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FRACTIONAL CHEBYSHEV OPERATIONAL MATRIX FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

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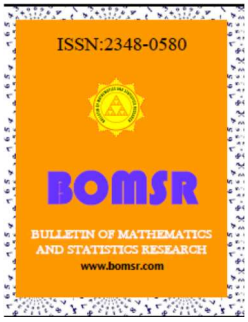

ABSTRACT
In this paper, we constructed the operational matrix of Shifted Fractional Chebyshev functions of the first kind and the second kind, applied to the problem for obtaining the numerical solution of the linear fractional differential equation of order $0<\nu<1$. The main advantage of the operational matrix is that the fractional derivatives of shifted fractional Chebyshev functions of the first kind and second kind can be indicated easily in terms of the same functions. By using fractional Chebyshev operational matrix, we can reduce the linear fractional differential equation to the solving of a system of linear algebraic equations. Through examples presented are shown the applicability of the presented method.
Keywords: linear fractional differential equation, shifted fractional Chebyshev functions, Caputo-type fractional derivative

1. INTRODUCTION

Fractional calculus has been collected the attention of many researchers in several areas of physics [1], control engineering [2], signal processing [3], electromagnetism [4] etc. Fractional differential equations(FDEs) modelled the real world usually do not have explicit solutions. So the finding numerical solution of FDEs is the important task. Recently the scientists have an interestin the theory of the existence and uniqueness of FDE [5,6]. The several methods for obtaining the numerical solution of FDEs is such as predictor-corrector method [7], Adomian's decomposition method [8], variational iteration method [9] and homotopyanalysis methods [10]. The operational matrix method based on orthogonal polynomials is one of the tools for obtaining the numerical

solution of high accuracy of FDEs [11-14]. The main advantage of operational matrix method reduces the problem for solving FDEs to those for solving a system of algebraic equations. For obtaining the numerical solutions of FDEs with initial conditions, A.H. Bhrawy et al. [15] proposed shifted fractional-order Jacobi orthogonal functions and S. Kazem et al. [16] fractional-order Legendre functions. Also A.H. Bhrawyet et al. [17] and E.H. Doha et al. [18] advanced shifted Chebyshev operational matrix.

In this paper, from the viewpoint of accuracy, we applied shifted fractional-order Chebyshev functions to FDEs with an initial value. We made theorems about error bound of approximative function based on shifted fractional order Chebyshev functions and improved the accuracy of the numerical solution of proposed problems.

This paper is constructed as follows. In Section 2, we describe several definitions and properties of fractional calculus. The properties of shifted fractional-order of Chebyshev functions of the first and second kind is introduced in Section 3. Section 4 is provided with the bounded error of approximative function based on shifted fractional Chebyshev operational matrix of the first kind and second kind. The application of fractional Chebyshev operational matrix of the first kind and the second kind is introduced in Section 5. The several examples used method presented is given in Sections 6. The conclusion is described in the last Section.

2. PRELIMINARIES

Definition 2.1 [19] The Riemann-Liouville fractional integral operator of order $\nu(\nu \geq 0)$ is defined as

$$(I^{\nu} f)(x) = \frac{1}{\Gamma(\nu)} \int_0^x \frac{f(t)}{(x-t)^{1-\nu}} dt, \nu > 0, x > 0, \ (I^0 f)(x) = f(x).$$

Definition 2.2 [19] The Caputo fractional derivatives of order ν is defined as

$${}^c D^{\nu} f(x) = I^{m-\nu} D^m f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} f^m(t) dt, m-1 < \nu \leq m, x > 0,$$

where D^m is the classical differential operator of order m.

