\xi_0\right)-\alpha\xi_1^{\,2}+\frac{K_0}{\lambda_c}\xi_1\left(\xi_1+\bar{\xi}\right)\left(\xi_1+2\bar{\xi}\right)=\bar{\xi}\xi_0 \tag{2}$$

$$\xi_0\left(0\right)=\xi_1\left(0\right)=\frac{d\xi_0\left(0\right)}{dT}=\frac{d\xi_1\left(0\right)}{dT}=0 \tag{3}$$

$$\lambda_c=\frac{K_1}{2},\ \omega_0=\left(\frac{K_0}{M_0}\right)^{\frac{1}{2}},\ \omega_1=\left(\frac{K_1}{M_1}\right)^{\frac{1}{2}},K_0>0,\ K_1>0$$

where ξ_0 and ξ_1 are distinct buckling modes with respective circular frequencies ω_0 and ω_1.
Here, $\bar{\xi}$ is the imperfection parameter satisfying the inequality $0<\bar{\xi}<1$, while in Danielson's case, $f\left(T\right)$ was a step load with amplitude λ and where $\alpha>0$.

In this work, we assume the following:

(a) The load $f\left(T\right)$, is an impulse given by $\bar{\xi}I\delta\left(T\right)$ i.e. the impulse loading is of the order of the imperfection parameter $\bar{\xi}$.

(b) Light viscous damping on the modes ξ_0 and ξ_1 .

(c) The modes ξ_0 and ξ_1 are continuous functions of the time T.

Substituting all these assumptions in (1) and (2) we get

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{dT^2}+C_0\frac{d\xi_0}{dT}+\xi_0-\frac{K_0}{\lambda_c}\xi_1\left(\xi_1+2\bar{\xi}\right)=\bar{\xi}I\delta(T) \tag{4}$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{dT^2}+C_0\frac{d\xi_1}{dT}+\xi_1\left(\xi_1-\xi_0\right)-\alpha\xi_1^2+\frac{K_0}{\lambda_c}\xi_1\left(\xi_1+\bar{\xi}\right)\left(\xi_1+2\bar{\xi}\right)=\bar{\xi}\xi_0 \tag{5}$$

$$\xi_0\left(0^-\right)=\frac{d\xi_0}{dT}\left(0^-\right)=\xi_1\left(0^-\right)=\frac{d\xi_1}{dT}\left(0^-\right)=0 \tag{6}$$

where $\left(0^-\right)$ is the time just before the action of the impulse.

Let

$$\hat{t}=\omega_0 T,\ \frac{d}{dT}=\omega_0\frac{d}{d\hat{t}},\quad \frac{d^2}{dT^2}=\omega_0^2\frac{d^2}{d\hat{t}^2}$$

Substituting all these in (3) and (4), we get

$$\frac{d^2\xi_0}{d\hat{t}^2}+C_0\omega_0\frac{d\xi_0}{d\hat{t}}+\xi_0-\frac{K_0}{\lambda_c}\xi_1\left(\xi_1+2\bar{\xi}\right)=\bar{\xi}I\delta\left(\frac{\hat{t}}{\omega_0}\right)$$

$$\frac{d^2\xi_1}{d\hat{t}^2}+Q^2C_0\omega_0\frac{d\xi_1}{d\hat{t}}+\xi_1\left(\xi_1-\xi_0\right)-Q^2\alpha\xi_1^2+Q^2\frac{K_0}{\lambda_c}\xi_1\left(\xi_1+\bar{\xi}\right)\left(\xi_1+2\bar{\xi}\right)=Q^2\bar{\xi}\xi_0$$

where

$$Q=\left(\frac{\omega_1}{\omega_0}\right),\quad 0<Q<1$$

$$\xi_0\left(0^-\right)=\frac{d\xi_0}{d\hat{t}}\left(0^-\right)=\xi_1\left(0^-\right)=\frac{d\xi_1}{d\hat{t}}\left(0^-\right)=0$$

Let

$$2\varepsilon=C_0\omega_0,\quad 0<\varepsilon<<1$$

Here, ε is a small damping parameter that is not related to and is independent of $\bar{\xi}$.

Thus we get

$$\frac{d^2\xi_0}{d\hat{t}^2}+2\varepsilon\frac{d\xi_0}{d\hat{t}}+\xi_0-\frac{K_0}{\lambda_c}\xi_1\left(\xi_1+2\bar{\xi}\right)=\bar{\xi}I\delta\left(\frac{\hat{t}}{\omega_0}\right) \tag{7}$$

$$\frac{d^2\xi_1}{d\hat{t}^2}+2\varepsilon Q^2\frac{d\xi_1}{d\hat{t}}+\xi_1\left(\xi_1-\xi_0\right)-Q^2\alpha\xi_1^2+Q^2\frac{K_0}{\lambda_c}\xi_1\left(\xi_1+\bar{\xi}\right)\left(\xi_1+2\bar{\xi}\right)=Q^2\bar{\xi}\xi_0 \tag{8}$$

$$\xi_0\left(0^-\right)=\frac{d\xi_0}{d\hat{t}}\left(0^-\right)=\xi_1\left(0^-\right)=\frac{d\xi_1}{d\hat{t}}\left(0^-\right)=0 \tag{9}$$

To get the equation of motion after the action of the impulse, we integrate (7) from $\left(0^-\right)$ to $\left(0^+\right)$, where $\left(0^+\right)$ is the time shortly after the action of the impulse. Thus, we get

$$\int_{0^-}^{0^+} \frac{d^2\xi_0}{d\hat{t}^2} d\hat{t} + 2\varepsilon \int_{0^-}^{0^+} \frac{d\xi_0}{d\hat{t}} d\hat{t} + \int_{0^-}^{0^+} \xi_0 d\hat{t} - \int_{0^-}^{0^+} \frac{k_0}{\lambda_c} \xi_1 \left(\xi_1 + 2\bar{\xi}\right) d\hat{t} = I\bar{\xi} \int_{0^-}^{0^+} \delta\left(\frac{\hat{t}}{\omega_0}\right) d\hat{t} \qquad (10)$$

Since the displacement is continuous, then its higher powers are also continuous hence the second, third and fourth integrals on the left hand side of (10) vanish. Thus, from (10) we finally get

$$\left.\frac{d\xi_0}{d\hat{t}}\right|_{0^-}^{0^+} = I\bar{\xi} \int_{0^-}^{0^+} \delta\left(\frac{\hat{t}}{\omega_0}\right) d\hat{t} = I\bar{\xi}$$

$$\Rightarrow \quad \frac{d\xi_0}{d\hat{t}}\left(0^+\right) = I\bar{\xi} \qquad (11)$$

The improvement made on Danielson's model is that the improved quadratic structure is trapped by an impulse which is of the order of the imperfection $\bar{\xi}$, and that the structure is viscously damped, where the damping coefficient ε is not in any way related to the imperfection $\bar{\xi}$.
Thus, the equations of motion after the action of impulse are

$$\frac{d^2\xi_0}{d\hat{t}^2} + 2\varepsilon \frac{d\xi_0}{d\hat{t}} + \xi_0 - \frac{K_0}{\lambda_c} \xi_1 \left(\xi_1 + 2\bar{\xi}\right) = 0 \qquad (12a)$$

$$\xi_0\left(0^+\right) = 0, \quad \frac{d\xi_0}{d\hat{t}}\left(0^+\right) = I\bar{\xi} \qquad (12b)$$

and

$$\frac{d^2\xi_1}{d\hat{t}^2} + 2\varepsilon Q^2 \frac{d\xi_1}{d\hat{t}} + Q^2\xi_1\left(\xi_1 - \xi_0\right) - \alpha Q^2 \xi_1^{\,2} + Q^2 \frac{k_0}{\lambda_c} \xi_1 \left(\xi_1 + \bar{\xi}\right)\left(\xi_1 + 2\bar{\xi}\right) = Q^2 \bar{\xi} \xi_0 \qquad (12c)$$

$$\xi_1\left(0^+\right) = 0, \quad \frac{d\xi_1}{d\hat{t}}\left(0^+\right) = 0 \qquad (12d)$$

.

Henceforth, the evaluation at 0^+ will be written simply as evaluation at '0'. We shall assume $0 < \bar{\xi} << 1$. The following procedures are the steps to be taken in this analysis:

a) We shall first obtain asymptotic expressions of the displacements $\xi_0\left(\hat{t}\right)$ and $\xi_1\left(\hat{t}\right)$ by means of a two – small parameter regular perturbation analysis.
b) We shall next determine the maximum values, say respectively η_a and φ_c of these displacements η and φ respectively and obtain the net displacement $M_a = \eta_a + \varphi_c$
c) We shall lastly determine the dynamic buckling impulse I_D , which is defined as the largest impulse value for the problem to have a bounded solution. This obtained from the maximization $\frac{dI}{dM_a} = 0$.

3.Assumption and Perturbation Solution:

3.1Perturbation Solution:

Let

$$\tau = \varepsilon \hat{t}, \quad t = \hat{t} + \left(\frac{\omega_2(\tau)\bar{\xi}^2 + \omega_3(\tau)\bar{\xi}^3}{\varepsilon} \right) \tag{13a}$$

$$\omega_i(0) = 0, \quad \omega_i = \omega_i(\tau), \; i = 1,2,3,\dots , \tag{13b}$$

$$\therefore \quad \frac{d\xi}{d\hat{t}} = \left(1 + \omega_2{}'\bar{\xi}^2 + \omega_3{}'\bar{\xi}^3 + \dots\right)\xi_{,t} + \varepsilon\xi_{,\tau} \tag{13c}$$

$$\frac{d^2\xi}{d\hat{t}^2} = \left(1 + \omega_2{}'\bar{\xi}^2 + \omega_3{}'\bar{\xi}^3 + \dots\right)^2 \xi_{,tt} + 2\varepsilon\left(1 + \omega_2{}'\bar{\xi}^2 + \omega_3{}'\bar{\xi}^3 + \dots\right)\xi_{,t\tau} + \varepsilon^2\xi_{,\tau\tau} + \varepsilon\left(\omega_2{}''\bar{\xi}^2 + \omega_3{}''\bar{\xi}^3 + \dots\right)\xi_{,t} \tag{13d}$$

where

$$(\;)' = \frac{d}{d\tau}$$

and a subscript following a comma indicates partial differentiation.

Let

$$\xi_0(\hat{t}) = \eta(t,\tau,\bar{\xi},\varepsilon) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \eta^{ij}(t,\tau)\bar{\xi}^i\varepsilon^j \tag{14a}$$

$$\xi_1(\hat{t}) = \varphi(t,\tau,\bar{\xi},\varepsilon) = \sum_{i=2}^{\infty}\sum_{j=0}^{\infty} \varphi^{ij}(t,\tau)\bar{\xi}^i\varepsilon^j \tag{14b}$$

We now substitute (14a),(13c,d) in (12a,b) and equate coefficients of powers of $\bar{\xi}$ and ε to get

$$\eta^{10}_{,tt} + \eta^{10} = 0 \tag{15a}$$

$$\eta^{11}_{,tt} + \eta^{11} = -2\eta^{10}_{,t} - 2\eta^{10}_{,t\tau} \tag{15b}$$

$$\eta^{12}_{,tt} + \eta^{12} = -2\eta^{11}_{,t} - 2\eta^{10}_{,t\tau} - 2\eta^{10}_{,\tau} - \eta^{10}_{,\tau\tau} \tag{15c}$$

$$\eta^{20}_{,tt} + \eta^{20} = 0 \tag{16a}$$

$$\eta^{21}_{,tt} + \eta^{21} = -2\left(\eta^{20}_{,t\tau} + \eta^{20}_{,t}\right) \tag{16b}$$

$$\eta^{22}_{,tt} + \eta^{22} = --2\eta^{21}_{,t\tau} - 2\eta^{21}_{,t} - 2\eta^{20}_{,\tau} - \eta^{20}_{,\tau\tau} \tag{16c}$$

$$\eta_{,tt}^{30} + \eta^{30} = \frac{2k_0}{\lambda_c}\varphi^{20} - 2\omega_2'\eta_{,tt}^{10} \quad (17a)$$

$$\eta_{,tt}^{31} + \eta^{31} = \frac{2k_0}{\lambda_c}\varphi^{21} - 2\omega_2'\eta_{,tt}^{11} - 2\eta_{,t\tau}^{30} - 2\omega_2'\eta_{,t\tau}^{10} - \omega_2''\eta_{,t}^{10} - 2\eta_{,t}^{30} - 2\omega_2'\eta_{,t}^{10} \quad (17b)$$

$$\eta_{,tt}^{32} + \eta^{32} = \frac{2k_0}{\lambda_c}\varphi^{22} - 2\omega_2'\eta_{,tt}^{12} - 2\eta_{,t\tau}^{31} - 2\omega_2'\eta_{,t\tau}^{11} - \omega_2''\eta_{,t}^{11} - 2\eta_{,t}^{31}$$
$$- 2\omega_2'\eta_{,t}^{11} - \eta_{,\tau\tau}^{30} - 2\eta_{,\tau}^{30} \quad (17c)$$

We next substitute (14b) and (13c,d) into (12c,d) and get

$$\varphi_{,tt}^{20} + Q^2\varphi^{20} = Q^2\eta^{10} \quad (18a)$$

$$\varphi_{,tt}^{21} + Q^2\varphi^{21} = Q^2\eta^{11} - 2\varphi_{,t\tau}^{20} - 2Q^2\varphi_{,t}^{20} \quad (18b)$$

$$\varphi_{,tt}^{22} + Q^2\varphi^{22} = Q^2\eta^{12} - 2\varphi_{,t\tau}^{21} - 2Q^2\varphi_{,t}^{21} - 2Q^2\varphi_{,\tau}^{20} - \varphi_{,\tau\tau}^{20} \quad (18c)$$

$$\varphi_{,tt}^{30} + Q^2\varphi^{30} = Q^2\eta^{10}\varphi^{20} + Q^2\eta^{20} \quad (18d)$$

$$\varphi_{,tt}^{31} + Q^2\varphi^{31} = Q^2\eta^{10}\varphi^{21} + Q^2\eta^{11}\varphi^{20} + Q^2\eta^{21} - 2\varphi_{,t\tau}^{30} - 2Q^2\varphi_{,t}^{30} \quad (18e)$$

$$\varphi_{,tt}^{32} + Q^2\varphi^{32} = Q^2\eta^{10}\varphi^{22} + Q^2\eta^{11}\varphi^{21} + Q^2\eta^{12}\varphi^{20} - 2\varphi_{,t\tau}^{31} - 2Q^2\varphi_{,t}^{31}$$
$$- 2Q^2\varphi_{,\tau}^{30} - \varphi_{,\tau\tau}^{30} + Q^2\eta^{22} \quad (18f)$$

The initial conditions which are evaluated at $t = \tau = \mathbf{0}$, are

$$\eta^{10} = 0, \eta_{,t}^{10} = I;\ \eta^{ij} = 0\ \forall\, i, j \quad \textbf{(19a)}$$

$$\eta_{,t}^{1k} + \eta_{,\tau}^{1p} = 0,\ p = k - 1,\ k = 1,2,3,.... \quad \textbf{(19b)}$$

$$\eta_{,t}^{20} = 0 \quad \textbf{(19c)}$$

$$\eta_{,t}^{2k} + \eta_{,\tau}^{2p} = 0,\ p = k - 1,\ k = 1,2,3,.... \quad \textbf{(19d)}$$

$$\eta_{,t}^{30} + \omega_2(0)\eta_{,t}^{10} = 0 \quad \textbf{(19e)}$$

$$\eta_{,t}^{31} + \omega_2'(0)\eta_{,t}^{11} + \eta_{,\tau}^{30} = 0 \quad (19f)$$

$$\eta_{,t}^{3k} + \omega_2'(0)\eta_{,t}^{1k} + \eta_{,\tau}^{3p} = 0,\quad k = 1,2,3,....,\ p = k - 1, \quad (19g)$$

The initial conditions for φ^{ij}, evaluated at $t = \tau = 0$, are

$$\varphi^{pj}(0,0) = 0,\ p = 2,3,...\ ,\ j = 0,1,2,... \qquad (20a)$$

$$\varphi^{20}_{,t}(0,0) = 0 \qquad (20b)$$

$$\varphi^{2r}_{,t} + \varphi^{2s}_{,\tau} = 0\ ,\ s = r - 1\ ,\ r = 1,2,3,... \qquad (20c)$$

$$\varphi^{3r}_{,t} + \varphi^{3s}_{,\upsilon} = 0\ ,\ s = r - 1\ ,\ r = 1,2,3,... \qquad (20d)$$

3.2 Solutions of equations of order $\bar{\xi}^{1j}$, $j = 0,1,2,...$

On solving (15a), we get

$$\eta^{10}(t,\tau) = \alpha_{10}(\tau)\cos t + \beta_{10}(\tau)\sin t \qquad \textbf{(21a)}$$

On applying the initial conditions (19a), we get

$$\alpha_{10}(0) = 0,\ \beta_{10}(0) = I \qquad \textbf{(21b)}$$

Next we substitute (21a) into (15b), and get

$$\eta^{11}_{,tt} + \eta^{11} = 2(\alpha'_{10}\sin t - \beta'_{10}\cos t) + 2(\alpha_{10}\sin t - \beta_{10}\cos t) \qquad (21c)$$

To ensure a uniformly valid solution in t we equate to zero in (21c), the coefficients of $\cos t$ and $\sin t$ respectively get

$$\beta'_{10} + \beta_{10} = 0,\ \alpha'_{10} + \alpha_{10} = 0 \qquad \textbf{(21d)}$$

The solutions of (21d) are

$$\beta_{10}(\tau) = Ie^{-\tau},\ \ \alpha_{10}(\tau) = 0 \qquad \textbf{(21e)}$$

It follows that

$$\eta^{\mathbf{10}} = \beta_{10}(\tau)\sin t = Ie^{-\tau}\sin t \qquad \textbf{(21f)}$$

The remaining equation in (21c) is

$$\left.\begin{aligned} &\eta^{11}_{,tt} + \eta^{11} = 0 \\ &\eta^{11}(0,0) = 0,\ \eta^{11}_{,t}(0,0) + \eta^{10}_{,\tau}(0,0) = 0 \end{aligned}\right\} \qquad \textbf{(21g)}$$