For the Caputo derivative, we have:

$${}^c D^{\nu} C = 0 \ (C \text{ is a constant}),$$

$${}^c D^{\nu} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \nu \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)} x^{\beta-\nu}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq \lceil \nu \rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor \nu \rfloor. \end{cases} \tag{1}$$

The ceiling function $\lceil \nu \rceil$ denotes the smallest integer greater than or equal to ν and the floor function $\lfloor \nu \rfloor$ denotes the largest integer less than or equal to ν. Also $\mathbb{N}_0 = \{1, 2, \cdots\}$ and $\mathbb{N} = \{0, 1, 2, \cdots\}$.

Theorem 1. [19] Similar to the differential operator of integer order, the Caputo's fractional differentiation is a linear operator, i.e.

$${}^c D^{\nu} (\lambda f(x) + \mu g(x)) = \lambda {}^c D^{\nu} f(x) + \mu {}^c D^{\nu} g(x), \tag{2}$$

where λ and μ are constants.

3. SHIFTED FRACTIONAL CHEBYSHEV FUNCTIONS OF FIRST KIND AND SECOND KIND

Shifted fractional Chebyshev functions of the first kind and the second kind are special cases of shifted fractional Jacobi functions. They have been used widely in mathematical analysis and practical applications. When $y(x)$ is differential, the advantages of using fractional Chebyshev functions is the good representation of a smooth function by finite Chebyshev expansion [20]. In this Section, we will introduce shifted fractional Chebyshev functions of the first kind and second kind in order to complete our aim. In this paper, we consider the case of $0<\lambda<1$. If $\lambda=1$, shifted fractional Chebyshev functions is equivalent to shifted Chebyshev polynomials.

3.1 Shifted fractional Chebyshev functions of the first kind $T_n^{(\lambda)}(x)$

Shifted fractional Chebyshev functions of the first kind $T_n^{(\lambda)}(x)$ are the eigenfunctions of the Sturm-Liouville differential equation

$$(1-x^{\lambda})x^{-\lambda+2}v''(x)+x(-\frac{\lambda}{2}+(1-\frac{\lambda}{2})(1-x^{\lambda})x^{-\lambda})v'(x)+\lambda^2n^2v(x)=0,\ x\in I=[0,1].$$

They satisfy the following recurrence formula on $[0,1]$

$$T_{n+1}^{(\lambda)}(x)=2(2x^{\lambda}-1)T_n^{(\lambda)}(x)-T_{n-1}^{(\lambda)}(x).$$

Then the analytic expression of shifted fractional Chebyshev functions of the first kind $T_n^{(\lambda)}(x)$ is as follow

$$T_n^{(\lambda)}(x)=\sum_{k=0}^{n}E_{1,k}^{(n)}x^{k\lambda}=\sum_{k=0}^{n}\frac{n(-1)^{n-k}(n+k-1)!2^{2k}}{(n-k)!(2k)!}x^{k\lambda},\ n=0,1,2,\cdots,\ x\in I,\ 0<\lambda<1.$$

Clearly we have

$$T_n^{(\lambda)}(0)=(-1)^n,\ T_n^{(\lambda)}(1)=1. \tag{3}$$

Let $w_1^{(\lambda)}(x)=\lambda x^{\frac{\lambda}{2}-1}(1-x^{\lambda})^{-\frac{1}{2}}$. The shifted fractional Chebyshev functions of the first kind form a complete $L^2_{w_1^{(\lambda)}}(I)$-orthogonal system, i.e.,

$$\int_I T_i^{(\lambda)}(x)T_j^{(\lambda)}(x)w_1^{(\lambda)}(x)dx=h_{1,j}\delta_{ij}, \tag{4}$$

where δ_{ij} is the Kronecker function and

$$h_{1,j}=\begin{cases}\pi, & j=0,\\ \frac{\pi}{2}, & j=1,2,\cdots.\end{cases} \tag{5}$$

Let $\mathcal{F}_{1,N}=span\{T_n^{(\lambda)}:0\le n\le N\}$ be the finite-dimensional fractional function space. By the orthogonality (4), we can expand any u belonging to $\mathcal{F}_{1,N}$ as

$$u(x)=\sum_{j=0}^{\infty}u_jT_j^{(\lambda)}(x),\ u_j=(h_{1,j})^{-1}\int_I u(x)T_j^{(\lambda)}(x)w_1^{(\lambda)}(x)dx. \tag{6}$$

3. 2. Shifted fractional Chebyshev functions of the second kind

Shifted fractional Chebyshev functions of the second kind $U_n^{(\lambda)}(x)$ are similar to shifted fractional Chebyshev functions of the first kind $T_n^{(\lambda)}(x)$ and are the eigenfunctions of the Sturm-Liouville differential equation as following

$$(1-x^{\lambda})x^{-\lambda+2}v''(x)+x(-\frac{3}{2}\lambda+(1+\frac{\lambda}{2})(1-x^{\lambda})x^{-\lambda})v'(x)+\lambda^2n(n+2)v(x)=0,\ x\in I,$$

where $I=[0,1]$.

Also the recurrence relations that $U_n^{(\lambda)}(x)$ satisfy on $[0,1]$ is given by

$$U_{n+1}^{(\lambda)}(x)=2(2x^{\lambda}-1)U_n^{(\lambda)}(x)-U_{n-1}^{(\lambda)}(x).$$

Moreover analytic expression of shifted fractional Chebyshev functions of the second kind $U_n^{(\lambda)}(x)$ can be expressed

$$U_n^{(\lambda)}(x)=\sum_{k=0}^{n}E_{2,k}^{(n)}x^{k\lambda}=\sum_{k=0}^{n}\frac{(-1)^{n-k}(n+k+1)!2^{2k}}{(n-k)!(2k+1)!}x^{k\lambda},\ n=0,1,2,\cdots,\ x\in I. \tag{7}$$

Then we can know

$$U_n^{(\lambda)}(0)=(-1)^n(n+1),\ U_n^{(\lambda)}(1)=(n+1).$$

Now let $w_2^{(\lambda)}(x)=\lambda x^{\frac{3}{2}\lambda-1}(1-x^{\lambda})^{\frac{1}{2}}$. The shifted fractional Chebyshev functions of second kind form a complete $L^2_{w_2^{(\lambda)}}(I)$-orthogonal system, i.e.,

$$\int_I U_i^{(\lambda)}(x)U_j^{(\lambda)}(x)w_2^{(\lambda)}(x)dx=h_{2,j}\delta_{ij}, \tag{8}$$

where δ_{ij} is the Kronecker function and $h_{2,j}=\frac{\pi}{8},\ j=0,1,2,\cdots.$

We can know $\mathcal{F}_{2,N}=span\{U_n^{(\lambda)}:0\le n\le N\}$ form the finite-dimensional fractional function space and by the orthogonality condition (8) expand any u belonging to $\mathcal{F}_{2,N}$ as

$$u(x)=\sum_{j=0}^{\infty}u_jU_j^{(\lambda)}(x),\ u_j=(h_{2,j})^{-1}\int_I u(x)U_j^{(\lambda)}(x)w_2^{(\lambda)}(x)dx. \tag{9}$$

4. FRACTIONAL CHEBYSHEV OPERATIONAL MATRIX OF CAPUTO'S DERIVATIVE

Here we introduce fractional operational matrix of the first kind and second kind of Caputo's derivative of order ν.

Theorem 2. Let $D^{k\lambda}u(x)\in C[0,1]$ for $k=0,1,\cdots,N$.

1) If $u_N(x)$ is the best approximation to $u(x)$ from $\mathcal{F}_{1,N}$, then the error bound is presented as follows

$$\|u(x)-u_N(x)\|_{w_1^{(\lambda)}}\le\frac{E_{1,\lambda}}{\Gamma((N+1)\lambda+1)}\sqrt{\frac{\Gamma(\frac{1}{2})\Gamma(2N+\frac{5}{2})}{\Gamma(2N+3)}}. \tag{10}$$

where $E_{1,\lambda}\ge|D^{(N+1)\lambda}u(x)|,\ x\in I$.