On solving (21g), we get

$$\eta^{11}(t,\tau) = \alpha_{11}(\tau)\cos t + \beta_{11}(\tau)\sin t \qquad (22a)$$

$$\alpha_{11}(0) = 0,\ \ \beta_{11}(0) = 0 \qquad (22b)$$

We next substitute (21f) and (22a) into (15c) and get

$$\eta^{12}_{,tt} + \eta^{12} = 2(\alpha'_{11}\sin t - \beta'_{11}\cos t) + 2(\alpha_{11}\sin t - \beta_{11}\cos t) - 2\beta'_{10}\sin t - \beta''_{10}\sin t \qquad \textbf{(23)}$$

To ensure a uniformly valid solution in t, we equate to zero in (23) the coefficients of $\cos t$ and $\sin t$ and respectively get

$$\beta'_{11} + \beta_{11} = 0,\ \ \alpha'_{11} + \alpha_{11} = \frac{1}{2}(\beta''_{10} + 2\beta'_{10}) \qquad \textbf{(24)}$$

From (24), we get

$$\beta_{11}(\tau) \equiv 0, \qquad \alpha_{11}(\tau) = \frac{1}{2} e^{-\tau} \int_0^\tau [\beta''_{10}(s) + 2\beta'_{10}(s)] e^s ds \qquad \textbf{(25a)}$$

$$\Rightarrow \alpha_{11}(\tau) = \frac{-I\tau e^{-\tau}}{2}$$

$$\therefore \eta^{11}(t,\tau) = \alpha_{11}(\tau)\cos t \qquad \textbf{(25b)}$$

It follows that

$$\alpha'_{11}(0) = \frac{-I}{2}, \quad \alpha''_{11}(0) = I \qquad \textbf{(25c)}$$

From (21d, f), we get

$$\beta'_{10}(0) = -I, \quad \beta''_{10}(0) = I, \quad \beta'''_{10}(0) = -I \qquad \textbf{(25d)}$$

From the remaining equation in (23), we get

$$\eta^{12}_{,tt} + \eta^{12} = 0 \qquad \textbf{(26a)}$$

$$\eta^{12}(0,0) = \eta^{12}_{,t}(0,0) + \eta^{11}_{,\tau}(0,0) = 0 \qquad \textbf{(26b)}$$

On solving (26a,b), we get

$$\left.\begin{array}{l} \eta^{12}(t,\tau) = \alpha_{12}(\tau)\cos t + \beta_{12}(\tau)\sin t \\ \alpha_{12}(0) = 0, \quad \beta_{12}(0) = \frac{I}{2} \end{array}\right\} \qquad \textbf{(27)}$$

3.3 Solutions of equations of order $\bar{\xi}^{2j}$, j = 0,1,2,3,...

We now substitute (21f), into (18a) and get

$$\varphi_{,tt}^{20} + Q^2\varphi^{20} = Q^2\eta^{10} = Q^2\beta_{10}\sin t \qquad (28a)$$

$$\varphi^{20}(0,0) = 0, \quad \varphi_{,t}^{20}(0,0) = 0 \qquad (28b)$$

On solving (28a,b), we get

$$\varphi^{20}(t,\tau) = \alpha_{20}(\tau)\cos Qt + \beta_{20}(\tau)\sin Qt + \frac{Q^2\beta_{10}\sin t}{Q^2 - 1}, \qquad (29a)$$

$$Q \neq 1, \ \alpha_{20}(0) = 0, \beta_{20}(0) = r_0 I, \ r_0 = \frac{Q}{1 - Q^2} \qquad (29b)$$

Next, we substitute (29a) and (25b) into (18b) and get

$$\varphi^{21}_{,tt} + Q^2\varphi^{21} = Q^2\alpha_{11}\cos t - 2\left(-Q\alpha'_{21}\sin Qt + Q\beta'_{20}\cos Qt + \frac{Q^2\beta'_{10}\cos t}{Q^2 - 1}\right)$$
$$- 2Q^2\left(-Q\alpha_{20}\sin Qt + Q\beta_{20}\cos Qt + \frac{Q^2\beta_{10}\cos t}{Q^2 - 1}\right) \qquad (30a)$$

$$\varphi^{21}(0,0)=0, \qquad \varphi_{,t}^{21}(0,0)+\varphi_{,\tau}{}^{20}(0,0)=0 \tag{30b}$$

We ensure a uniformly valid solution in (30a) by equating to zero the coefficients of $\cos Qt$ and $\sin Qt$ and get

$$\beta_{20}'+Q^2\beta_{20}=0, \qquad \alpha_{20}'+Q^2\alpha_{20}=0 \tag{31}$$

On solving (31), we get

$$\beta_{20}(\tau)=\beta_{20}(0)e^{-Q^2\tau}, \quad \alpha_{20}(\tau)=0 \tag{32}$$

Therefore, from (29a), we get

$$\varphi^{20}(t,\tau)=\beta_{20}(\tau)\sin Qt+\frac{Q^2\beta_{10}\sin t}{Q^2-1} \tag{33}$$

Then we get

$$\beta_{20}'(0)=-Q^2r_0I, \quad \beta_{20}''(0)=Q^4r_0I, \quad \beta_{20}'''(0)=-Q^6r_0I$$

The remaining equation in (30a,b) is solved to get

$$\varphi^{21}(t,\tau)=\alpha_{21}(\tau)\cos Qt+\beta_{21}(\tau)\sin Qt+\frac{Q^2\alpha_{11}\cos t}{Q^2-1}-\frac{2Q^2\beta_{10}'\cos t}{(Q^2-1)^2}-\frac{2Q^4\beta_{10}\cos t}{(Q^2-1)^2} \tag{34a}$$

From (30b), we get

$$\varphi^{21}(0,0)=0\Rightarrow \alpha_{21}(0)=\frac{2Q^2\beta_{10}'(0)+2Q^4\beta_{10}(0)}{(Q^2-1)^2}$$

Thus, we get

$$\alpha_{21}(0)=-2Qr_0I, \quad \beta_{21}(0)=0 \tag{34b}$$

Substituting (34a) in (18c), we have

$$\varphi_{,tt}^{22}+Q^2\varphi^{22}=Q^2\beta_{12}\sin t-2\left(-Q\alpha_{21}'\sin Qt+Q\beta_{21}'\cos Qt-\frac{Q^2\alpha_{11}'\sin t}{Q^2-1}\right.$$
$$\left.+\frac{2Q^2\beta_{10}''\sin t}{(Q^2-1)^2}+\frac{2Q^4\beta_{10}{}'\sin t}{(Q^2-1)^2}\right)-\left(\beta_{20}''\sin Qt+\frac{Q^2\beta_{10}''\sin t}{Q^2-1}\right)$$
$$-2Q^2\left(-Q\alpha_{21}\sin Qt+Q\beta_{21}\cos Qt-\frac{Q^2\alpha_{11}\sin t}{Q^2-1}+\frac{2Q^2\beta_{10}'\sin t}{(Q^2-1)^2}+\frac{2Q^4\beta_{10}\sin t}{(Q^2-1)^2}\right)$$

To ensure a uniformly valid solution in t, we equate to zero the coefficients of $\cos Qt$ and $\sin Qt$ to zero in (35a) to get

$$\beta_{21}'+Q^2\beta_{21}=0, \quad \alpha_{21}'+Q^2\alpha_{21}=\frac{1}{2Q}\left(2Q^2\beta_{20}'+\beta_{20}''\right)=h_1(\tau) \tag{35b}$$

Solving (35b), we get

$$\beta_{21}(\tau)=0, \qquad \alpha_{21}(\tau)=e^{-Q^2\tau}\left(\int_0^\tau h_1(s)e^{Q^2s}ds+\alpha_{21}(0)\right) \tag{35c}$$

Thus, from (34a), we get

$$\varphi^{21}(t,\tau)=\alpha_{21}(\tau)\cos Qt+\frac{Q^2\alpha_{11}\cos t}{Q^2-1}-\frac{2Q^2\beta'_{10}\cos t}{\left(Q^2-1\right)^2}-\frac{2Q^4\beta_{10}\cos t}{\left(Q^2-1\right)^2} \qquad \textbf{(36a)}$$

The solution of the remaining part of (35a) is

$$\varphi^{22}(t,\tau)=\alpha_{22}(\tau)\cos Qt+\beta_{22}(\tau)\sin Qt+R_1(\tau)\sin t \qquad \textbf{(36b)}$$

$$R_1(\tau)=\frac{Q^2\beta_{12}}{Q^2-1}+\frac{2Q^2\alpha'_{11}}{\left(Q^2-1\right)^2}-\frac{4Q^2\beta''_{10}}{\left(Q^2-1\right)^3}-\frac{8Q^4\beta'_{10}}{\left(Q^2-1\right)^3}-\frac{Q^2\beta''_{10}}{\left(Q^2-1\right)^2}+\frac{2Q^4\alpha_{11}}{\left(Q^2-1\right)^2}$$
$$-\frac{4Q^6\beta_{10}}{\left(Q^2-1\right)^3}-\frac{2Q^4\beta'_{10}}{\left(Q^2-1\right)^2} \qquad \textbf{(37a)}$$

$$R_1(0)=Ir_1 \qquad \textbf{(37b)}$$

$$r_1=\frac{Q^2}{2\left(Q^2-1\right)}-\frac{2Q^2}{\left(Q^2-1\right)^2}-\frac{4Q^2}{\left(Q^2-1\right)^3}+\frac{8Q^4}{\left(Q^2-1\right)^3}-\frac{4Q^6}{\left(Q^2-1\right)^2}+\frac{2Q^4}{\left(Q^2-1\right)^2} \qquad \textbf{(37c)}$$

On solving (16a,b,c) with the appropriate initial conditions as in (19a – c), we get

$$\eta^{2j}\equiv 0 \quad \forall \mathbf{j}$$

Substituting (21f) and (33) into (17a), we have

$$\eta^{30}_{,tt}+\eta^{30}=\frac{2K_0}{\lambda_c}\left(\beta_{20}\sin Qt+\frac{Q^2\beta_{10}\sin t}{Q^2-1}\right)+2\omega'_2\beta_{10}\sin t \qquad (38a)$$

$$\eta^{30}(0,0)=0, \quad \eta^{30}_{,t}(0,0)+\omega'_2\eta^{10}_{,t}(0,0)=0 \qquad (38b)$$

To ensure a uniformly valid solution in t in (38a), we equate to zero the coefficients of $\sin t$ and get

$$\frac{2K_0Q^2\beta_{10}}{\lambda_c\left(Q^2-1\right)}+2\omega'_2\beta_{10}=0 \qquad \textbf{(38c)}$$

Thus,

$$\omega'_2(\tau)=\frac{-K_0Q^2}{\lambda_c\left(Q^2-1\right)} \quad \Rightarrow \omega_2(\tau)=\frac{-K_0Q^2\tau}{\lambda_c\left(Q^2-1\right)} \qquad \textbf{(38d)}$$

Solving the remaining part of (38a), we have

$$\eta^{30}(t,\tau)=\alpha_{30}(\tau)\cos t+\beta_{30}(\tau)\sin t+\frac{2K_0\beta_{20}\sin Qt}{\lambda_c\left(1-Q^2\right)} \qquad \textbf{(38e)}$$

From (38b), we have

$$\alpha_{30}(0)=0, \qquad \beta_{30}(0)=\frac{k_0r_0I\left[Q-2r_0\right]}{\lambda_c} \qquad \textbf{(38f)}$$

Substituting (38e), (36), (21f) and (25b) into (17b), we get

$$\eta_{,tt}^{31}+\eta^{31}=\frac{2k_0}{\lambda_c}\left[\alpha_{21}\cos Qt+\frac{\alpha_{11}\cos t}{Q^2-1}-\frac{2Q^4\beta_{10}\cos t}{\left(Q^2-1\right)^2}-\frac{2Q^2\beta'_{10}\cos t}{\left(Q^2-1\right)^2}\right]$$
$$-2\omega'_2\left(-\alpha_{11}\cos t\right)-2\left[-\alpha'_{30}\sin t+\beta'_{30}\cos t+\frac{2Qk_0\beta'_{20}\cos Qt}{\lambda_c\left(1-Q^2\right)}\right]-2\omega'_2\beta_{10}\cos t \qquad \textbf{(39a)}$$

We ensure a uniformly valid solution in time t in (39a) by equating the coefficients of $\cos t$ and $\sin t$ to zero, and having

$$\cos t:\ \beta'_{\mathbf{30}}+\beta_{\mathbf{30}}=\frac{Q^2K_0\alpha_{11}}{\lambda_c\left(Q^2-1\right)}-\frac{2Q^4K_0\beta_{10}}{\lambda_c\left(Q^2-1\right)^2}-\frac{2Q^2K_0\beta'_{10}}{\lambda_c\left(Q^2-1\right)^2}+\omega'_2\alpha_{11}$$
$$-\omega'_2\beta'_{10}-\omega''_2\beta_{10}-\omega'_2\beta_{10} \qquad \textbf{(39b)}$$

$$\sin t: 2\alpha'_{30}+2\alpha_{30}=0 \qquad \textbf{(39c)}$$

Solving (39b,c), we have

$$\beta_{30}(\tau)=e^{-\tau}\int_0^{\tau}e^{s}H_1(s)ds+C_1e^{-\tau},\ \alpha_{\mathbf{30}}(\tau)=C_2e^{-\tau},\ C_1,C_2=cons\tan t \qquad \textbf{(39d)}$$

where

$$H_1(\tau)=\frac{Q^2K_0\alpha_{11}(\tau)}{\lambda_c\left(Q^2-1\right)}-\frac{2Q^4K_0\beta_{10}(\tau)}{\lambda_c\left(Q^2-1\right)^2}-\frac{2Q^2K_0\beta'_{10}(\tau)}{\lambda_c\left(Q^2-1\right)^2}+\omega'_2(\tau)\left[\alpha_{11}(\tau)-\beta'_{10}(\tau)-\beta_{10}(\tau)\right] \qquad \textbf{(39e)}$$

Let

$$H_1(0)=Ir_0r_2,\ r_2=\frac{2K_0Q}{\lambda_c} \qquad \textbf{(39f)}$$

From (38f) and (39d), we have

$$\beta_{30}(\tau)=\frac{2K_0QI\tau r_0}{\lambda_c}+\frac{K_0r_0I}{\lambda_c}\left(Q-2r_0\right)e^{-\tau} \qquad (39g)$$

$$\alpha_{30}(\tau)=0 \qquad \textbf{(39h)}$$

Hence

$$\eta^{\mathbf{30}}(t,\tau)=\beta_{30}(\tau)\sin t+\frac{2K_0\beta_{20}(\tau)\sin Qt}{\lambda_c\left(1-Q^2\right)} \qquad \textbf{(39i)}$$

The remaining part of (39a) gives

$$\eta_{,tt}^{31}+\eta^{\mathbf{31}}=\frac{2K_0\alpha_{21}(\tau)\cos Qt}{\lambda_c}-\frac{4QK_0\beta'_{20}(\tau)\cos Qt}{\lambda_c\left(1-Q^2\right)}-\frac{4QK_0\beta_{20}(\tau)\cos Qt}{\lambda_c\left(1-Q^2\right)} \qquad \textbf{(40a)}$$

Solving (40a) gives

$$\eta^{31}(t,\tau)=\alpha_{31}(\tau)\cos t+\beta_{31}(\tau)\sin t+\frac{2K_0\alpha_{21}(\tau)\cos Qt}{\lambda_c(1-Q^2)}-\frac{4QK_0\beta'_{20}(\tau)\cos Qt}{\lambda_c(1-Q^2)^2}-\frac{4QK_0\beta_{20}(\tau)\cos Qt}{\lambda_c(1-Q^2)^2} \quad \textbf{(40b)}$$

Applying initial conditions in (19a,g), we have

$$\alpha_{31}(0)=Ir_3 \quad , \quad \beta_{\mathbf{31}}(0)=0 \quad \textbf{(40c)}$$

where

$$r_3=\frac{4k_0r_0}{\lambda_c(1-Q^2)}\left[r_0+Q-Q^2r_0\right] \quad \textbf{(40d)}$$

Substituting (40b) in (17c), we have

$$\eta_{,tt}^{32}+\eta^{32}=\frac{2K_0}{\lambda_c}\left[\alpha_{22}(\tau)\cos Qt+\beta_{22}(\tau)\sin Qt+R_1\sin t+2\omega_2{}'\beta_{12}\sin t\right]$$
$$-2\left(-\alpha'_{31}(\tau)\sin t+\beta'_{31}(\tau)\cos t-\frac{2Q\alpha'_{21}(\tau)\sin Qt}{\lambda_c(1-Q^2)}+\frac{4Q^2K_0\beta''_{20}(\tau)\sin Qt}{\lambda_c(1-Q^2)^2}+\frac{4Q^2K_0\beta'_{20}(\tau)\sin Qt}{\lambda_c(1-Q^2)^2}\right)$$
$$+2\omega'_2\alpha'_{11}(\tau)\sin t-2\left(-\alpha_{31}(\tau)\sin t+\beta_{31}(\tau)\cos t-\frac{2Q\alpha_{21}(\tau)\sin Qt}{\lambda_c(1-Q^2)}+\frac{4Q^2K_0\beta'_{20}(\tau)\sin Qt}{\lambda_c(1-Q^2)^2}+\frac{4Q^2K_0\beta_{20}(\tau)\sin Qt}{\lambda_c(1-Q^2)^2}\right)$$
$$+2\omega'_2\alpha_{11}(\tau)\sin t-\beta''_{30}(\tau)\sin t-\frac{2K_0\beta''_{20}(\tau)\sin Qt}{\lambda_c(1-Q^2)}-2\beta'_{30}(\tau)\sin t-\frac{4K_0\beta'_{20}(\tau)\sin Qt}{\lambda_c(1-Q^2)} \quad (41a)$$

To ensure a uniformly valid solution in t weequate the coefficients of $\cos t$ and $\sin t$ to zero, we have

$$\cos t:\beta'_{31}+\beta_{31}=0 \quad (41b)$$

$$\sin t:\alpha'_{31}+\alpha_{31}=\frac{1}{2}\left(\beta''_{30}+2\beta'_{30}-\frac{2K_0R_1}{\lambda_c}-2\omega'_2\beta_{12}-2\omega'_2\alpha'_{11}-2\omega'_2\alpha_{11}\right)$$

$$\therefore \alpha'_{31}+\alpha_{31}=\frac{1}{2}H_2(\tau),\ H_2(\tau)=\beta''_{30}+2\beta'_{30}-\frac{2K_0R_1}{\lambda_c}-2\omega'_2\beta_{12}-2\omega'_2\alpha'_{11}-2\omega'_2\alpha_{11} \quad (41c)$$