2) If $u_N(x)$ is the best approximation to $u(x)$ from $\mathcal{F}_{2,N}$, then the error bound is presented as follows

$$\| u(x)-u_N(x)\|_{w_2^{(\lambda)}} \le \frac{E_{2,\lambda}}{\Gamma((N+1)\lambda+1)}\sqrt{\frac{\Gamma(\frac{3}{2})\Gamma(2N+\frac{7}{2})}{\Gamma(2N+5)}}. \tag{11}$$

where $E_{2,\lambda} \ge | D^{(N+1)\lambda}u(x)|,\ x\in I$.

Proof: We will consider for the error bound of approximation of shifted fractional Chebyshev functions of the first kind. Since $u_N(x)$ is the best approximation to $u(x)$ from $\mathcal{F}_{1,N}$, by the definition of the best approximation

$$\forall v(x)\in \mathcal{F}_{1,N},\ \| u(x)-u_N(x)\|_{w_1^{(\lambda)}} \le \| u(x)-v(x)\|_{w_1^{(\lambda)}}.$$

Now let $v(x)$ is considered the generalized Taylor formula of $u(x)$, then

$$v(x)=\sum_{k=0}^{N}\frac{x^{k\lambda}}{\Gamma(k\lambda+1)}D^{k\nu}u(0^+).$$

So we have

$$\left|u(x)-\sum_{k=0}^{N}\frac{x^{k\lambda}}{\Gamma(k\lambda+1)}D^{k\nu}u(0^+)\right| \le E_{1,\lambda}\frac{x^{(N+1)\lambda}}{\Gamma((N+1)\lambda+1)},$$

$$\begin{aligned}\| u(x)-u_N(x)\|^2_{w_1^{(\lambda)}} &\le \left\| u(x)-\sum_{k=0}^{N}\frac{x^{k\lambda}}{\Gamma(k\lambda+1)}D^{k\nu}u(0^+)\right\|^2_{w_1^{(\lambda)}}\\ &\le \frac{E^2_{1,\lambda}}{\Gamma((N+1)\lambda+1)^2}\int_0^1 x^{2(N+1)\lambda}w_1^{(\lambda)}(x)dx\\ &\le \frac{E^2_{1,\lambda}}{\Gamma((N+1)\lambda+1)^2}\frac{\Gamma(\frac{1}{2})\Gamma(2N+\frac{5}{2})}{\Gamma(2N+3)}.\end{aligned}$$

Similarly, we can prove the error bound of approximation of shifted fractional Chebyshev of the second kind. Therefore, we had proved this theorem.

Theorem 3. Let $0<\nu<1$ and $\Phi(x)$ is the shifted fractional Chebyshev vector. Then Caputo fractional derivatives operator of order ν expressas

$$D^{\nu}\Phi(x)\simeq D^{(\nu)}\Phi(x), \tag{12}$$

where $D^{(\nu)}$ is the $(N+1)\times(N+1)$ fractional Chebyshev operational matrix of first kind or second kind of derivative of order ν and is defined by:

$$D^{(\nu)}=\begin{pmatrix} 0 & 0 & \dots & 0\\ S_\nu^\lambda(\lceil\nu\rceil,0) & S_\nu^\lambda(\lceil\nu\rceil,1) & \dots & S_\nu^\lambda(\lceil\nu\rceil,N)\\ \vdots & \vdots & \ddots & \vdots\\ S_\nu^\lambda(i,0) & S_\nu^\lambda(i,1) & \dots & S_\nu^\lambda(i,N)\\ \vdots & \vdots & \ddots & \vdots\\ S_\nu^\lambda(N,0) & S_\nu^\lambda(N,1) & \dots & S_\nu^\lambda(N,N)\end{pmatrix},$$

where in case of shifted fractional Chebyshev functions of the first kind, $S_\nu^\lambda(i,j)=S_{1,\nu}(i,j,\alpha,\beta,\lambda)$,

$$S_{1,\nu}(i,j,\lambda)=\sum_{k=1}^{i}\frac{(-1)^{i-k}\,i\,\Gamma(i+k)}{\Gamma(k+\frac{1}{2})(i-k)!k!}\frac{\Gamma(\lambda k+1)}{\Gamma(\lambda k+1-\nu)}\frac{\pi^{3/2}}{h_{1,j}}\sum_{s=0}^{j}\frac{(-1)^{j-s}\,j\,\Gamma(j+s)}{\Gamma(s+\frac{1}{2})(j-s)!s!}\frac{\Gamma(k+s+\frac{1}{2}-\frac{\nu}{\lambda})}{\Gamma(k+s+1-\frac{\nu}{\lambda})}, \tag{13}$$

in case of shifted fractional Chebyshev functions of the second kind, $S_\nu^\lambda(i,j)=S_{2,\nu}(i,j,\alpha,\beta,\lambda)$,

$$S_{2,\nu}(i,j,\lambda)=\sum_{k=1}^{i}\frac{(-1)^{i-k}\Gamma(i+k+2)}{\Gamma(k+\frac{3}{2})(i-k)!k!}\frac{\pi^{1/2}\Gamma(\lambda k+1)}{\Gamma(\lambda k+1-\nu)}\times\sum_{s=0}^{j}\frac{(-1)^{j-s}\Gamma(j+s+2)}{\Gamma(s+\frac{3}{2})(j-s)!s!}\frac{\Gamma(k+s+\frac{3}{2}-\frac{\nu}{\lambda})}{\Gamma(k+s+3-\frac{\nu}{\lambda})}. \tag{14}$$

Proof: We only will consider for shifted fractional Chebyshev functions of the first kind. Using (1), (2) and (3) yields

$$D^{\nu}T_i^{\lambda}(x)=\sum_{k=0}^{i}E_{1,k}^{(i)}D^{\nu}x^{k\lambda}=\sum_{k=1}^{i}E_{1,k}^{(i)}\frac{\Gamma(\lambda k+1)}{\Gamma(\lambda k-\nu+1)}x^{\lambda k-\nu}.$$

Also approximating $x^{\lambda k-\nu}$ by $N+1$ terms of shifted fractional order Chebyshev series, we can obtain

$$x^{\lambda k-\nu}\simeq\sum_{j=0}^{N}b_{k,j}T_j^{(\lambda)}(x),$$

where by Eq. (6), $b_{k,j}$ is given as

$$\begin{aligned}b_{k,j}&=\frac{1}{h_{1,j}}\int_0^1 x^{\lambda k-\nu}T_j^{(\lambda)}(x)w_1^{(\lambda)}(x)dx\\&=\frac{1}{h_{1,j}}\sum_{s=0}^{j}\frac{(-1)^{j-s}j(j+s-1)!2^{2s}}{(j-s)!(2s)!}\int_0^1 x^{\lambda k-\nu}x^{\lambda\nu}w_1^{(\lambda)}(x)dx\\&=\frac{1}{h_{1,j}}\sum_{s=0}^{j}\frac{(-1)^{j-s}j(j+s-1)!2^{2s}}{(j-s)!(2s)!}\frac{\Gamma(\frac{1}{2})\Gamma(k+s+\frac{1}{2}-\frac{\nu}{\lambda})}{\Gamma(k+s+1-\frac{\nu}{\lambda})}.\end{aligned}$$