Solving (41b,c) using (40c) and substituting for $\beta''_{30}, \beta'_{30}, R_1, \omega'_2, \beta_{12}, \alpha_{11}$ and α'_{11} in $H_2(\tau)$, the simplification gives

$$\beta_{31}(\tau) = 0 \tag{41d}$$

$$\alpha_{31}(\tau) = \frac{e^{-\tau}}{2}\int_0^{\tau} e^{s} H_2(s)ds + \theta(\tau) \tag{41e}$$

where

$$H_2(\tau) = \frac{K_0 r_0 I}{\lambda_c}\left[Q - (Q - 2r_0)e^{-\tau} + 2Qe^{-\tau}R_2(\tau) + Q\tau e^{-\tau}\right], \tag{41d}$$

$$\theta(\tau) = \frac{4K_0 r_0{}^2 I}{\lambda_c(1-Q^2)}\left(2 - Q^2 - Qr_0\right)e^{-\tau}$$

$$R_2(\tau) = \frac{1-\tau}{2} - \frac{2+\tau}{Q^2-1} - \frac{4}{(Q^2-1)^2} + 8r_0{}^2 + Qr_0{}^2\tau - 4Q^2 r_0{}^2 - 2Qr_0 \tag{41e}$$

Hence from (40b) and (41d,e) we have

$$\eta^{31}(t,\tau) = \alpha_{31}(\tau)\cos t + F_1(\tau)\cos Qt \tag{41f}$$

$$F_1(\tau) = \frac{2K_0\alpha_{21}(\tau)}{\lambda_c(1-Q^2)} - \frac{4QK_0\beta'_{20}(\tau)}{\lambda_c(1-Q^2)^2} - \frac{4QK_0\beta_{20}(\tau)}{\lambda_c(1-Q^2)^2}$$

The remaining part of (41a) is

$$\eta^{32}_{,tt} + \eta^{32} = \frac{2K_0\alpha_{22}(\tau)\cos Qt}{\lambda_c} + F_2(\tau)\sin Qt \tag{41g}$$

$$F_2(\tau) = \frac{2K_0\beta_{22}(\tau)}{\lambda_c} + \frac{2K_0}{\lambda_c(1-Q^2)}\left[2Q\alpha'_{21} - \frac{4Q^2\beta''_{20}(\tau)}{(1-Q^2)} - \frac{8Q^2\beta'_{20}(\tau)}{(1-Q^2)} + 2Q\alpha_{21}(\tau) - \frac{4Q^2\beta_{20}(\tau)}{(1-Q^2)} - \beta''_{20}(\tau) - 2\beta'_{20}(\tau)\right]$$

Solving (41g) gives

$$\eta^{32}(t,\tau) = \alpha_{32}(\tau)\cos t + \beta_{32}(\tau)\sin t + \frac{2K_0\alpha_{22}(\tau)\cos Qt}{\lambda_c(1-Q^2)} + \frac{2K_0}{\lambda_c(1-Q^2)}\left[\beta_{22}(\tau) + \frac{2Q\alpha'_{21}(\tau)}{(1-Q^2)} - \frac{4Q^2\beta''_{20}(\tau)}{(1-Q^2)^2} - \frac{8Q^2\beta'_{20}(\tau)}{(1-Q^2)^2} + \frac{2Q\alpha_{21}(\tau)}{(1-Q^2)} - \frac{4Q^2\beta_{20}(\tau)}{(1-Q^2)^2} - \frac{\beta''_{20}(\tau)}{(1-Q^2)} - \frac{2\beta'_{20}(\tau)}{(1-Q^2)}\right] \tag{41h}$$

Applying the initial conditions (19a,e), we have

$$\eta^{32}(0,0) = 0 \Rightarrow \alpha_{32}(0) = -\frac{2K_0\alpha_{22}(0)}{\lambda_c(1-Q^2)} = 0 \tag{41i}$$

$$\eta^{32}_{,tt}(0,0)+\omega_2'\eta^{12}_{,t}(0,0)+\eta^{31}_{,\tau}(0,0)=0 \quad (41j)$$

Similarly applying the initial conditions (20a,c) to (36b), we get

$$\alpha_{22}(0)=0,\ \beta_{22}(0)=Ir_4,\ \mathrm{r}_4=\frac{Q^2r_0}{2}-\frac{2r_0}{Q^2-1}-2r_0^{\ 2}Q-\frac{r_1}{Q}-2Q^2r_0-\frac{r_0}{2} \quad (41k)$$

From (41a), we get by equating the coefficients of $\sin t$ to zero

$$\alpha_{31}{}'(0)=Ir_5,\quad r_5=\frac{3K_0r_0Q}{2\lambda_c}+\frac{K_0r_0^{\ 2}}{\lambda_c}-\frac{K_0r_1}{\lambda_c}-r_3 \quad (42a)$$

From (41j), we have

$$\beta_{32}(0)=Ir_6$$

$$r_6=\frac{8Q^4K_0r_0^{\ 4}}{\lambda_c}-\frac{16Q^2K_0r_0^{\ 4}}{\lambda_c}-\frac{2K_0r_0r_4}{\lambda_c}-\frac{6Q^3r_0^{\ 3}}{\lambda_c}+\frac{8Qr_0^{\ 3}}{\lambda_c}+\frac{8K_0r_0^{\ 4}}{\lambda_c}+\frac{6Q^3K_0r_0^{\ 3}}{\lambda_c}$$
$$-\frac{8QK_0r_0^{\ 3}}{\lambda_c}-\frac{QK_0r_0}{2\lambda_c}-r_5-\frac{3Q^2K_0r_0^{\ 2}}{\lambda_c} \quad (42b)$$

We now simplify the following term to be used in (18d).

$$\eta^{10}\varphi^{20}=Q^2\beta_{10}(\tau)\beta_{20}(\tau)\left[\frac{\cos(Q-1)t}{2}-\frac{\cos(Q+1)t}{2}\right]+\frac{Q^4\beta_{10}^2(\tau)}{Q^2-1}(1-\cos 2t) \quad (42c)$$

Substituting (42c) in (18d), we get after simplifying

$$\varphi^{30}_{,tt}+Q^2\varphi^{30}=R_7(\tau)[\cos(Q-1)t-\cos(Q+1)t]+R_8(\tau)(1-\cos 2t) \quad (43a)$$

where

$$R_7(\tau)=\frac{Q^2\beta_{10}(\tau)\beta_{20}(\tau)}{2},\qquad R_8(\tau)=\frac{Q^4\beta_{10}^2(\tau)}{Q^2-1} \quad (43b)$$

Solving (43b), we have

$$\varphi^{30}(t,\tau)=\alpha_{40}(\tau)\cos Qt+\beta_{40}(\tau)\sin Qt+R_7(\tau)\left\{\frac{\cos(Q-1)t}{2Q-1}+\frac{\cos(Q+1)t}{2Q+1}\right\}$$
$$+R_8(\tau)\left\{\frac{1}{Q^2}-\frac{\cos 2t}{Q^2-4}\right\} \quad (43c)$$

valid for $Q\neq 2$

From the initial conditions in (20a), we have

$$\varphi^{30}(0,0)=0\Rightarrow\alpha_{40}(0)=-R_7(0)\left\{\frac{1}{2Q-1}+\frac{1}{2Q+1}\right\}-R_8(0)\left\{\frac{1}{Q^2}-\frac{1}{Q^2-4}\right\}$$

Thus

$$\alpha_{40}(0) = -2Qr_0 I^2\left(\frac{Q^2}{4Q^2-1}+\frac{1}{Q^2-4}\right) \tag{43e}$$

From (19d), we have

$$\varphi_{,t}^{30}(0,0) = 0 \Rightarrow \beta_{40}(0) = 0$$

We next simplify the following multiplication to be substituted into (18e).

$$\eta^{10}\varphi^{21} = \frac{\beta_{10}(\tau)\alpha_{21}(\tau)}{2}\{\sin(1+Q)t+\sin(1-Q)t\}+\frac{\beta_{10}(\tau)\alpha_{11}(\tau)Q^2\sin 2t}{Q^2-1}$$
$$-\frac{2Q^2\beta_{10}(\tau)\beta_{10}(\tau)'\sin 2t}{(Q^2-1)^2}-\frac{2Q^4\beta_{10}{}^2(\tau)\sin 2t}{(Q^2-1)^2} \tag{43f}$$

Therefore, the substitutions into (18e) and further simplification using (43f) gives,

$$\varphi_{,tt}^{31}+Q^2\varphi^{31} = \frac{Q^2\beta_{10}(\tau)\alpha_{11}(\tau)}{2}\{\sin(1+Q)t+\sin(1-Q)t\}+\frac{Q^4\beta_{10}(\tau)\alpha_{11}(\tau)\sin 2t}{2(Q^2-1)}$$
$$-\frac{Q^4\beta_{10}(\tau)\beta_{10}'(\tau)\sin 2t}{(Q^2-1)^2}-\frac{Q^6\beta_{10}^2(\tau)\sin 2t}{(Q^2-1)^2}+\frac{Q^2\alpha_{11}(\tau)\beta_{20}(\tau)}{2}\{\sin(Q+1)t+\sin(Q-1)t\}$$
$$+\frac{Q^4\alpha_{11}(\tau)\beta_{10}(\tau)\sin 2t}{2(Q^2-1)}+2Q\alpha_{40}'(\tau)\sin Qt-2Q\beta_{40}'(\tau)\sin Qt+\frac{2(Q-1)R_7'(\tau)\sin(Q-1)t}{2Q-1}$$
$$+\frac{2(Q+1)R_7'(\tau)\sin(Q+1)t}{2Q+1}-\frac{4R_8'(\tau)\sin 2t}{Q^2-4}+2Q^3\alpha_{40}(\tau)\sin Qt-2Q^3\beta_{40}(\tau)\cos Qt$$
$$+\frac{2Q^2(Q-1)R_7(\tau)\sin(Q-1)t}{2Q-1}+\frac{2Q^2(Q+1)R_7(\tau)\sin(Q+1)t}{2Q+1}-\frac{4Q^2R_8(\tau)\sin 2t}{Q^2-4} \tag{44a}$$

We ensure uniformly valid solution in (44a) in terms of t by equating to zero the coefficients of $\cos Qt$ and $\sin Qt$ and get,

For $\cos Qt$ and $\sin Qt$

$$\beta_{40}'+Q^2\beta_{40} = 0 \quad , \quad \boldsymbol{\alpha_{40}'+Q^2\alpha_{40} = 0} \tag{44b}$$

Solving (44b) using appropriate initial conditions, we get respectively,

$$\beta_{40}(\tau) = 0 \quad \text{and} \quad \alpha_{40}(\tau) = \alpha_{40}(0)e^{-Q^2\tau} = -2Qr_0I^2\left(\frac{Q^2}{4Q^2-1}+\frac{1}{Q^2-4}\right)e^{-Q^2\tau} \tag{44c}$$

Hence, (43d) becomes

$$\varphi^{30}(t,\tau) = \alpha_{40}(\tau)\cos Qt+R_7(\tau)\left\{\frac{\cos(Q-1)t}{2Q-1}+\frac{\cos(Q+1)t}{2Q+1}\right\}+R_8(\tau)\left\{\frac{1}{Q^2}-\frac{\cos 2t}{Q^2-4}\right\} \tag{44d}$$

The remaining equation in (44a) is

$$\varphi_{,tt}^{31}+Q^2\varphi^{31} = R_9(\tau)\sin(1+Q)t+R_{10}(\tau)\sin(1-Q)t+R_{11}(\tau)\sin 2t \tag{44e}$$

where

$$\left.\begin{aligned}
R_9(\tau) &= \frac{Q^2\beta_{10}(\tau)\alpha_{11}(\tau)}{2} + \frac{Q^2\alpha_{11}(\tau)\beta_{20}(\tau)}{2} - \frac{2(Q+1)R_7'}{2Q+1} - \frac{2Q^2(Q+1)R_7}{2Q+1} \\
R_{10}(\tau) &= \frac{Q^2\beta_{10}(\tau)\alpha_{11}(\tau)}{2} + \frac{Q^2\alpha_{11}(\tau)\beta_{20}(\tau)}{2} - \frac{2(Q-1)R_7'}{2Q-1} - \frac{2Q^2(Q-1)R_7}{2Q-1} \\
R_{11}(\tau) &= \frac{Q^4\beta_{10}(\tau)\alpha_{11}(\tau)}{2(Q^2-1)} - \frac{Q^4\beta_{10}(\tau)\beta_{10}'(\tau)}{2(Q^2-1)^2} - \frac{Q^6\beta_{10}^{\,2}(\tau)}{(Q^2-1)^2} + \frac{Q^4\alpha_{11}(\tau)\beta_{10}}{2(Q^2-1)} \\
&\quad - \frac{4R_8'}{Q^2-4} - \frac{4Q^2R_8}{Q^2-4}
\end{aligned}\right\} \qquad (44f)$$

Solving (44e) with (20a,d), we have

$$\varphi^{31}(t,\tau) = \alpha_{41}(\tau)\cos Qt + \beta_{41}(\tau)\sin Qt - \frac{R_9\sin(1+Q)t}{2Q+1} + \frac{R_{10}\sin(1-Q)t}{2Q-1} + \frac{R_{11}\sin 2t}{Q^2-4} \qquad (44g)$$

$$Q \neq \frac{1}{2},\ Q \neq 2$$

where

$$\alpha_{41}(0) = 0 \qquad (44h)$$

Hence,

$$\beta_{41}(0) = I^2 r_7 \qquad (44i)$$

$$r_7 = Qr_0\left\{-\frac{(Q+1)^2}{(2Q+1)^2} + \frac{(Q-1)^2}{(2Q-1)^2} - \frac{2}{Q^2-4}\left(r_0 - Q^2r_0 - \frac{4Q}{Q^2-4} + \frac{2Q^3}{Q^2-4}\right)\right.$$
$$\left. - \left(\frac{2Q^3}{4Q^2-1} + \frac{2Q}{Q^2-4}\right) + \frac{1}{2} + \frac{Q^2}{4Q-2} + \frac{Q^2}{4Q+2} + \frac{1}{Q} - \frac{Q}{Q^2-4}\right\} \quad \textbf{(44j)}$$

Substituting $\eta^{10}, \eta^{11}, \eta^{12}, \varphi^{20}, \varphi^{21}, \varphi^{22}, \varphi^{30}$ and φ^{31} in (18f), we have

$$\begin{aligned}
\varphi_{,tt}^{32} + Q^2\varphi^{32} &= \frac{Q^2\beta_{10}\beta_{22}}{2}\{\sin(1+Q)t + \sin(1-Q)t\} + \frac{Q^2\beta_{10}R_1\sin 2t}{2} \\
&+ \frac{Q^2\alpha_{11}\alpha_{21}}{2}\{\cos(1+Q)t + \cos(1-Q)t\} + \frac{Q^4\alpha_{11}^2\cos 2t}{2(Q^2-1)} - \frac{Q^4\alpha_{11}\beta_{10}'\cos 2t}{(Q^2-1)^2} \\
&- \frac{Q^6\alpha_{11}\beta_{10}\cos 2t}{(Q^2-1)^2} + \frac{Q^2\beta_{12}\beta_{20}}{2}\{\sin(1+Q)t + \sin(1-Q)t\} + \frac{Q^4\beta_{12}\beta_{10}\sin 2t}{2(Q^2-1)} \\
&+ 2Q\alpha_{41}'\sin Qt - 2Q\beta_{41}'\cos Qt - \frac{(1+Q)R_9'\cos(1+Q)t}{2Q+1} + \frac{(1-Q)R_{10}'\cos(1-Q)t}{2Q-1} \\
&+ \frac{2R_{11}'\cos 2t}{Q^2-4} + 2Q^3\alpha_{41}\sin Qt - 2Q^3\beta_{41}\cos Qt + \frac{2Q^2(1+Q)R_9\cos(1+Q)t}{2Q+1} \\
&- \frac{2Q^2(1-Q)R_{10}\cos(1-Q)t}{2Q-1} - \frac{4Q^2R_{11}\cos 2t}{Q^2-4} - 2Q^2\alpha_{40}'\cos Qt
\end{aligned}$$

$$-2Q^2R_7'\left\{\frac{\cos(Q-1)t}{2Q-1}+\frac{\cos(Q+1)t}{2Q+1}\right\}-2Q^2R_8'\left\{\frac{1}{Q^2}-\frac{\cos 2t}{Q^2-4}\right\}-\alpha_{40}''\cos Qt$$

$$-R_7''\left\{\frac{\cos(Q-1)t}{2Q-1}+\frac{\cos(Q+1)t}{2Q+1}\right\}-R_8''\left\{\frac{1}{Q^2}-\frac{\cos 2t}{Q^2-4}\right\} \quad \textbf{(45a)}$$

We ensure a uniformly valid solution in terms of t in (45a) by equating the coefficients of $\cos Qt$ and $\sin Qt$ to zero. Thus, we have, for $\cos Qt$

$$\beta_{41}'+Q^2\beta_{41}=-\left(Q\alpha_{40}'+\frac{1}{2Q}\alpha_{40}''\right)=h_8(\tau) \quad \textbf{(45b)}$$

$$\therefore \beta_{41}(\tau)=e^{-Q^2\tau}\left[\int_0^\tau h_8(s)e^{Q^2s}ds+\beta_{41}(0)\right] \quad \textbf{(45c)}$$

Similarly, for $\sin Qt$

$$\alpha_{41}'+Q^2\alpha_{41}=0 \quad (45d)$$

On solving (45d) using (44h), we have

$$\alpha_{41}(\tau)=0 \quad (45e)$$

Consequently, we have

$$\varphi^{31}(t,\tau)=\beta_{41}(\tau)\sin Qt-\frac{R_9(\tau)\sin(1+Q)t}{2Q+1}+\frac{R_{10}(\tau)\sin(1-Q)t}{2Q-1}+\frac{R_{11}(\tau)\sin 2t}{Q^2-4} \quad (45f)$$