Therefore

$$D^{\nu}T_i^{(\lambda)}(x)=\sum_{j=0}^{N}\sum_{k=1}^{i}\frac{(-1)^{i-k}2^{i}\Gamma(i+k)}{\Gamma(k+\frac{1}{2})(i-k)!k!}\frac{\Gamma(\lambda k+1)}{\Gamma(\lambda k+1-\nu)}\frac{\pi^{3/2}}{h_{1,j}}\cdot\sum_{s=0}^{j}\frac{(-1)^{j-s}2^{j}\Gamma(j+s)}{\Gamma(s+\frac{1}{2})(j-s)!s!}\frac{\Gamma(k+s+\frac{1}{2}-\frac{\nu}{\lambda})}{\Gamma(k+s+1-\frac{\nu}{\lambda})}T_j^{(\lambda)}(x),$$

that is,

$$D^{\nu}T_i^{(\lambda)}(x)=\sum_{j=0}^{N}S_{1,\nu}(i,j,\lambda)T_j^{(\lambda)}(x).$$

5. THE APPLICATION OF FRACTIONAL CHEBYSHEV OPERATIONAL MATRIX OF FIRST KIND AND SECOND KIND

In this section, we introduce the tau method based on the fractional derivative of shifted fractional order Chebyshev functions for solving numerically FDEs. Consider the linear FDEs

$$\begin{aligned}&D^{\nu}u(x)+u(x)=f(x),\quad x\in I=[0,1],\\&u(0)=u_0.\end{aligned} \tag{15}$$

where $0<\nu<1$ is constant, D^{ν} is the sense of Caputo fractional derivative and $f(x)$ is given function.

Now we first consider the approximation based on shifted fractional order Chebyshev functions of the first kind. To solving the initial value problem (15), we approximate $u(x)$ and $f(x)$ by shifted fractional Chebyshev functions of the first kind. By using typical tau method to Eq. (15), we obtain as follow

$$\begin{aligned}&(D^{\nu}u_N,T_k^{(\lambda)}(x))_{w_1^{(\lambda)}}+(u_N,T_k^{(\lambda)}(x))_{w_1^{(\lambda)}}=(f,T_k^{(\lambda)}(x))_{w_1^{(\lambda)}},\ k=0,1,\cdots,N-1,\\&u_N(0)=u_0.\end{aligned}$$

Let

$$u_N(x)=\sum_{j=0}^{N}a_jT_j^{(\lambda)},\ \mathbf{a}=(a_0,a_1,\cdots,a_N)^T,$$
$$\mathbf{f}=(f_0,f_1,\cdots,f_N)^T, \tag{16}$$
$$f_k=(f,T_k^{(\lambda)}(x))_{w_1^{(\lambda)}},\ k=0,1,\cdots,N-1.$$

It is clear that numerically solving linear FDEs is equivalent to

$$\sum_{j=0}^{N}a_j\left[(D^\nu T_j^{(\lambda)}(x),T_k^{(\lambda)}(x))_{w_1^{(\lambda)}}+(T_j^{(\lambda)}(x),T_k^{(\lambda)}(x))_{w_1^{(\lambda)}}\right]=(f,\ T_k^{(\lambda)}(x))_{w_1^{(\lambda)}},$$
$$\sum_{j=0}^{N}a_jT_j^{(\lambda)}(0)=u_0,\ k=0,1,\cdots,N-1.$$

By using matrix form, we can rewrite the above equation as

$$A=(a_{kj})_{0\le k,j\le N},\ B=(b_{kj})_{0\le k,j\le N},$$
$$(A+B)\mathbf{a}=\mathbf{f}.$$

Similarly, we can consider approximation based on shifted fractional Chebyshev functions of the second kind

$$(D^\nu u_N,U_k^{(\lambda)}(x))_{w_2^{(\lambda)}}+(u_N,\ U_k^{(\lambda)}(x))_{w_2^{(\lambda)}}=(f,\ U_k^{(\lambda)}(x))_{w_2^{(\lambda)}},\ k=0,1,\cdots,N-1,$$
$$u_N(0)=u_0.$$

Theorem 4. The coefficients of the above matrix equation are presented as

$$a_{kj}=\begin{cases}h_{1,k}S_{1,\nu}(j,k,\lambda), & 0\le k\le N-1, 0\le j\le N,\\ (-1)^j, & k=N,\ 0\le j\le N,\end{cases} \tag{17}$$
$$b_{kj}=h_{1,k},\ 0\le k=j\le N-1,$$

when $u(x)$ approximate based on shifted fractional order Chebyshev functions of the first kind

$$a_{kj}=\begin{cases}h_{2,k}S_{2,\nu}(j,k,\lambda), & 0\le k\le N-1, 0\le j\le N,\\ (-1)^j(j+1), & k=N,\ 0\le j\le N,\end{cases} \tag{18}$$
$$b_{kj}=h_{2,k}, 0\le k=j\le N-1,$$

when $u(x)$ approximate based on shifted fractional order Chebyshev functions of the second kind.

Proof: The proof easily is proved by using Eq. (16) and Theorem 3.

6. ILLUSTRATIVE EXAMPLES

In this Section, we discuss several examples by using the proposed method for solving FDEs.

Example 1. Let's consider FDE with initial conditions as

$$\begin{cases}\left({}^cD_{0^+}^\nu u\right)(x)=au(x)+(x+1),\ 0<\nu\le 1,\\ u(0)=0.5.\end{cases} \tag{19}$$

Here, we suppose that $a=-1$, then the exact solution is presented as

$$y(x)=0.5E_{\nu,1}[-x^\nu]+\int_0^x(x-t)^{\nu-1}E_{\nu,\nu}[-(x-t)^\nu](t+1)dt, \tag{20}$$

where $0<\nu\le 1$, $E_{\nu,u}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\nu k+u)}$ is the generalized Mittag-Leffler function.

From Table 1, we can know that when $\nu=\lambda$, is gained a good approximation solution by using the proposed method. Also, when $m=9$, absolute errors based on shifted Chebyshev polynomials and shifted fractional order Chebyshev functions of the first kind and second kind is shown in Table 1 for difference value ν of Example 1.

Example 2. Let's consider the inhomogeneous linear FDEs

$$\begin{cases} D_{0^+}^{\nu} y)(x) + y(x) = x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-\nu)}x^{(3-\nu)} + \frac{24}{\Gamma(5-\nu)}x^{(4-\nu)}, \\ y(0) = -1. \end{cases} \tag{21}$$

The exact solution of this equation is presented as

$$y(x) = -E_{\nu,1}[-x^{\nu}] + \int_0^x (x-t)^{\nu-1} E_{\nu,\nu}[-(x-t)^{\nu}](t^4 - \frac{1}{2}t^3 - \frac{3}{\Gamma(4-\nu)}t^{(3-\nu)} + \frac{24}{\Gamma(5-\nu)}t^{(4-\nu)})dt,\ 0 < \nu \le 1. \tag{22}$$

Similarly, we can gain a numerical solution of Example 2.

In Table 2, we can know that gain a good approximate solution with high accuracy by using the proposed method. As example 1, when $\nu = \lambda$, the numerical solution obtained by using shifted fractional order Chebyshev functions is high accuracy than the one obtained by using shifted Chebyshev polynomials.

Examples 3. Let's consider the fuzzy fractional oscillation equation

$$\begin{cases} ({}^{c}D_{0^+}^{\nu} y)(x) + y(x) = xe^{-x},\ 0 \le x \le 1, \\ y(0) = 1. \end{cases} \tag{23}$$

The exact solution of Example 3 is presented as

$$y(x) = (-1+r)E_{\nu,1}[-x^{\nu}] + \int_0^x (x-t)^{\nu-1} E_{\nu,\nu}[-(x-t)^{\nu}]te^{-t}dt,\ 0 < \nu \le 1. \tag{24}$$

As Example 1 and 2, the numerical solution of Example 3 is obtained by using the fractional Chebyshev operational matrix. Table 3 is an absolute error of numerical solution obtained by using shifted fractional order Chebyshev functions of the first kind and second kind.