The remaining part of (45a) is

$$\varphi_{,tt}^{32}+Q^2\varphi^{32}=R_{12}(\tau)\{\sin(1+Q)t+\sin(1-Q)t\}+R_{13}(\tau)\sin 2t+R_{14}(\tau)\cos(1+Q)t$$
$$+R_{15}(\tau)\cos(1-Q)t+R_{16}(\tau)\cos 2t+R_{17}(\tau) \quad (45g)$$

where

$$R_{12}(\tau)=\frac{Q^2\beta_{10}(\tau)\beta_{22}(\tau)}{2}+\frac{Q^2\beta_{12}(\tau)\beta_{20}(\tau)}{2} \quad (45h)$$

$$R_{13}(\tau)=\frac{Q^2\beta_{10}(\tau)R_1(\tau)}{2}+\frac{Q^4\beta_{12}(\tau)\beta_{10}(\tau)}{2(Q^2-1)} \quad (45i)$$

$$R_{14}(\tau)=\frac{Q^2\alpha_{11}(\tau)\alpha_{21}(\tau)}{2}-\frac{(Q+1)R_9'(\tau)}{2Q+1}+\frac{2Q^2(Q+1)R_9(\tau)}{2Q+1}-\frac{Q^2R_7'(\tau)}{2Q-1}-\frac{R_7''(\tau)}{2Q+1} \quad (45j)$$

$$R_{15}(\tau)=\frac{Q^2\alpha_{11}(\tau)\alpha_{21}(\tau)}{2}-\frac{(1-Q)R_{10}'(\tau)}{2Q-1}-\frac{2Q^2(1-Q)R_{10}(\tau)}{2Q-1}+\frac{2Q^2R_7'(\tau)}{2Q-1}+\frac{R_7''(\tau)}{2Q+1} \quad (45k)$$

$$R_{16}(\tau) = \frac{Q^4\alpha_{11}{}^2(\tau)}{2(Q^2-1)} - \frac{Q^4\alpha_{11}(\tau)\beta_{10}(\tau)}{(Q^2-1)^2} - \frac{Q^6\alpha_{11}(\tau)\beta_{10}(\tau)}{(Q^2-1)^2} + \frac{2R'_{11}}{Q^2-4}$$
$$-\frac{4Q^2R_{11}}{Q^2-4} + \frac{2Q^2R'_8}{Q^2-4} + \frac{2R''_8}{Q^2-4} \quad (45l)$$

$$R_{17}(\tau) = -2R'_8(\tau) - \frac{R''_8(\tau)}{Q^2} \quad (45m)$$

Solving (45g), with the necessary initial conditions, we get

$$\varphi^{32}(t,\tau) = \alpha_{42}(\tau)\cos Qt + \beta_{42}(\tau)\sin Qt - \frac{R_{12}(\tau)\sin(1+Q)t}{2Q+1} + \frac{R_{12}(\tau)\sin(1-Q)t}{2Q-1}$$
$$+\frac{R_{13}(\tau)\sin 2t}{Q^2-4} - \frac{R_{14}(\tau)\cos(1+Q)t}{2Q+1} + \frac{R_{15}(\tau)\cos(1-Q)t}{2Q-1} + \frac{R_{16}(\tau)\cos 2t}{Q^2-4} + \frac{R_{17}(\tau)}{Q^2} \quad (46a)$$

where

$$\alpha_{42}(0) = \frac{R_{14}(0)}{2Q+1} - \frac{R_{15}(0)}{2Q-1} - \frac{R_{16}(0)}{Q^2-4} - \frac{R_{17}(0)}{Q^2} \quad (46b)$$

We find it necessary to obtain the following values which will be useful later

From (43d), (44f) and (45j – m), we obtain

$$R'_7(0) = \frac{-Q^2r_0I^2}{2}\frac{-Q^4r_0I^2}{2}, \qquad R''_7(0) = Q^2r_0I^2\left(Q^2 + \frac{1}{2} + \frac{Q^2}{2}\right) \quad (47a)$$

$$R'_8(0) = \frac{-Q^4I^2}{Q^2-1}, \quad R''_8(0) = \frac{2Q^4I^2}{Q^2-1}, \; R_9(0) = -Q^2r_0I^2\left(\frac{Q+1}{2Q+1}\right) \quad (47b)$$

$$R'_9(0) = I^2r_8, \; r_8 = \frac{-Q^2}{4} - \frac{Q^2r_0}{4} - \frac{2(Q+1)}{2Q+1}\left[Q^4r_0 + \frac{Q^2r_0}{2} + \frac{Q^6r_0}{2}\right]$$
$$+\frac{(Q+1)}{2Q+1}\left[Q^4r_0 + Q^6r_0\right] \quad (47c)$$

$$R_{10}(0) = Q^2r_0I^2\left(\frac{Q-1}{2Q-1}\right), \; R'_{10}(0) = I^2r_{10} \quad (47d)$$

$$r_{10} = \frac{Q^2r_0}{4} - \frac{Q^2}{4} - \frac{2(Q-1)}{2Q-1}\left[Q^4r_0 + \frac{Q^2r_0}{2} + \frac{Q^6r_0}{2}\right] + \frac{(Q-1)}{2Q-1}\left[Q^4r_0 + Q^6r_0\right] \quad (47e)$$

$$R_{11}(0) = I^2r_{12}, \; r_{12} = \frac{Q^4}{(Q^2-1)^2} - \frac{Q^6}{(Q^2-1)^2} + \frac{4Q^4}{Q^2-4} - \frac{2Q^6}{(Q^2-4)(Q^2-1)} \quad (47f)$$

$$R'_{11}(0) = I^2r_{13}, \; r_{13} = -\frac{3Q^4}{4(Q^2-1)} - \frac{2Q^4}{(Q^2-1)^2} + \frac{2Q^6}{(Q^2-1)^2} - \frac{8Q^4}{(Q^2-4)(Q^2-1)} + \frac{4Q^6}{Q^2-4} \quad (47g)$$

$$R_{14}(0) = I^2r_9,$$

$$r_9=\frac{-(1+Q)r_8}{2Q+1}-\frac{2(1+Q)^2Q^4r_0}{(2Q+1)^2}+\frac{Q^4r_0}{2Q-1}+\frac{Q^6r_0}{2Q-1}-\frac{Q^2r_0}{2Q+1}\left(Q^2+\frac{1}{2}+\frac{Q^4}{2}\right) \tag{47h}$$

$$R_{15}(0)=I^2r_{11},\ r_{11}=\frac{(1-Q)r_{10}}{2Q-1}+\frac{2(Q-1)^2Q^4r_0}{(2Q-1)^2}-\frac{Q^4r_0}{2Q-1}-\frac{Q^6r_0}{2Q-1}+\frac{Q^2r_0}{2Q-1}\left(Q^2+\frac{1}{2}+\frac{Q^4}{2}\right) \tag{47i}$$

$$R_{16}(0)=I^2r_{14},\ r_{14}=\frac{2r_{13}}{Q^2-4}-\frac{4Q^2r_{12}}{Q^2-4}-\frac{2Q^6}{(Q^2-4)(Q^2-1)}+\frac{2Q^4}{(Q^2-4)(Q^2-1)} \tag{47j}$$

$$R_{17}(0)=I^2r_{15},\quad r_{15}=\frac{2Q^4}{Q^2-1}-\frac{4Q^4}{Q^2(Q^2-1)} \tag{47k}$$

Hence

$$\alpha_{42}(0)=I^2r_{16},\ \ r_{16}=\frac{r_9}{2Q+1}-\frac{r_{11}}{2Q-1}-\frac{r_{14}}{Q^2-4}-\frac{r_{15}}{Q^2} \tag{47l}$$

From (20d), we have

$$\beta_{42}(0)=0 \tag{48a}$$

Summary :

The displacement components so far derived are as follows:

$$\xi_0(t)=\eta(t,\tau,\bar{\xi},\varepsilon)=\bar{\xi}\left(\eta^{10}+\varepsilon\eta^{11}+\varepsilon^2\eta^{12}+\ldots\right)+\bar{\xi}^2\left(\eta^{20}+\varepsilon\eta^{21}+\varepsilon^2\eta^{22}+\ldots\right)+\bar{\xi}^3\left(\eta^{30}+\varepsilon\eta^{31}+\varepsilon^2\eta^{32}+\ldots\right)+\ldots \tag{48b}$$

$$\xi_1(t)=\varphi(t,\tau,\bar{\xi},\varepsilon)=\bar{\xi}^2\left(\varphi^{20}+\varepsilon\varphi^{21}+\varepsilon^2\varphi^{22}+\ldots\right)+\bar{\xi}^3\left(\varphi^{30}+\varepsilon\varphi^{31}+\varepsilon^2\varphi^{32}+\ldots\right)+\ldots \tag{48c}$$

3.4 Values of variables of Maximum Displacements

We shall now determine values of the independent variables at maximum displacement.

These will help us determine the maximum displacement which is necessary for determining the dynamic buckling impulse.

Let $\hat{t}_a$, t_a , τ_a be the values of $\hat{t}$, t and τ for η at maximum and now expand them asymptotically as

$$\hat{t}_a=\hat{t}_0+\hat{t}_{01}\varepsilon+\hat{t}_{02}\varepsilon^2+\ldots+\bar{\xi}\left(\hat{t}_{10}+\hat{t}_{11}\varepsilon+\hat{t}_{12}\varepsilon^2+\ldots\right)+\bar{\xi}^2\left(\hat{t}_{20}+\hat{t}_{21}\varepsilon+\hat{t}_{22}\varepsilon^2+\ldots\right)+\ldots \tag{49a}$$

$$t_a=t_0+t_{01}\varepsilon+t_{02}\varepsilon^2+\ldots+\bar{\xi}\left(t_{10}+t_{11}\varepsilon+t_{12}\varepsilon^2+\ldots\right)+\bar{\xi}^2\left(t_{20}+t_{21}\varepsilon+t_{22}\varepsilon^2+\ldots\right)+\ldots \tag{49b}$$

$$\tau_a=\varepsilon\hat{t}_a=\varepsilon\left[\hat{t}_0+\hat{t}_{01}\varepsilon+\hat{t}_{02}\varepsilon^2+\ldots+\bar{\xi}\left(\hat{t}_{10}+\hat{t}_{11}\varepsilon+\hat{t}_{12}\varepsilon^2+\ldots\right)+\bar{\xi}^2\left(\hat{t}_{20}+\hat{t}_{21}\varepsilon+\hat{t}_{22}\varepsilon^2+\ldots\right)+\ldots\right] \tag{49c}$$

The condition for maximum for η is

$$\frac{d\xi_0}{dt}=0\Rightarrow\left(1+\omega_2'(\tau)\bar{\xi}^2+\omega_3'(\tau)\bar{\xi}^3+\ldots\right)\eta_{,t}+\varepsilon\eta_{,\tau}=0 \tag{50}$$

Expanding (50) in Taylor series about $(t_0,0)$, using (49a – c), we have

$$\bar{\xi}\left[\eta^{10}_{,t} + \left\{t_{01}\varepsilon + t_{02}\varepsilon^2 + \ldots + \bar{\xi}\left(t_{10} + t_{11}\varepsilon + t_{12}\varepsilon^2 + \ldots\right) + \bar{\xi}^2\left(t_{20} + t_{21}\varepsilon + t_{22}\varepsilon^2 + \ldots\right)\right\}\eta^{10}_{,tt}\right.$$

$$+ \varepsilon\left\{\hat{t}_0 + \hat{t}_{01}\varepsilon + \ldots + \bar{\xi}\left(\hat{t}_{10} + \hat{t}_{11}\varepsilon + \ldots\right) + \bar{\xi}^2\left(\hat{t}_{20} + \hat{t}_{21}\varepsilon + \ldots\right)\right\}\eta^{10}_{,t\tau} + \frac{1}{2}\left\{\left\{\left[t_{01}\varepsilon + t_{02}\varepsilon^2 + \ldots\right.\right.\right.$$

$$\left.+ \bar{\xi}\left(t_{10} + t_{11}\varepsilon + \ldots\right)\right]^2\eta^{10}_{,ttt} + 2\varepsilon\left\{t_{01}\varepsilon + \ldots + \bar{\xi}\left(t_{10} + t_{11}\varepsilon + \ldots\right) + \bar{\xi}^2\left(t_{20} + t_{21}\varepsilon + \ldots\right)\right\}\left\{\hat{t}_0 + \hat{t}_{01}\varepsilon + \ldots\right.$$

$$\left.\left.\left.+ \bar{\xi}\left(\hat{t}_{10} + \hat{t}_{11}\varepsilon + \ldots\right) + \bar{\xi}^2\left(\hat{t}_{20} + \hat{t}_{21}\varepsilon + \ldots\right)\right\}\eta^{10}_{,tt\tau} + \varepsilon^2\left\{\hat{t}_0 + \ldots + \bar{\xi}\hat{t}_{10} + \ldots + \bar{\xi}^2\hat{t}_{20}\right\}^2\eta^{10}_{,t\tau\tau}\right\}\right\}\right]$$

$$+ \bar{\xi}\varepsilon\left[\eta^{10}_{,\tau} + \left\{t_{01}\varepsilon + \ldots + \bar{\xi}\left(t_{10} + t_{11}\varepsilon + \ldots\right) + \bar{\xi}^2\left(t_{20} + t_{21}\varepsilon + \ldots\right) + \right\}^2\eta^{10}_{,\pi}\right.$$

$$+ \varepsilon\left\{\hat{t}_0 + \ldots + \bar{\xi}\hat{t}_{10} + \ldots + \bar{\xi}^2\hat{t}_{20}\right\}\eta^{10}_{,\tau\tau}$$

$$\left.+ 2\varepsilon\left\{\ldots + \bar{\xi}t_{10} + \bar{\xi}^2 t_{20} + \bar{\xi}^2 t_{21}\varepsilon + \ldots\right\}\left\{\hat{t}_0 + \ldots + \bar{\xi}\hat{t}_{10} + \ldots + \bar{\xi}^2\hat{t}_{20} + \ldots\right\}\eta^{10}_{,\pi\tau}\right\}\right\}\Big]$$

$$+ \bar{\xi}\varepsilon^2\left[\eta^{12}_{,t} + \left\{\ldots + \bar{\xi}t_{10} + \ldots + \bar{\xi}^2 t_{20}\right\}\eta^{12}_{,tt} + \frac{1}{2}\left\{\ldots + \bar{\xi}t_{10}\right\}^2\eta^{12}_{,ttt}\right]$$

$$+ \bar{\xi}^3\left[\eta^{30}_{,t} + \left\{t_{01}\varepsilon + t_{02}\varepsilon^2 + \ldots\right\}\eta^{30}_{,tt} + \varepsilon\left\{\hat{t}_0 + \hat{t}_{01}\varepsilon + \ldots\right\}\eta^{30}_{,t\tau} + \frac{t_{01}{}^2\varepsilon^2}{2}\eta^{30}_{,ttt}\right.$$

$$\left.+ \varepsilon\left\{t_{01}\varepsilon + t_{02}\varepsilon^2 + \ldots\right\}\left\{\hat{t}_0 + \hat{t}_{01}\varepsilon + \ldots\right\}\eta^{30}_{,tt\tau} + \frac{\hat{t}_0{}^2\varepsilon^2}{2}\eta^{30}_{,t\tau\tau}\right]$$

$$+ \bar{\xi}^3\varepsilon\left[\eta^{31}_{,t} + t_{01}\varepsilon\eta^{31}_{,tt} + \hat{t}_0\varepsilon\eta^{31}_{,t\tau}\right] + \bar{\xi}^3\varepsilon^2\eta^{32}_{.t}$$

$$+ \bar{\xi}^3\left[\omega_2'\eta^{10}_{,t} + \left\{t_{01}\varepsilon + t_{02}\varepsilon^2 + \ldots\right\}\omega_2'\eta^{10}_{,tt} + \varepsilon\left\{\hat{t}_0 + \hat{t}_{01}\varepsilon + \ldots\right\}\left(\omega_2'\eta^{10}_{,t}\right)_{\tau}\right.$$

$$\left.+ \frac{1}{2}\left\{\left\{t_{01}{}^2\varepsilon^2\omega_2'\eta^{10}_{,ttt} + 2\varepsilon\left\{t_{01}\varepsilon + \ldots\right\}\left\{\hat{t}_0 + \ldots\right\}\left(\omega_2'\eta^{10}_{,t}\right)_{,t\tau} + \varepsilon^2\hat{t}_0{}^2\left(\omega_2'\eta^{10}_{,t}\right)_{,\tau\tau}\right\}\right\}\right]$$

$$+ \bar{\xi}\varepsilon^2\left[\eta^{11}_{,\tau} + \left\{\ldots + \bar{\xi}t_0 + \ldots + \bar{\xi}^2 t_{20}\right\}\eta^{11}_{,\pi} + \frac{\bar{\xi}^2 t_{10}{}^2\eta^{11}_{,\pi t}}{2}\right] = 0 \qquad \textbf{(51)}$$

Equating relevant powers of orders of $\bar{\xi}$ and ε from (51), we have the following

$$O\left(\bar{\xi}.1\right): \eta^{10}_{,t}\left(t_0, 0\right) = 0 \qquad (52a)$$

$$O\left(\bar{\xi}.\varepsilon\right): t_{01}\eta^{10}_{,tt} + \hat{t}_0\eta^{10}_{,t\tau} + \eta^{11}_{,t} + \eta^{10}_{,\tau} = 0 \qquad (52b)$$

$$O\left(\bar{\xi}.\varepsilon^2\right): t_{02}\eta^{10}_{,tt} + \hat{t}_{01}\eta^{10}_{,t\tau} + \frac{t_{01}^2\eta^{10}_{,tt}}{2} + t_{01}\hat{t}_0\eta^{10}_{,tt\tau} + \frac{\hat{t}_0{}^2\eta^{10}_{,t\tau\tau}}{2} + t_{01}\eta^{11}_{,tt} + \hat{t}_0\eta^{11}_{,t\tau}$$

$$+ \eta^{12}_{,t} + t_{01}\eta^{10}_{,\pi} + \hat{t}_0\eta^{10}_{,\tau\tau} + \eta^{11}_{,\tau} = 0 \qquad (52c)$$

$$O\left(\bar{\xi}^3.1\right): t_{20}\eta^{10}_{,tt} + \eta^{30}_{,t} + \omega_2'\eta^{10}_{,t} = 0 \qquad (52d)$$

$$O\left(\bar{\xi}^3.\varepsilon\right): t_{21}\eta^{10}_{,tt} + \hat{t}_{20}\eta^{10}_{,t\tau} + t_{20}\hat{t}_0\eta^{10}_{,tt\tau} + t_{10}\hat{t}_{10}\eta^{10}_{,tt\tau} + t_{20}\eta^{11}_{,tt} + \frac{t_{10}^2\eta^{11}_{,ttt}}{2} + t_{01}\eta^{30}_{,tt}$$

$$+ \hat{t}_0\eta^{30}_{,t\tau} + \eta^{31}_{,t} + \omega' t_{01}\eta^{10}_{,tt} + \hat{t}_0\left(\omega_2'\eta^{10}_{,t}\right)_{,\tau} + t_{20}\eta^{10}_{,\pi} + \frac{t_{10}^2\eta^{10}_{,\pi t}}{2} = 0 \qquad (52e)$$

From (21f), we have

$$\eta^{10}(t_0,0) = I\sin t_0 \tag{53a}$$

$$\therefore \eta_{,t}^{10}(t_0,0) = I\cos t_0 \tag{53b}$$

Hence (52a) implies

$$\cos t_0 = 0 \tag{53c}$$

$$\therefore t_0 = \frac{\pi}{2} \tag{53d}$$

From (52b), we have

$$t_{01} = \frac{-1}{\eta_{,tt}^{10}}\left[\hat{t}_0\eta_{,t\tau}^{10} + \eta_{,t}^{11} + \eta_{,\tau}^{10}\right]_{(t_0,0)} \tag{53e}$$

From (21f), (53e) and (25b), we have

$$t_{01} = \frac{-1}{I}\left[-I\right] = 1 \tag{53f}$$

From (52c), we have

$$t_{02} = \frac{-1}{\eta_{,tt}^{10}}\left[t_{01}\hat{t}_0\eta_{,tt\tau}^{10} + \hat{t}_0\eta_{,t\tau}^{11} + \eta_{,t}^{12} + \hat{t}_0\eta_{,\tau\tau}^{10}\right]_{(t_0,0)} \tag{53g}$$

$$= \frac{1}{I}\left[t_{01}\hat{t}_0 I - \frac{I\hat{t}_0}{2} + \hat{t}_0 I\right] = \frac{3\hat{t}_0}{2} \tag{53h}$$

(52d) implies

$$t_{20} = -\frac{1}{\eta_{,tt}^{10}}\left[\eta_{,t}^{30} + \omega_2'\eta_{,t}^{10}\right]_{(t_0,0)}$$

On simplifying, we get

$$\therefore t_{20} = P_1 = \frac{2K_0 Q r_0 \cos\left(\frac{Q\pi}{2}\right)}{\lambda_c\left(1-Q^2\right)} \tag{53i}$$

From (52e) we have

$$t_{21} = -\frac{1}{\eta_{,tt}^{10}}\left[t_{20}\hat{t}_0\eta_{,tt\tau}^{10} + t_{01}\eta_{,tt}^{30} + \hat{t}_0\eta_{,t\tau}^{30} + \eta_{,t}^{31} + \omega_2' t_{01}\eta_{,tt}^{10}\right]_{(t_0,0)} \tag{53j}$$

We shall now determine $\hat{t}_0, \hat{t}_{01}, \hat{t}_{02}, ..., \hat{t}_{20}, \hat{t}_{21}$, which are necessary in determining t_{21} and the maximum displacement. We know that

$$t_a = \hat{t}_a + \frac{\omega_2(\tau_a)\bar{\xi}^2 + \omega_3(\tau_a)\bar{\xi}^3 + ...}{\varepsilon} \tag{54a}$$

and

$$\omega_2(\tau_a) = \omega_2(0) + \tau_a \omega_2'(0) + \frac{\tau_a{}^2 \omega_2''(0)}{2} + ... \tag{54b}$$

From (13a), we know that

$$\tau_a = \varepsilon \hat{t}_a$$

$$\therefore \ t_a = \hat{t}_a + \frac{1}{\varepsilon}\left(\omega_2(0) + \varepsilon \hat{t}_a \omega_2'(0) + \frac{(\varepsilon \hat{t}_a)^2 \omega_2''(0)}{2} + ... \right)\bar{\xi}^2 + ...$$

$$t_a = \hat{t}_a + \hat{t}_a \omega_2'(0)\bar{\xi}^2 + ... = \left(1 + \omega_2'(0)\bar{\xi}^2\right)\hat{t}_a + ... \tag{54c}$$

Substituting for t_a from (49b) and for $\hat{t}_a$ from (49a) in (54c), we get

$$t_0 + t_{01}\varepsilon + t_{02}\varepsilon^2 + ... + \bar{\xi}^2\left(t_{20} + t_{21}\varepsilon + t_{22}\varepsilon^2 + ...\right) + ...$$
$$= \left[1 + \omega_2'(0)\bar{\xi}^2\right]\left\{\hat{t}_0 + \hat{t}_{01}\varepsilon + \hat{t}_{02}\varepsilon^2 + ... + \bar{\xi}^2\left(\hat{t}_{20} + \hat{t}_{21}\varepsilon + \hat{t}_{22}\varepsilon^2 + ...\right)\right\} + ...$$

Equating relevant orders of powers of $\bar{\xi}$ and ε, we have

$$O(1): t_0 = \hat{t}_0 = \frac{\pi}{2} \tag{55a}$$

$$O(\varepsilon.1): t_{01} = \hat{t}_{01} = 1 \tag{55b}$$

$$O(\varepsilon^2.1): t_{02} = \hat{t}_{02} \tag{55c}$$

$$O(\bar{\xi}^2.1): t_{20} = \hat{t}_{20} + \omega_2'(0)\hat{t}_0 \Rightarrow \hat{t}_{20} = t_{20} - \omega_2'(0)t_0 \tag{55d}$$

$$O(\bar{\xi}^2.\varepsilon): t_{21} = \hat{t}_{21} + \omega_2'(0)\hat{t}_{01} \Rightarrow \hat{t}_{21} = t_{21} - \omega_2'(0)t_{01} \tag{55e}$$

From (55d), we have

$$\hat{t}_{20} = P_1 + \frac{K_0 Q^2 \pi}{2\lambda_c(Q^2 - 1)} = P_3 \tag{55f}$$

Similarly, (55e) implies

$$\hat{t}_{21} = P_2 + \frac{K_0 Q^2}{\lambda_c(Q^2 - 1)} = P_4 \tag{55g}$$

On substituting $\hat{t}_0$, t_{01} , t_{20} in (53j), we get

$$t_{21} = \frac{\pi P_1}{2} + r_{17} + \frac{\pi r_{18}}{2} + r_{19} + \frac{K_0 Q^2}{\lambda_c(Q^2 - 1)} = P_2 \tag{55h}$$

where

$$r_{17}=\frac{2K_0{r_0}^2\cos(Q\pi/2)}{\lambda_c},\ r_{18}=\frac{K_0Qr_0}{\lambda_c}\left\{\left(1+\frac{2r_0}{Q}\right)-\left(2r_0\sin\left(\frac{Q\pi}{2}\right)\right)\right\},\ r_{19}=\frac{-2Q^2K_0{r_0}^2\cos\left(\frac{Q\pi}{2}\right)}{\lambda_c}$$

We shall now determine the values of the independent variables at the maximum value of $\varphi(\mathrm{t},\tau,\varepsilon,\bar{\xi})$ and now

let t_c, $\hat{t}_c$ and τ_c be the values of t, $\hat{t}$ and τ for $\varphi(\mathrm{t},\tau,\varepsilon,\bar{\xi})$ at maximum. Hence

$$t_c=T_0+\varepsilon T_{01}+\varepsilon^2T_{02}+...+\bar{\xi}(T_{10}+\varepsilon T_{11}+...)+\bar{\xi}^2(T_{20}+\varepsilon T_{21}+...)+... \tag{56a}$$

$$\hat{t}_c=\hat{T}_0+\varepsilon\hat{T}_{01}+\varepsilon^2\hat{T}_{02}+...+\bar{\xi}(\hat{T}_{10}+\varepsilon\hat{T}_{11}+...)+\bar{\xi}^2(\hat{T}_{20}+\varepsilon\hat{T}_{21}+...)+... \tag{56b}$$

$$\tau_c=\varepsilon t_c=\varepsilon\left[T_0+\varepsilon T_{01}+\varepsilon^2T_{02}+...+\bar{\xi}(T_{10}+\varepsilon T_{11}+...)+\bar{\xi}^2(T_{20}+\varepsilon T_{21}+...)+...\quad\right] \tag{56c}$$

The condition for maximum for φ is

$$\frac{d\xi_1}{dt}=0\Rightarrow\left(1+\omega_2'\bar{\xi}^2+\omega_3'\bar{\xi}^3+...\right)\varphi_{,t}+\varepsilon\varphi_{,\tau}=0 \tag{57a}$$

But

$$\varphi(\mathrm{t},\tau,\varepsilon,\bar{\xi})=\bar{\xi}^2\left(\varphi^{20}+\varepsilon\varphi^{21}+\varepsilon^2\varphi^{22}+...\right)+\bar{\xi}^3\left(\varphi^{30}+\varepsilon\varphi^{31}+\varepsilon^2\varphi^{32}+...\right)+... \tag{57b}$$

Hence, substituting in (57a) from (57b), we get

$$\begin{aligned}&\left(1+\omega_2'\bar{\xi}^2+\omega_3'\bar{\xi}^3+...\right)\left[\bar{\xi}^2\left(\varphi^{20}+\varepsilon\varphi^{21}+\varepsilon^2\varphi^{22}+...\right)+\bar{\xi}^3\left(\varphi^{30}+\varepsilon\varphi^{31}+\varepsilon^2\varphi^{32}+...\right)\right]_{,t}\\&\varepsilon\left[\bar{\xi}^2\left(\varphi^{20}+\varepsilon\varphi^{21}+\varepsilon^2\varphi^{22}+...\right)+\bar{\xi}^3\left(\varphi^{30}+\varepsilon\varphi^{31}+\varepsilon^2\varphi^{32}+...\right)\right]_{,\tau}=0\end{aligned} \tag{58}$$

Similarly, expanding (58) in Taylor series about $(T_0,0)$, we obtain

$$\begin{aligned}&\bar{\xi}^2\Big[\varphi^{20}_{,t}+\left\{\varepsilon T_{01}+\varepsilon^2T_{02}+...+\bar{\xi}\left(T_{10}+\varepsilon T_{11}+\varepsilon^2T_{12}+...\right)\right\}\varphi^{20}_{,tt}\\&+\varepsilon\left\{\hat{T}_0+\varepsilon\hat{T}_{01}+...+\bar{\xi}\left(\hat{T}_{10}+\varepsilon\hat{T}_{11}+...\right)\right\}\varphi^{20}_{,t\tau}+\frac{1}{2}\Big\{\left(\varepsilon T_{01}+...+\bar{\xi}T_{10}\right)^2\varphi^{20}_{,ttt}\\&+2\varepsilon\left\{\varepsilon T_{01}+...+\bar{\xi}\left(T_{10}+\varepsilon T_{11}+...\right)\right\}\left\{\hat{T}_0+\varepsilon T_{01}+...+\bar{\xi}\left(\hat{T}_{10}+\varepsilon\hat{T}_{11}+...\right)\right\}\varphi^{20}_{,tt\tau}\\&+\varepsilon^2\left(\hat{T}_0+...+\bar{\xi}T_{10}\right)^2\varphi^{20}_{,t\tau\tau}\Big\}\Big]\\&\quad+\bar{\xi}^2\varepsilon\Big[\varphi^{21}_{,t}+\left\{\varepsilon T_{01}+...+\bar{\xi}\left(T_{10}+\varepsilon T_{11+...}\right)\right\}\varphi^{21}_{,tt}+\varepsilon\left\{\hat{T}_0+...+\bar{\xi}\hat{T}_{10}\right\}\varphi^{21}_{,t\tau}\\&\quad+\frac{1}{2}\Big\{\left(\varepsilon T_{01}+...+\bar{\xi}T_{10}\right)^2\varphi^{21}_{,ttt}+2\varepsilon\left\{...+\bar{\xi}T_{10}+...\right\}\left\{\hat{T}_0+\bar{\xi}\hat{T}_{10}\right\}\varphi^{21}_{,tt\tau}\Big\}\Big]\\&\quad\bar{\xi}^2\varepsilon\left[\varphi^{22}_{,tt}+\left\{...+\bar{\xi}T_{10}\right\}\varphi^{22}_{,ttt}\right]\\&\quad+\bar{\xi}^3\left[\varphi^{30}_{,t}+\left\{T_{01}\varepsilon+\varepsilon^2T_{02}\right\}\varphi^{30}_{,tt}+\varepsilon\left\{\hat{T}_0+\varepsilon\hat{T}_{01}+...\right\}\varphi^{30}_{,t\tau}\right.\\&+\frac{1}{2}\Big\{\varepsilon^2{T_{01}}^2\varphi^{30}_{,ttt}+2\varepsilon\left\{\varepsilon T_{01}\right\}\left\{\hat{T}_0+\varepsilon\hat{T}_{01}\right\}\varphi^{30}_{,tt\tau}+\varepsilon^2{\hat{T}_0}^2\varphi^{30}_{,t\tau\tau}\Big\}\Big]\end{aligned}$$

$$+\bar{\xi}^3\varepsilon\left[\varphi_{,t}^{31}+\varepsilon T_{01}\varphi_{,tt}^{31}+\varepsilon\hat{T}_0\varphi_{,t\tau}^{31}\right]+\bar{\xi}^3\varepsilon^2\varphi_{,t}^{32}$$

$$+\bar{\xi}^2\varepsilon\left[\varphi_{,\tau}^{20}+\left\{\varepsilon T_{01}+\bar{\xi}\left(T_{10}+\varepsilon T_{11}\right)\right\}\varphi_{,t\tau}^{20}+\varepsilon\left\{\hat{T}_0+\bar{\xi}\hat{T}_{10}\right\}\varphi_{,\tau\tau}^{20}\right.$$

$$+\bar{\xi}^3\varepsilon\left[\varphi_{,\tau}^{30}+\varepsilon T_{01}\varphi_{,t\tau}^{30}+\varepsilon\hat{T}_0\varphi_{,\tau\tau}^{30}\right]+\bar{\xi}^3\varepsilon^2\varphi_{,\tau}^{31}=0 \quad (59)$$

$$+\frac{1}{2}\left\{\left(\varepsilon T_{01}+\bar{\xi}T_{10}\right)^2\varphi_{,\tau tt}^{20}+2\varepsilon\bar{\xi}T_{10}\hat{T}_0\varphi_{,\tau t\tau}^{20}\right\}\Big]+\bar{\xi}^2\varepsilon^2\left[\varphi_{,\tau}^{21}+\bar{\xi}T_{10}\varphi_{,t\tau}^{21}\right]$$

We now equate coefficients of powers of $\bar{\xi}^i\varepsilon^j$ to get.

$$O\left(\bar{\xi}^2.1\right):\varphi_{,t}^{20}=0 \quad \textbf{(60a)}$$

$$O\left(\bar{\xi}^2.\varepsilon\right):T_{01}\varphi_{,tt}^{20}+\hat{T}_0\varphi_{,t\tau}^{20}+\varphi_{,\tau}^{20}+\varphi_{,t}^{21}=0 \quad \textbf{(60b)}$$

$$O\left(\bar{\xi}^2.\varepsilon^2\right):T_{02}\varphi_{,tt}^{20}+\frac{T_{01}{}^2\varphi_{,ttt}^{20}}{2}+T_{01}\hat{T}_0\varphi_{,tt\tau}^{20}+\hat{T}_0{}^2\varphi_{,t\tau\tau}^{20}+\hat{T}_{01}\varphi_{,t\tau}^{20}+T_{01}\varphi_{,tt}^{21}$$
$$+\hat{T}_0\varphi_{,t\tau}^{21}+\varphi_{,t}^{22}+T_{01}\varphi_{,t\tau}^{20}+\hat{T}_0\varphi_{,\tau\tau}^{20}+\varphi_{,\tau}^{21} \quad \textbf{(60c)}$$

$$O\left(\bar{\xi}^3.1\right):T_{10}\varphi_{,tt}^{20}+\varphi_{,t}^{30}=0 \quad \textbf{(60d)}$$

$$O\left(\bar{\xi}^3.\varepsilon\right):T_{11}\varphi_{,tt}^{20}+\hat{T}_{10}\varphi_{,t\tau}^{20}+T_{01}T_{10}\varphi_{,ttt}^{20}+T_{10}\hat{T}_0\varphi_{,tt\tau}^{20}+T_{10}\varphi_{,tt}^{21}+T_{01}\varphi_{,tt}^{30}$$
$$+\hat{T}_0\varphi_{,t\tau}^{30}+\varphi_{,t}^{31}+T_{10}\varphi_{,t\tau}^{20}+\varphi_{,\tau}^{30}=0 \quad \textbf{(60e)}$$

$$O\left(\bar{\xi}^3.\varepsilon^2\right):T_{12}\varphi_{,tt}^{20}+\hat{T}_{11}\varphi_{,t\tau}^{20}+T_{11}\hat{T}_0\varphi_{,tt\tau}^{20}+T_{01}\hat{T}_{10}\varphi_{,tt\tau}^{20}+T_{10}\hat{T}_{01}\varphi_{,tt\tau}^{20}+\hat{T}_0\hat{T}_{10}\varphi_{,t\tau\tau}^{20}$$
$$+T_{11}\varphi_{,tt}^{21}+\hat{T}_{10}\varphi_{,t\tau}^{21}+T_{01}T_{10}\varphi_{,tt}^{21}+\hat{T}_0T_{10}\varphi_{,tt\tau}^{21}+T_{10}\varphi_{,tt}^{22}+T_{02}\varphi_{,tt}^{30}$$
$$+\hat{T}_{01}\varphi_{,t\tau}^{30}+\frac{T_{01}{}^2\varphi_{,ttt}^{30}}{2}+T_{01}\hat{T}_0\varphi_{,tt\tau}^{30}+\hat{T}_{02}{}^2\varphi_{,t\tau\tau}^{30}+T_{01}\varphi_{,tt}^{31}+\hat{T}_0\varphi_{,t\tau}^{31}$$
$$+\varphi_{,t}^{32}+T_{11}\varphi_{,t\tau}^{20}+\hat{T}_{10}\varphi_{,\tau\tau}^{20}+T_{01}T_{10}\varphi_{,\tau tt}^{20}+T_{10}\hat{T}_0\varphi_{,\tau t\tau}^{20}+T_{10}\varphi_{,t\tau}^{21}$$
$$+T_{01}\varphi_{,t\tau}^{30}+\hat{T}_0\varphi_{,\tau\tau}^{30}+\varphi_{,\tau}^{31}=0 \quad \textbf{(60f)}$$

Substituting in (60a), we get

$$Q\beta_{20}(0)\cos QT_0+\frac{Q^2\beta_{10}(0)\cos T_0}{Q^2-1}=0 \quad (61a)$$

On simplifying (61a), we get

$$\cos QT_0-\cos T_0=0 \quad (61b)$$

Taking the first few terms of the Taylor series expansion of (61b), we have

$$1-\frac{(QT_0)^2}{2}+\frac{(QT_0)^4}{4!}+\ldots-\left\{1-\frac{T_0{}^2}{2}+\frac{T_0{}^4}{4!}+\ldots\right\}=0 \quad (61c)$$

$$\Rightarrow\ \frac{\left(Q^4-1\right)T_0{}^4}{4!}+\frac{\left(1-Q^2\right)T_0{}^2}{2}=0$$

$$\Rightarrow T_0^{\ 2}\left[\frac{(Q^2+1)(Q^2-1)T_0^{\ 2}}{4!}-\frac{(Q^2-1)}{2}\right]=0 \tag{61d}$$

On solving (61d), we get

$$T_0 \approx \pm 2\sqrt{\frac{3}{(Q^2+1)}} \tag{61e}$$

We shall however take only the positive sign.
Solving (60b), we have

$$T_{01} = -\frac{1}{\varphi_{,tt}^{\ 20}}\left[\hat{T}_0\varphi_{,t\tau}^{\ 20}+\varphi_{,\tau}^{\ 20}+\varphi_{,t}^{\ 21}\right]_{(T_0,0)} \tag{62a}$$

$$\therefore\ T_{01} = -\frac{1}{IS_2}\left[\hat{T}_0 IS_1 + IS_2 + IS_3\right] = -\left(1+\frac{\hat{T}_0 S_1}{S_2}+\frac{S_3}{S_2}\right) \tag{62b}$$

where

$$S_1 = Qr_0\sin T_0 - Q^2 r_0 \sin QT_0\ ,\quad S_2 = S_3 = Qr_0\cos T_0 - Q^3 r_0\cos QT_0$$

Solving (60c), we have

$$T_{02} = -\frac{1}{\varphi_{,tt}^{\ 20}}\left[\frac{T_{01}^{\ 2}\varphi_{,ttt}^{\ 20}}{2}+T_{01}\hat{T}_0\varphi_{,tt\tau}^{\ 20}+\hat{T}_0^{\ 2}\varphi_{,t\tau\tau}^{\ 20}+\hat{T}_{01}\varphi_{,t\tau}^{\ 20}+T_{01}\varphi_{,tt}^{\ 21}+\hat{T}_0\varphi_{,t\tau}^{\ 21}\right.$$
$$\left.+\varphi_{,t}^{\ 22}+T_{01}\varphi_{,t\tau}^{\ 20}+\hat{T}_0\varphi_{,\tau\tau}^{\ 20}+\varphi_{,\tau}^{\ 21}\right]_{(T_0,0)}$$

$$\therefore\ T_{02} = -\frac{1}{S_2}\left[\frac{T_{01}^{\ 2}S_4}{2}+T_{01}\hat{T}_0S_5+\hat{T}_0^{\ 2}S_6+\hat{T}_{01}S_1+T_{01}S_8+\hat{T}_0S_9+S_{10}+T_{01}S_7+\hat{T}_0S_5+S_{11}\right] \tag{62c}$$

where

$$S_4 = Q^4 r_0\sin QT_0 - Qr_0\sin T_0\ ,\ S_5 = Q^5 r_0\cos QT_0 - Qr_0\cos T_0$$
$$S_6 = 2r_0\left(Q^2\sin QT_0 + Q^2 r_0\sin T_0 - r_0\sin T_0\right),\ S_7 = 2r_0\left(Q^3\cos QT_0 - r_0\cos T_0 + Q^2 r_0\cos T_0\right)$$
$$S_8 = r_0\left(\frac{-3Q^4\sin(QT_0)}{2}-\frac{Q\sin T_0}{2}+2r_0\sin T_0 - 2Q^2 r_0\sin T_0\right),$$
$$S_9 = 2r_0\left(-Q^4\sin(QT_0)+r_0\sin T_0 - Q^2 r_0\sin T_0\right)$$
$$S_{10} = r_0\left(\frac{-3Q^5\cos(QT_0)}{5}-\frac{Q\cos T_0}{2}+2r_0\cos T_0 - 2Q^2 r_0\cos T_0\right)$$
$$S_{11} = Qr_4\cos(QT_0)+r_1\cos T_0,\quad S_{12} = -\left(Q^2 r_4\sin(QT_0)+r_1\sin T_0\right)$$

Solving (60d), we have

$$T_{10} = -\frac{\varphi_{,t}^{\ 30}}{\varphi_{,tt}^{\ 20}}\Big|_{(T_0,0)} = \frac{-I^2 S_{13}}{IS_1} = -\frac{IS_{13}}{S_1} \tag{62d}$$

$$S_{13} = Q^3 r_0\left\{\frac{-r_{33}\sin(QT_0)}{2}-\frac{\sin 2T_0}{Q^2-4}\right\}+\frac{Q^2 r_0}{2}\left\{\frac{(1-Q)\sin((Q-1)T_0)}{2Q-1}-\frac{(Q+1)\sin((Q+1)T_0)}{2Q+1}\right\}$$

Solving (60e), we have

$$T_{11} = -\frac{1}{\varphi_{,tt}^{20}}\left[\hat{T}_{10}\varphi_{,t\tau}^{20} + T_{01}T_{10}\varphi_{,ttt}^{20} + T_{10}\hat{T}_0\varphi_{,tt\tau}^{20} + T_{10}\varphi_{,tt}^{21} + \varphi_{,tt}^{30} + \hat{T}_0\varphi_{,t\tau}^{30} + \varphi_{,t}^{31} + T_{10}\varphi_{,t\tau}^{20} + \varphi_{,\tau}^{30}\right.$$

$$= -\frac{1}{IS_2}\left[\hat{T}_{10}IS_1 + T_{01}T_{10}IS_4 + T_{10}\hat{T}_0 IS_5 + T_{10}IS_8 + I^2 S_{13} + \hat{T}_0 I^2 S_{14} + I^2 S_{16} + T_{10}IS_1 + I^2 S_{15}\right]$$

where

$$S_{14} = \frac{Q^2 r_0}{2}\left\{-Q^2 r_{33}\cos(QT_0) - \frac{(1-Q)^2\cos((Q-1)T_0)}{2Q-1} - \frac{(1+Q)^2\cos((Q+1)T_0)}{2Q+1} - \frac{4Q\cos 2T_9}{Q^2-4}\right\}$$

$$S_{15} = \frac{Q^2 r_0}{2}\left\{-Q^3 r_{33}\sin(QT_0) - (Q^2+1)\left[\frac{(1-Q)\sin((Q-1)T_0)}{2Q-1} - \frac{(Q+1)\sin((Q+1)T_0)}{2Q+1}\right] + \frac{4Q\sin 2T_0}{Q^2-4}\right\}$$

$$S_{16} = \frac{Q^2 r_0}{2}\left\{-Q^3 r_{33}\sin(QT_0) + \left[\frac{-(1-Q)^3\sin((Q-1)T_0)}{2Q-1} + \frac{(Q+1)^3\sin((Q+1)T_0)}{2Q+1}\right] + \frac{8Q\sin 2T_0}{Q^2-4}\right\}$$

$$S_{17} = \frac{Q^2 r_0}{2}\left\{Q^4 r_{33}\cos(QT_0) - (Q^2+1)\left[\frac{-(Q-1)^2\cos((Q-1)T_0)}{2Q-1} - \frac{(Q+1)^2\cos((Q+1)T_0)}{2Q+1}\right] + \frac{8Q\cos 2T_0}{Q^2-4}\right\}$$

$$S_{18} = \frac{-Q^7 r_0 r_{33}\sin(QT_0)}{2} + r_{34}\left\{\frac{(1-Q)\sin((Q-1)T_0)}{2Q-1} - \frac{(Q+1)\sin((Q+1)T_0)}{2Q+1}\right\} - \frac{4Q^3 r_0\sin 2T_0}{Q^2-4}$$

$$S_{19} = Qr_7\cos(QT_0) + \left(\frac{1+Q}{2Q+1}\right)^2 Q^2 r_0\cos((1+Q)T_0) - \left(\frac{1-Q}{2Q-1}\right)^2 Q^2 r_0\cos((1-Q)T_0) + \frac{r_{12}\sin 2T_0}{Q^2-4}$$

$$S_{20} = -Q^2 r_7\sin(QT_0) + \frac{(1+Q)^3}{(2Q+1)^2}Q^2 r_0\sin((1+Q)T_0) + \frac{(1-Q)^3}{(2Q-1)^2}Q^2 r_0\sin((1-Q)T_0) - \frac{4r_{12}\sin 2T_0}{Q^2-4}$$

$$S_{21} = -Qr_{16}\sin(QT_0) - \frac{(1+Q)}{2Q+1}\left\{\frac{Q^2 r_4}{2} + \frac{Q^2 r_0}{4}\right\}\cos((1+Q)T_0)$$

$$+\frac{(1-Q)}{2Q-1}\left\{\frac{Q^2 r_4}{2} + \frac{Q^2 r_0}{4}\right\}\cos((1-Q)T_0) + \frac{1}{Q^2-4}\left\{Q^2 r_1 - Q^3 r_0\right\}\cos 2T_0$$

$$S_{22} = Qr_0(\sin T_0 - Q\sin(QT_0)), \qquad S_{23} = Q^4 r_0 \sin(QT_0) - Qr_0\sin(QT_0)$$

$$S_{24} = Qr_0\left(\frac{3Q^2\cos(QT_0)}{2} + \frac{\cos T_0}{2}\right) + 2r_0{}^2\left(Q^2\cos T_0 - \cos T_0\right)$$

$$S_{25} = \frac{Q^2 r_0}{2}\left\{-Q^2 r_{33}\cos(QT_0) - (Q^2+1)\left[\frac{\cos((Q-1)T_0)}{2Q-1} + \frac{\cos((Q+1)T_0)}{2Q+1}\right]\right\}$$

$$+Q^3 r_0\left\{\frac{1}{Q^2} - \frac{\cos((Q+1)T_0)}{2Q+1}\right\}$$

$$S_{26} = \frac{Q^6 r_0 r_{33}\cos(QT_0)}{2} + r_{34}\left[\frac{\cos((Q-1)T_0)}{2Q-1} + \frac{\cos((Q+1)T_0)}{2Q+1}\right] - 2Q^3 r_0\left\{\frac{1}{Q^2} - \frac{\cos 2T_0}{Q^2-4}\right\}$$

$$S_{27} = r_{35}\sin(QT_0) - \frac{r_8\sin((1+Q)T_0)}{2Q+1} + \frac{r_{10}\sin((1-Q)T_0)}{2Q-1} + \frac{r_{13}\sin 2T_0}{Q^2-4}$$

$$S_{28} = r_0\sin(QT_0) - Qr_0\sin T_0,\quad S_{29} = 2r_0\left(r_0\cos T_0 - Q\cos(QT_0) - Q^2 r_0\cos T_0\right)$$

$$S_{30} = r_4\sin(QT_0) + r_1\sin T_0$$

$$S_{31} = \frac{Q^2 r_0}{2}\left\{Q^2 r_{33}\cos(QT_0) + \left[\frac{\cos((Q-1)T_0)}{2Q-1} + \frac{\cos((Q+1)T_0)}{2Q+1}\right] - \frac{1}{Q^2} - \frac{Q\cos 2T_0}{Q^2-4}\right\}$$

$$S_{32} = r_7\sin(QT_0) + \frac{Q^2 r_0(Q+1)\sin((1+Q)T_0)}{(2Q+1)^2} + \frac{Q^2 r_0(Q-1)\sin((1-Q)T_0)}{(2Q-1)^2} + \frac{r_{12}\sin 2T_0}{Q^2-4}$$

$$S_{33} = r_{16}\cos(QT_0) - \left(\frac{Q^2 r_4}{2} + \frac{Q^2 r_0}{4}\right)\frac{\sin((1+Q)T_0)}{2Q+1} + \left(\frac{Q^2 r_4}{2} + \frac{Q^2 r_0}{4}\right)\frac{\sin((1-Q)T_0)}{2Q-1}$$

$$+\left(\frac{Q^2 r_1}{2} - \frac{Q^3 r_0}{2}\right)\frac{\sin 2T_0}{Q^2-4} - \frac{r_9\cos((1+Q)T_0)}{2Q+1} + \frac{r_{11}\cos((1-Q)T_0)}{2Q-1} + \frac{r_{14}\cos 2T_0}{Q^2-4} + \frac{r_{15}}{Q^2}$$

where

$$r_{20} = \frac{K_0 r_0}{\lambda_c}\left\{4r_0 - 2Qr_0 + Q - 2Qr_0\sin\left(\frac{Q\pi}{2}\right)\right\},$$

$$r_{21} = \frac{K_0 r_0}{\lambda_c}\left\{4Q^2 r_0 - 6r_0 - \frac{3Q}{2}\right\} - \frac{K_0}{2\lambda_c(Q^2-1)} + \frac{2K_0 r_0{}^2 Q^3\sin\left(\frac{Q\pi}{2}\right)}{\lambda_c}$$

$$r_{22}=\frac{K_0 r_0}{\lambda_c}\left\{2Q^3\sin\left(\frac{Q\pi}{2}\right)-4r_0+2Qr_0-Q\right\},\quad r_{23}=\frac{2Q^4K_0{r_0}^2\cos\left(\frac{Q\pi}{2}\right)}{\lambda_c},$$

$$r_{24}=\frac{-2Q^2K_0r_0\cos\left(\frac{Q\pi}{2}\right)}{\lambda_c},\quad r_{25}=\frac{K_0r_0}{\lambda_c}\left\{\frac{2\sin\left(\frac{Q\pi}{2}\right)}{\left(1-Q^2\right)}-Q-2r_0\right\},$$

$$r_{26}=\frac{2K_0{r_0}^2\cos\left(\frac{Q\pi}{2}\right)}{\lambda_c}\left\{2Qr_0-\frac{2}{1-Q^2}-1\right\}$$

$$r_{27}=\frac{2K_0r_0}{\lambda_c\left(1-Q^2\right)}\left\{2Q^2\sin\left(\frac{Q\pi}{2}\right)+2Qr_0\sin\left(\frac{Q\pi}{2}\right)-2Q^3r_0\sin\left(\frac{Q\pi}{2}\right)+r_1-2Q\right\}$$

$$r_{28}=\frac{4QK_0{r_0}^2\cos\left(\frac{Q\pi}{2}\right)}{\lambda_c}\left\{Q-Q^2r_0+r_0\right\},\quad r_{29}=\frac{Q^2K_0{r_0}^2\sin\left(\frac{Q\pi}{2}\right)}{\lambda_c}\left\{4Q^2r_0-3Q-4r_0\right\}-r_5$$

$$r_{30}=\frac{QK_0{r_0}^2\cos\left(\frac{Q\pi}{2}\right)}{\lambda_c}\left\{3Q-4Q^2r_0+4r_0\right\},$$

$$r_{31}=\frac{2K_0{r_0}^3\sin\left(\frac{Q\pi}{2}\right)}{\lambda_c}\left\{Q^2-4Q^3r_0+8Qr_0-\frac{4}{1-Q^2}\right\}+r_6$$

$$r_{32}=\frac{2K_0r_0\cos\left(\frac{Q\pi}{2}\right)}{\lambda_c}\left\{r_4+2Q^3{r_0}^2-4Q^4{r_0}^3+8Q^2{r_0}^3-2Q{r_0}^2-4{r_0}^3\right\}$$

$$r_{33}=\left\{-\frac{1}{2Q-1}-\frac{1}{2Q+1}+\frac{1}{Q}-\frac{Q}{Q^2-4}\right\},\quad r_{34}=Q^2r_0\left(\frac{Q^4}{2}+Q^2+\frac{1}{2}\right)$$

$$r_{35}=Q^4r_0\left(\frac{Q^2}{4Q^2-1}+\frac{1}{Q^2-4}\right)-Q^2r_7$$

where the r_i and S_i , $i=0,1,2,...,35$ are gotten from the values of η and φ and their partial derivatives or $\alpha(\tau)$ and $\beta(\tau)$ and their derivatives evaluated at the critical values of t and τ.
We shall now determine $\hat{T}_0$, $\hat{T}_{01}$, $\hat{T}_{10}$, $\hat{T}_{11}$ that shall be needed in obtaining the maximum displacement φ_c of φ.

From (54c), we have

$$t_c = \left(1+\omega'(0)\bar{\xi}^2\right)\hat{t}_c + ... \quad (63a)$$

Substituting in (63a) for t_c and $\hat{t}_c$ from (56a,b), we have

$$T_0 + \varepsilon T_{01} + \varepsilon^2 T_{02} + ... + \bar{\xi}\left(T_{10} + \varepsilon T_{11} + \varepsilon^2 T_{12} + ...\right) + \bar{\xi}^2\left(T_{20} + \varepsilon T_{21} + ...\right)$$
$$= \left(1+\omega'(0)\bar{\xi}^2\right)\left[\hat{T}_0 + \varepsilon\hat{T}_{01} + \varepsilon^2\hat{T}_{02} + ... + \bar{\xi}\left(\hat{T}_{10} + \varepsilon T_{11} + \varepsilon^2 T_{12} + ...\right) + ..\right.$$
.
$$\left. + \bar{\xi}^2\left(\hat{T}_{20} + \varepsilon T_{21} + ...\right)\right] \quad (63b)$$

Comparing coefficients of orders of ε and $\bar{\xi}$, we obtain

$$O(1): T_0 = \hat{T}_0,\ O(\varepsilon): T_{01} = \hat{T}_{01},\ O(\varepsilon.\bar{\xi}): T_{11} = \hat{T}_{11},\ O(\bar{\xi}.1): T_{10} = \hat{T}_{10} \quad (63c)$$

3.5 Maximum Displacement :

The maximum displacement for $\eta(t,\tau,\varepsilon,\bar{\xi})$ is obtained by evaluating (48b) at the critical values $t = t_a,\ \hat{t} = \hat{t}_a,\ \tau = \tau_a = \varepsilon\hat{t}_a$. Thereafter, we expand $\eta(t_a,\tau_a)$ in a Taylor series using (49a – c), similar to the expansion (51) to get

$$\xi_0(t_a) = \eta(t_a,\tau_a) = \bar{\xi}\Big[\ \eta^{10} + \{\varepsilon t_{01} + \varepsilon t_{01} + \varepsilon^2 t_{02} + ... + \bar{\xi}\left(t_{10} + \varepsilon t_{11} + \varepsilon^2 t_{12} + ...\right)$$
$$+ \bar{\xi}^2\left(t_{20} + \varepsilon t_{21} + \varepsilon^2 t_{22} + ...\ \right)\Big\}\eta^{10}_{,t} + \varepsilon\Big\{\ \hat{t}_0 + \varepsilon\hat{t}_{01} + ... + \bar{\xi}\left(\hat{t}_{10} + \varepsilon\hat{t}_{11} + ...\right)$$
$$+ \bar{\xi}^2\left(\hat{t}_{20} + \varepsilon\hat{t}_{21} + ...\ \right)\Big\}\eta^{10}_{,\tau} + \frac{1}{2}\Big\{\Big\{\Big[\ \varepsilon t_{01} + \varepsilon^2 t_{02} + ... + \bar{\xi}\left(t_{10} + \varepsilon t_{11} + ...\right)$$
$$+ \bar{\xi}^2\left(t_{20} + \varepsilon t_{21} + ...\ \right)\Big]^2 \eta^{10}_{,tt} + 2\varepsilon\Big\{\ \varepsilon t_{01} + ... + \bar{\xi}\left(t_{10} + \varepsilon t_{11} + ...\ \right)$$
$$+ \bar{\xi}^2\left(t_{20} + \varepsilon t_{21} + ...\ \right)\Big\}\Big\{\hat{t}_0 + \varepsilon\hat{t}_{01} + ... + \bar{\xi}\left(\hat{t}_{10} + \varepsilon\hat{t}_{11} + ...\ \right) + \bar{\xi}^2\left(\hat{t}_{20} + \varepsilon\hat{t}_{21} + ...\ \right)\Big\}\eta^{10}_{,t\tau}$$
$$+ \varepsilon^2\Big\{\hat{t}_0 + ... + \bar{\xi}\hat{t}_{10} + \bar{\xi}^2\hat{t}_{20}\ \Big\}^2 \eta^{10}_{,\tau\tau}\Big\}\Big\}\Big] + \bar{\xi}\varepsilon\Big[\eta^{11} + \Big\{\varepsilon t_{01} + ... + \bar{\xi}\left(t_{10} + \varepsilon t_{11} + ...\ \right)$$
$$+ \bar{\xi}^2\left(t_{20} + \varepsilon t_{21} + ...\ \right)\Big\}\eta^{11}_{,t} + \varepsilon\Big\{\hat{t}_0 + ... + \bar{\xi}\hat{t}_{10} + \bar{\xi}^2\hat{t}_{20} + ...\ \Big\}\eta^{11}_{,\tau}$$
$$+ \frac{1}{2}\Big\{\Big\{\Big[\varepsilon t_{01} + ... + \bar{\xi}t_{10} + \bar{\xi}\varepsilon t_{11} + ... + \bar{\xi}^2 t_{20} + \bar{\xi}^2\varepsilon t_{21} + ...\ \Big]\eta^{11}_{,tt}$$
$$+ 2\varepsilon\Big\{\ \varepsilon t_{01} + ... + \bar{\xi}t_{10} + \bar{\xi}\varepsilon t_{11} + ...$$
$$+ \bar{\xi}^2\left(t_{20} + \varepsilon t_{21} + ...\right)\ \Big\}\Big\{\hat{t}_0 + ... + \varepsilon\hat{t}_{01} + ... + \bar{\xi}\hat{t}_{10} + \bar{\xi}^2\hat{t}_{20} + ...\Big\}\eta^{11}_{,t\tau}\ \Big\}\Big\}\Big]$$

$$+\bar{\xi}\varepsilon^2\left[\eta^{12}+\left\{\bar{\xi}t_{10}+\bar{\xi}^2t_{20}+\ldots\right\}\eta^{12}_{,t}+\frac{\bar{\xi}^2t_{10}^2\eta^{12}_{,tt}}{2}\right]+\bar{\xi}^3\left[\eta^{30}+\left\{\varepsilon t_{01}+\varepsilon^2t_{02}+\ldots\right\}\eta^{30}_{,t}\right.$$

$$+\varepsilon\left\{\hat{t}_0+\varepsilon\hat{t}_{01}+\ldots\right\}\eta^{30}_{,\tau}+\frac{1}{2}\left\{\varepsilon t_{01}^2\eta^{30}_{,tt}+2\varepsilon^2t_{01}\hat{t}_0\eta^{30}_{,t\tau}+\varepsilon^2\hat{t}_0^2\right\}\eta^{30}_{,\tau\tau}\Big]$$

$$+\bar{\xi}^3\varepsilon\left[\eta^{31}+\varepsilon t_{01}\eta^{31}_{,t}+\varepsilon\hat{t}_0\eta^{31}_{,\tau}\right]+\bar{\xi}^3\varepsilon^2\eta^{32} \qquad (64)$$

On substituting in (64) and simplifying, we get (65a)

$$\eta_a=\bar{\xi}\left[I-\frac{\varepsilon\pi I}{2}+\varepsilon^2I\left(1-\frac{\pi^2}{4}\right)\right]+\bar{\xi}^3\left[Ir_{25}+\varepsilon I\left\{P_1-P_3-r_{17}+\frac{\pi r_{20}}{2}+r_{26}\right\}\right.$$

$$\left.+\varepsilon^2I\left\{\pi P_3+P_2-P_4-\frac{3\pi P_1}{4}+\frac{\pi r_{17}}{4}-r_{20}+\frac{r_{18}}{2}-\pi r_{19}-r_{27}+\frac{\pi r_{30}}{2}+r_{31}+\frac{r_{21}}{2}\right\}\right] \qquad (65a)$$

Thus, we have

$$\eta_{\mathrm{a}}=\bar{\xi}I\left(A_{10}+A_{11}\varepsilon+A_{12}\varepsilon^2+\ldots\right)+\bar{\xi}^3\left(A_{30}+A_{31}\varepsilon+A_{32}\varepsilon^2+\ldots\right) \qquad (65b)$$

where

$$A_{10}=1,\ A_{11}=\frac{-\pi}{2},\ A_{12}=1-\frac{\pi^2}{4},\ A_{30}=r_{25},\ A_{31}=P_1-P_2-r_{17}+\frac{\pi r_{20}}{2}+r_{26}$$

$$A_{32}=\pi P_3+P_2-P_4-\frac{3\pi P_1}{4}+\frac{\pi r_{17}}{4}-r_{20}+\frac{r_{18}}{2}-\pi r_{19}-r_{27}+\frac{\pi r_{30}}{2}+r_{31}+\frac{r_{21}}{2}$$

Also the maximum displacement for $\varphi(t,\tau,\varepsilon,\bar{\xi})$ is obtained by evaluating (48c) at the critical values $t=t_c,\ \hat{t}=\hat{t}_c,\ \tau=\tau_c=\varepsilon\hat{t}_c$. Thereafter, we expand $\varphi(t_c,\tau_c)$ in a Taylor series using (49a _ c), similar to expansion (51) to get

$$\xi_1(t_c)=\varphi(t_c,\tau_c,\bar{\xi},\varepsilon)=\bar{\xi}^2\left[\varphi_{20}+\left\{\varepsilon T_{01}+\varepsilon^2T_{02}+\ldots+\bar{\xi}\left(T_{10}+\varepsilon T_{11}+\varepsilon^2T_{12}+\ldots\right)\right\}\varphi^{20}_{,t}\right.$$

$$+\varepsilon\left\{\hat{T}_0+\varepsilon\hat{T}_0+\ldots+\bar{\xi}\left(\hat{T}_{10}+\varepsilon\hat{T}_{11}+\ldots\right)\right\}\varphi^{20}_{,\tau}+\frac{1}{2}\left\{\left\{\left[\varepsilon T_{01}+\ldots+\bar{\xi}\left(T_{10}+\varepsilon T_{11}+\ldots\right)\right]^2\varphi^{20}_{,tt}\right.\right.$$

$$2\varepsilon\left\{\varepsilon T_{01}+\ldots\bar{\xi}\left(T_{10}+\varepsilon T_{11}+\ldots\right)\right\}\left\{\hat{T}_0+\varepsilon\hat{T}_{01}+\ldots+\bar{\xi}\left(\hat{T}_{10}+\varepsilon\hat{T}_{11}+\ldots\right)\right\}\varphi^{20}_{,t\tau}$$

$$\left.\left.\left.+\varepsilon^2\left\{\hat{T}_0+\bar{\xi}\hat{T}_{10}\right\}^2\varphi^{20}_{,\tau\tau}\right\}\right\}\right]+\bar{\xi}^2\varepsilon\left[\varphi^{21}+\left\{\varepsilon T_{01}+\ldots+\bar{\xi}\left(T_{10}+\varepsilon T_{11}\right)\right\}\varphi^{21}_{,t}\right.$$

$$\left.+\varepsilon^2\left\{\hat{T}_0+\bar{\xi}\hat{T}_{10}\right\}\varphi^{21}_{,\tau}+\frac{1}{2}\left\{\left\{\left[\varepsilon T_{01}+\bar{\xi}\left(T_{10}+\varepsilon T_{11}+\ldots\right)\right]^2\varphi^{21}_{,tt}+2\varepsilon\bar{\xi}T_{10}\hat{T}_0{}^2\varphi^{21}_{,t\tau}\right\}\right\}\right]$$

$$+\bar{\xi}^2\varepsilon^2\left[\varphi^{22}+\bar{\xi}T_{10}\varphi^{22}_{,t}\right]+\bar{\xi}^3\left[\varphi^3+\left\{\varepsilon T_{01}+\varepsilon^2 T_{02}\right\}\varphi^{30}_{,t}+\varepsilon\left\{\hat{T}_0+\varepsilon\hat{T}_{01}\right\}\varphi^{30}_{,\tau}\right.$$
$$\left.+\frac{1}{2}\left\{\left\{\varepsilon^2 T_{01}^2\varphi^{30}_{,tt}+2\varepsilon T_{01}\hat{T}_0\varphi^{30}_{,t\tau}+\varepsilon^2\hat{T}_0^2\varphi^{30}_{,\tau\tau}\right\}\right\}\right]$$
$$+\bar{\xi}^3\varepsilon\left[\varphi^{31}+\varepsilon T_{01}\varphi^{31}_{,t}+\varepsilon\hat{T}_0\varphi^{31}_{,\tau}\right]+\bar{\xi}^3\varepsilon^2\varphi^{32} \qquad (65c)$$

On substituting the relevant values in (65c) and simplifying, we get

$$\varphi_c=\bar{\xi}^2 I\left(B_{20}+\varepsilon B_{21}+\varepsilon^2 B_{22}+\ldots\right)+\bar{\xi}^3 I^2\left(B_{30}+\varepsilon B_{31}+\varepsilon^2 B_{32}+\ldots\right) \qquad (65d)$$

where

$$B_{20}=S_{28},\ B_{21}=T_0S_{22}+S_{29},$$
$$B_{22}=T_{01}S_{22}+\frac{T_{01}^2S_1}{2}+T_{01}T_0S_2+\frac{T_0^2S_{23}}{2}+T_{01}S_6+T_0S_{24}+S_{30}$$
$$B_{30}=S_{31},\ B_{31}=X_{10}S_{22}+T_{01}X_{10}S_1+T_0X_{10}S_2+X_{10}S_6+T_{01}S_{13}+T_0S_{25}+T_0T_{01}S_{15}+S_{32}$$
$$B_{32}=X_{11}S_{22}+T_{01}X_{11}S_1+T_0X_{11}S_2+2T_{01}X_{10}S_2+T_0X_{10}S_{23}+X_{11}S_6+X_{10}S_{24}+T_{01}X_{10}S_7$$
$$+T_0X_{10}S_8+X_{10}S_{11}+T_{02}S_{13}+T_{01}S_{25}+\frac{T_{01}^2S_{14}}{2}+\frac{T_0^2S_{26}}{2}+T_{01}S_{19}+T_0S_{27}+S_{33}$$

$$T_{10}=IX_{10}\ \text{and}\ T_{11}=IX_{11} \qquad (66b)$$

$$X_{10}=\frac{S_{13}}{S_1},\ X_{11}=\frac{1}{S_2}\left[2X_{10}S_1+T_{01}X_{10}S_4+T_0X_{10}S_5+X_{10}S_8+S_{13}+T_0S_{14}+S_{16}+S_{15}\right]$$

Thus

$$\varphi_c=I\bar{\xi}^2\left[\varepsilon\left\{S_2T_{01}+S_1T_0+S_3+S_2\right\}+\varepsilon^2\left\{\begin{array}{l}S_2T_{02}+S_1T_{01}+\dfrac{S_4T_{01}{}^2}{2}+S_5T_{01}T_0+\dfrac{S_6T_{01}{}^2}{2}\\+S_8T_{01}+S_9T_0+S_{10}+S_7T_{01}+S_5T_0+S_{11}\end{array}\right\}\right]$$
$$+I^2\bar{\xi}^3\left[S_2X_{10}+S_{12}+\varepsilon\left\{S_2X_{11}+S_1X_{10}+S_4X_{10}T_{01}+S_5X_{10}T_0+S_8X_{10}\right.\right.$$
$$\left.+S_{13}T_{01}+S_{14}T_0+S_{16}+S_7X_{10}+S_{15}\right\}$$
$$+\varepsilon^2\left\{S_1X_{11}+S_5X_{11}T_0+2S_5X_{10}T_{01}+S_6X_{10}T_0+S_8X_{11}+S_9X_{10}+S_{17}X_{10}T_{01}\right.$$
$$+S_{27}X_{10}T_0+S_{18}X_{10}+S_{13}T_{02}+S_{14}T_{01}+\frac{S_{19}T_{01}{}^2}{2}+S_{20}T_0T_{01}+\frac{S_{22}T_{01}{}^2}{2}$$
$$+S_{23}T_{01}+S_{24}T_0+S_{26}+S_7X_{11}+S_5X_{10}+S_5X_{10}T_{01}+S_6X_{10}T_0+S_9X_{10}$$
$$\left.\left.+S_{14}T_{01}+S_{21}T_0+S_{25}\right\}\right] \qquad (66c)$$

With further simplification, we get

$$\varphi_c = I\bar{\xi}^2\left(B_{21}\varepsilon + B_{22}\varepsilon^2 + \ldots\right) + I^2\bar{\xi}^3\left(B_{30} + B_{31}\varepsilon + B_{32}\varepsilon^2 + \ldots\right) \tag{66d}$$

where

$$B_{21} = S_2T_{01} + S_1T_0 + S_3 + S_2 \tag{66e}$$

$$B_{22} = S_2T_{02} + S_1T_{01} + \frac{S_4T_{01}^{\ 2}}{2} + S_5T_{01}T_0 + \frac{S_6T_{01}^{\ 2}}{2} + S_8T_{01} + S_9T_0 + S_{10} + S_7T_{01} + S_5T_0 + S_{11} \tag{66f}$$

$$B_{30} = S_2X_{10} + S_{12} \tag{66g}$$

$$B_{31} = S_2X_{11} + S_1X_{10} + S_4X_{10}T_{01} + S_5X_{10}T_0 + S_8X_{10} + S_{13}T_{01} + S_{14}T_0 + S_{16} + S_7X_{10} + S_{15} \tag{66h}$$

$$B_{32} = S_1X_{11} + S_5X_{11}T_0 + 2S_5X_{10}T_{01} + S_6X_{10}T_0 + S_8X_{11} + S_9X_{10} + S_{17}X_{10}T_{01} + S_{27}X_{10}T_0 + S_{18}X_{10} + S_{13}T_{02} + S_{14}T_{01} + \frac{S_{19}T_{01}^{\ 2}}{2} + S_{20}T_{01}T_0 + \frac{S_{22}T_{01}^{\ 2}}{2} + S_{23}T_{01} + S_{24}T_0 + S_{26} + S_7X_{11} + S_5X_{10} + S_5X_{10}T_{01} + S_6X_{10}T_0 \quad S_9X_{10} + S_{14}T_{01} + S_{21}T_0 + S_{25} \tag{66i}$$

Net maximum displacement M_a is

$$M_a = \quad \eta_a + \varphi_c \tag{66j}$$

Thus

$$M_a = \bar{\xi}I\left(A_{10} + A_{11}\varepsilon + A_{12}\varepsilon^2 + \ldots\right) + \bar{\xi}^3I\left(A_{30} + A_{31}\varepsilon + A_{32}\varepsilon^2 + \ldots\right) + \bar{\xi}^2I\left(B_{20} + B_{21}\varepsilon + B_{22}\varepsilon^2 + \ldots\right) + \bar{\xi}^3I^2\left(B_{30} + B_{31}\varepsilon + B_{32}\varepsilon^2 + \ldots\right) \tag{67a}$$

Let

$$C_1 = I\left(A_{10} + A_{11}\varepsilon + A_{12}\varepsilon^2 + \ldots\right) \equiv IQ_1 \tag{67b}$$

$$C_2 = I\left(B_{20} + B_{21}\varepsilon + B_{22}\varepsilon^2 + \ldots\right) \equiv IQ_2 \tag{67c}$$

$$C_3 = I\left(A_{30} + A_{31}\varepsilon + A_{32}\varepsilon^2 + \ldots\right) + I^2\left(B_{30} + B_{31}\varepsilon + B_{32}\varepsilon^2 + \ldots\right)$$

$$\Rightarrow \quad C_3 = \left(IA_{30} + I^2B_{30}\right) + \varepsilon\left(IA_{31} + I^2B_{31}\right) + \varepsilon^2\left(IA_{32} + I^2B_{32}\right)$$

$$= \left(IA_{30} + I^2B_{30}\right)\left[1 + \frac{1}{IA_{30} + I^2B_{30}}\left\{\varepsilon\left(IA_{31} + I^2B_{31}\right) + \varepsilon^2\left(IA_{32} + I^2B_{32}\right)\right\}\right] \tag{67d}$$

$$\equiv \left(IA_{30} + I^2B_{30}\right)Q_3(I) \tag{67e}$$

where

$$Q_1 = A_{10} + A_{11}\varepsilon + A_{12}\varepsilon^2 + \ldots \quad , \quad Q_2 = B_{20} + B_{21}\varepsilon + B_{22}\varepsilon^2 + \ldots$$

$$Q_3(I) = 1 + \frac{1}{IA_{30} + I^2B_{30}}\left\{\varepsilon\left(IA_{31} + I^2B_{31}\right) + \varepsilon^2\left(IA_{32} + I^2B_{32}\right)\right\}$$

Hence

$$M_a = C_1\bar{\xi} + C_2\bar{\xi}^2 + C_3\bar{\xi}^3 + \ldots \tag{68}$$

By reversal of series Amazigo [17], we have

$$\bar{\xi} = M_a e_1 + M_a{}^2 e_2 + M_a{}^3 e_3 + ... \tag{69}$$

Substituting for M_a from (68) into (69), we have

$$\bar{\xi} = e_1\left(C_1\bar{\xi} + C_2\bar{\xi}^2 + C_3\bar{\xi}^3\right) + e_2\left(C_1\bar{\xi} + C_2\bar{\xi}^2 + C_3\bar{\xi}^3\right)^2 + e_3\left(C_1\bar{\xi} + C_2\bar{\xi}^2 + C_3\bar{\xi}^3\right)^3 \tag{70a}$$

Equating coefficients of $\bar{\xi}, \bar{\xi}^2$ and $\bar{\xi}^3$ in (70a), we have

$$e_1 = \frac{1}{C_1},\ e_2 = \frac{-C_2}{C_1^{\ 3}},\ \ e_3 = \frac{2C_2^{\ 2} - C_1C_3}{C_1^{\ 5}} \tag{70b}$$

4 Dynamic Buckling Impulse, I_D

The condittion for dynamic buckling is

$$\frac{dI}{dM_a}\left(I_D\right) = 0 \tag{71}$$

Thus, from (69), we have

$$e_1 + 2M_{aD}e_2 + 3M_{aD}{}^2 e_3 = 0 \tag{72a}$$

where

$$M_{aD} = M_a\left(I_D\right) \tag{72b}$$

From (72a), we have

$$M_{aD} = \frac{1}{3e_3}\left\{-e_2 \pm \left(e_2{}^2 - 3e_1e_3\right)^{\frac{1}{2}}\right\} \tag{73}$$

Now, with a view to simplifying M_{aD} in (73), we have

$$e_2{}^2 - 3e_1e_3 = \frac{3C_1C_3 - 5C_2{}^2}{C_1{}^6}$$

$$= \frac{3C_1C_3}{C_1{}^6}\left[1 - \frac{5C_2{}^2}{3C_1C_3}\right] = \frac{3C_3}{C_1{}^5}\left[1 - \frac{5C_2{}^2}{3C_1C_3}\right] \tag{74a}$$

Thus, after substituting and simplifying terms in (74a), we get

$$e_2{}^2 - 3e_1e_3 = \frac{3\left(IA_{30} + I^2B_{30}\right)Q_3}{I^5Q_1{}^5}\left[1 - \frac{5I^2Q_2{}^2}{3Q_3IQ_1\left(IA_{30} + I^2B_{30}\right)}\right]$$

$$= \frac{3Q_3\left(A_{30} + IB_{30}\right)}{I^4Q_1{}^5}\left[1 - \frac{5Q_2{}^2}{3Q_3Q_1\left(A_{30} + IB_{30}\right)}\right] = \frac{3Q_3\left(A_{30} + IB_{30}\right)}{I^4Q_1{}^5}K_2$$

$$K_2 = \left[1 - \frac{5Q_2{}^2}{3Q_3Q_1\left(A_{30} + IB_{30}\right)}\right]$$

$$\therefore\ e_2{}^2 - 3e_1e_3 = \frac{3Q_3\left(A_{30} + IB_{30}\right)}{I^4Q_1{}^5}K_2 \tag{74b}$$

$$e_2 = \frac{-C_2}{C_1^{\ 3}} = \frac{-IQ_2}{I^3 Q_1^{\ 3}} = \frac{-Q_2}{I^2 Q_1^{\ 3}} \tag{74c}$$

$$-e_2 \pm \left(e_2^{\ 2} - 3e_1e_3\right)^{\frac{1}{2}} = \left(e_2^{\ 2} - 3e_1e_3\right)^{\frac{1}{2}} \left[\frac{\pm e_2}{\left(e_2^{\ 2} - 3e_1e_3\right)^{\frac{1}{2}}} - 1 \right] \tag{75a}$$

$$= \frac{\sqrt{3Q_3\left(A_{30} + IB_{30}\right)K_1}}{I^2 Q_1^{\frac{5}{2}}} \left[\frac{\pm Q_2}{I^2 Q_1^{\ 3}} \left(\frac{I^4 Q_1^{\ 5}}{3Q_3\left(A_{30} + IB_{30}\right)K_1} \right)^{\frac{1}{2}} - 1 \right] \tag{75b}$$

$$= \frac{\sqrt{3}}{I^2} \left\{ \frac{Q_3\left(A_{30} + IB_{30}\right)K_1}{Q_1^{\ 5}} \right\}^{\frac{1}{2}} \left[\frac{\pm Q_2}{I^2 Q_1^{\ 3}} \left(\frac{I^4 Q_1^{\ 5}}{3Q_3\left(A_{30} + IB_{30}\right)K_1} \right)^{\frac{1}{2}} - 1 \right] \tag{75c}$$

Taking the negative part and let

$$-e_2 - \left(e_2^{\ 2} - 3e_1e_3\right)^{\frac{1}{2}} = \frac{-\sqrt{3}}{I^2} \left\{ \frac{Q_3\left(A_{30} + IB_{30}\right)K_1}{Q_1^{\ 5}} \right\}^{\frac{1}{2}} K_2 \tag{75d}$$

$$K_2 = \left[\frac{Q_2}{I^2 Q_1^{\ 3}} \left(\frac{I^4 Q_1^{\ 5}}{3Q_3\left(A_{30} + IB_{30}\right)K_1} \right)^{\frac{1}{2}} - 1 \right] \tag{75e}$$

Now

$$e_3 = \frac{2C_2^{\ 2} - C_1C_3}{C_1^{\ 5}} = \frac{-C_1C_3}{C_1^{\ 5}} \left[1 - \frac{2C_2^{\ 2}}{C_1C_3} \right]$$

$$= \frac{-C_3}{C_1^{\ 4}} \left[1 - \frac{2C_2^{\ 2}}{C_1C_3} \right] \tag{76a}$$

$$= \frac{-\left(IA_{30} + I^2 B_{30}\right)Q_3}{I^4 Q_1^{\ 4}} \left[1 - \frac{2I^2 Q_2^{\ 2}}{IQ_1\left(IA_{30} + I^2 B_{30}\right)Q_3} \right] \tag{76b}$$

$$= \frac{-\left(A_{30} + IB_{30}\right)Q_3}{I^3 Q_1^{\ 4}} \left[1 - \frac{2Q_2^{\ 2}}{Q_1Q_3\left(A_{30} + IB_{30}\right)} \right] \tag{76c}$$

Let

$$K_3 = \left[1 - \frac{2Q_2^{\ 2}}{Q_1Q_3\left(A_{30} + IB_{30}\right)} \right] \tag{76d}$$

$$e_3 = \frac{-(A_{30} + IB_{30})Q_3}{I^3 Q_1^{\,4}} K_3 \tag{76e}$$

Substituting (76e) in (73), we have

$$M_{aD} = \frac{I^3 Q_1^{\,4}}{3(A_{30} + IB_{30})Q_3 K_3}\left[\frac{-\sqrt{3}}{I^2}\left\{\frac{Q_3(A_{30} + IB_{30})K_1}{Q_1^{\,5}}\right\}^{\frac{1}{2}} K_2\right] \tag{77a}$$

$$\Rightarrow M_{aD} = I_D Q_1^{3/2}\left(\frac{K_2}{K_3}\right)\frac{1}{\sqrt{3}}\frac{\sqrt{K_1}}{\sqrt{(A_{30} + IB_{30})Q_3}} \tag{77b}$$

Evaluating (69) at buckling, we get

$$\bar{\xi} = M_{aD}e_1 + M_{aD}^{\;2}e_2 + M_{aD}^{\;3}e_3 + \ldots \tag{77c}$$

Where the e_1, e_2 and e_3 in (77c) are now evaluated at buckling. This implies

$$\bar{\xi} = M_{aD}\left[e_1 + M_{aD}e_2 + M_{aD}^{\;2}e_3\right] \tag{77d}$$

Multiplying (77d) by 3, we get

$$3\bar{\xi} = M_{aD}\left[3e_1 + 3M_{aD}e_2 + 3M_{aD}^{\;2}e_3\right] \tag{77e}$$

But from (72a), we get

$$3M_{aD}^{\;2} = -e_1 - 2M_{aD}e_2 \tag{77f}$$

Substituting (77f) in (77e), we get

$$3\bar{\xi} = M_{aD}\left[3e_1 + 3M_{aD}e_2 - e_1 - 2M_{aD}e_2\right]$$

$$3\bar{\xi} = M_{aD}\left[2e_1 + M_{aD}e_2\right] \tag{77g}$$

$$3\bar{\xi} = M_{aD}\left[\frac{2}{C_1} - \frac{C_2 M_{aD}}{C_1^{\,3}}\right] = \frac{2M_{aD}}{C_1}\left[1 - \frac{C_2 M_{aD}}{2C_1^{\,2}}\right] \tag{77h}$$

We now substitute in (77b) for M_{aD}, C_1 and C_2

$$3\bar{\xi} = \frac{2I_D Q_1^{3/2}\left(\frac{K_2}{K_3}\right)\frac{1}{\sqrt{3}}\frac{\sqrt{K_1}}{\sqrt{(A_{30} + I_D B_{30})Q_3}}}{I_D Q_1}\left[1 - \frac{I_D Q_2 I_D Q_1^{3/2}\left(\frac{K_2}{K_3}\right)\frac{1}{\sqrt{3}}\frac{\sqrt{K_1}}{\sqrt{(A_{30} + I_D B_{30})Q_3}}}{2I_D^{\,2}Q_1^{\,2}}\right]$$

$$3\bar{\xi} = 2Q_1^{1/2}\left(\frac{\mathrm{K}_2}{\mathrm{K}_3}\right)\frac{1}{\sqrt{3}}\frac{\sqrt{\mathrm{K}_1}}{\sqrt{(A_{30} + I_D B_{30})Q_3}}\left[1 - \frac{Q_2 Q_1^{-1/2}\left(\frac{K_2}{K_3}\right)\frac{1}{\sqrt{3}}\frac{\sqrt{K_1}}{(A_{30} + I_D B_{30})Q_3}}{2}\right]$$

$$3\bar{\xi} = \frac{2Q_1^{1/2}\left(\frac{K_2}{K_3}\right)}{\sqrt{3}} \left\{\frac{K_1}{(A_{30}+I_D B_{30})Q_3}\right\}^{\frac{1}{2}} \left[1 - \frac{\frac{Q_2 Q_1^{-1/2}}{\sqrt{3}}\left(\frac{K_2}{K_3}\right)\left\{\frac{K_1}{(A_{30}+I_D B_{30})Q_3}\right\}^{\frac{1}{2}}}{2}\right] \qquad (77i)$$

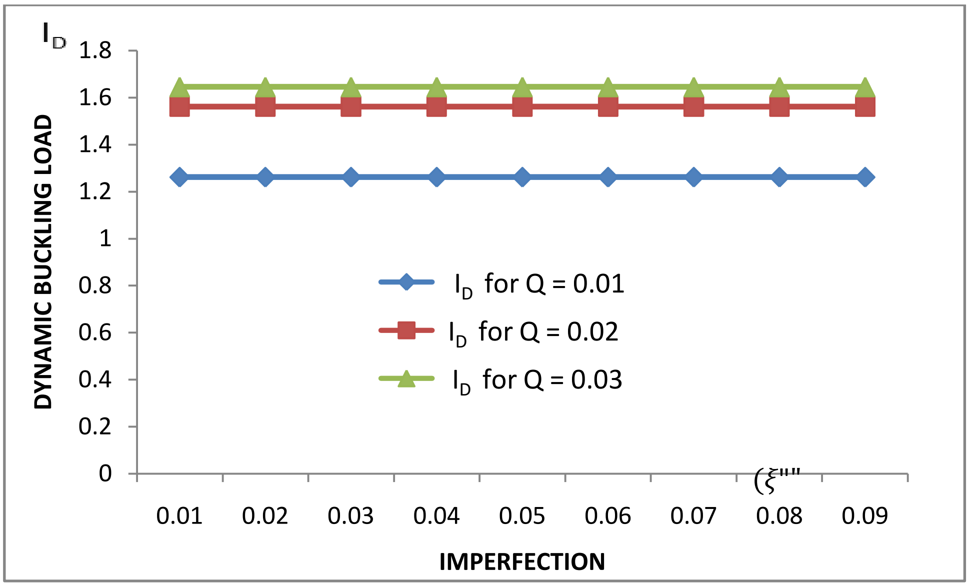

Fig. 1: Graph of Dynamic buckling impulse load I_D against Damping coefficient ε.

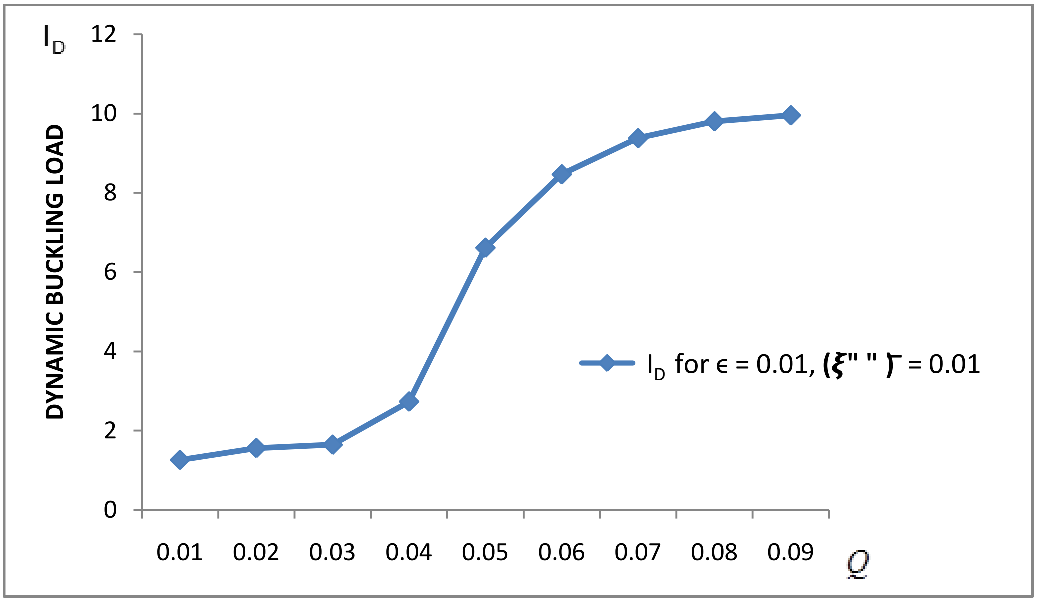

Fig. 2: Graph of Dynamic buckling impulse load I_D against the ratio of the two modes Q.

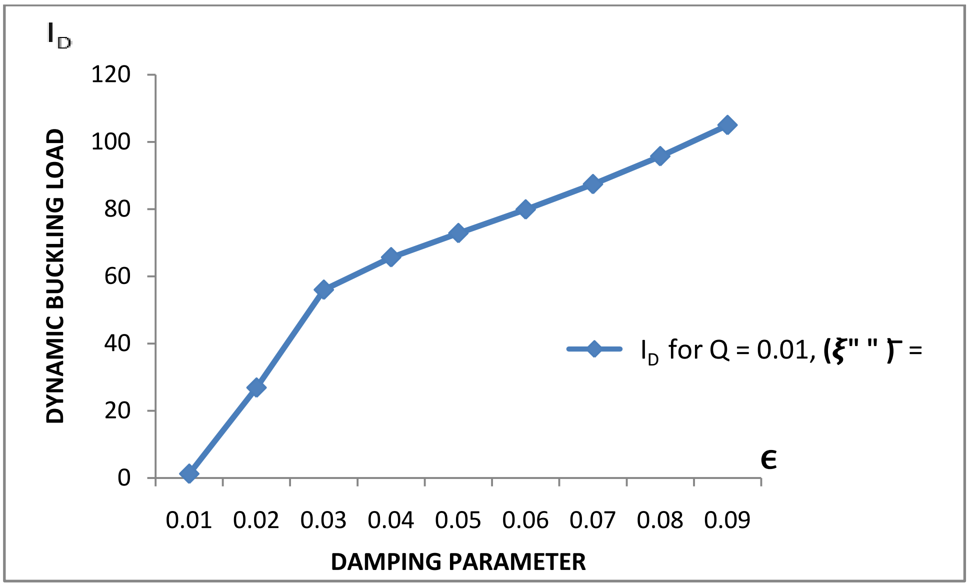

Fig. 3 : Graph of Dynamic buckling impulse load I_D against Damping coefficient ε

Table 1

$\bar{\xi}$	I_D for $Q = 0.01$	I_D for $Q = 0.02$	I_D for $Q = 0.03$
0.01	1.26098	1.56166	1.64484
0.02	1.26098	1.56166	1.64484
0.03	1.26098	1.56166	1.64484
0.04	1.26098	1.56166	1.64484
0.05	1.26098	1.56166	1.64484
0.06	1.26098	1.56166	1.64484
0.07	1.26098	1.56166	1.64484
0.08	1.26098	1.56166	1.64484
0.09	1.26098	1.56166	1.64484

Table 2

ε	I_D
0.01	1.26098
0.02	26.8773
0.03	55.9803
0.04	65.6337
0.05	72.8184
0.06	79.8541
0.07	87.3791
0.08	95.6784
0.09	104.951

Table 3

Q	I_D
0.01	1.26098
0.02	1.56166
0.03	1.64484
0.04	2.73426
0.05	6.61334
0.06	8.46683
0.07	9.37906
0.08	9.80319
0.09	9.95558

Table 3:

5. Discussion:

Equation (77j) gives an implicit formula for determining the dynamic impulse I_D. The result is asymptotic and is valid as the small parameters $\bar{\xi}$ and ε become increasing small relative to unity. Guided by the fact that the impulse is of the order of the small parameter $\bar{\xi}$, we don't expect much variation in the value of the impulse buckling load. However, it is important to notice the effect of the viscous light damping on the impulse buckling load as shown in table 2 and figure 3. A little change in the damping gives a remarkable change in the buckling load.It is essential that the two parameters $\bar{\xi}$ and ε be not mathematically related otherwise the problem becomes a one – small parameter analysis which is not what we intended originally.

6. Conclusion

We have successfully carried out a two – small parameter regular perturbation analysis of some coupled non homogeneous viscously damped nonlinear system in a dynamical setting and obtained an asymptotic result. This result which is strictly asymptotic, is also made possibly by employing multi – timing (two – timing) perturbation procedures. Perhaps, the salient novelty in the analysis is the inclusion of light viscous damping which coefficient ε serves as one of the perturbation parameters. It is our sincere desire that this procedure be applied, perhaps with some slight modifications, to real – life engineering structures, including columns, plates and even shells.

5. References:

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