Table 1: The absolute error of Example 1 with a different value of ν

	in case of the first kind			in case of the second kind		
x	ν =0.65	ν =0.75	ν =0.95	ν =0.65	ν =0.75	ν =0.95
0.0	1.110e-16	2.220e-16	3.886e-16	2.220e-16	1.110e-16	0
0.1	5.3234e-6	3.9869e-6	1.8443e-6	2.1227e-7	3.6989e-6	1.0890e-06
0.2	7.2824e-7	1.0342e-5	1.2936e-6	6.7896e-7	2.9589e-6	1.7169e-06
0.3	8.1269e-6	9.5942e-6	3.5813e-6	2.4401e-6	2.1507e-6	2.3033e-07
0.4	3.9919e-6	5.6360e-6	2.1660e-6	1.3404e-6	1.6881e-6	1.4018e-06
0.5	7.7827e-6	1.0312e-5	1.3895e-6	1.9350e-6	2.6212e-6	1.2352e-06
0.6	2.8075e-6	5.1786e-6	3.1485e-6	1.9235e-6	1.8577e-7	9.6466e-08
0.7	7.4930e-6	9.0416e-6	1.0761e-6	1.6425e-6	3.0652e-6	8.0583e-07
0.8	2.4420e-6	2.4537e-6	1.9665e-6	2.7334e-6	1.4369e-6	1.0393e-06
0.9	4.2536e-6	6.9973e-6	1.5338e-6	1.1456e-6	5.6254e-7	2.1483e-08
1.0	8.4242e-7	1.3792e-6	8.1595e-8	7.5382e-6	6.0599e-6	1.8363e-07

Table 2: The absolute error of Example 2 with a different value of ν

	in case of the first kind			in case of the second kind		
x	ν =0.6	ν =0.75	ν =0.85	ν =0.6	ν =0.75	ν =0.85
0.0	1.110e-16	1.110e-16	1.110e-16	2.220e-16	0	1.110e-16
0.1	2.6789e-8	3.9780e-8	4.9751e-7	6.5405e-9	3.2799e-8	2.1704e-7
0.2	1.2361e-8	6.8259e-8	5.0790e-7	7.509e-10	2.8295e-8	6.8503e-9
0.3	3.3619e-8	4.4140e-8	1.6120e-7	1.5155e-8	1.3788e-8	6.3481e-8

0.4	2.9566e-8	5.2073e-8	6.0428e-7	1.5848e-8	2.2611e-8	2.5515e-7
0.5	3.5341e-8	5.7896e-8	7.7641e-8	1.1506e-8	2.6324e-8	6.501e-11
0.6	2.5327e-8	4.7322e-8	5.7870e-7	2.1180e-8	9.6595e-9	1.4344e-7
0.7	4.1795e-8	5.2866e-8	1.4933e-7	1.2861e-8	3.0199e-8	1.8103e-7
0.8	1.2550e-8	1.7898e-8	3.6570e-7	1.8065e-8	1.9713e-8	6.7496e-8
0.9	5.1672e-9	6.1148e-8	3.4380e-7	3.1406e-8	2.1759e-8	1.1885e-7
1.0	2.9747e-8	9.9875e-9	8.2304e-8	5.2583e-8	9.0055e-8	2.7107e-7

Table 3: The absolute error of Example 3 with a different value of ν

	in case of the first kind			in case of the second kind		
x	ν =0.5	ν =0.7	ν =0.95	ν =0.5	ν =0.7	ν =0.95
0.0	0	0	2.220e-16	0	1.110e-16	2.220e-16
0.1	2.2507e-6	1.8145e-6	1.9599e-6	3.0468e-9	1.9564e-6	1.1446e-6
0.2	2.6789e-6	5.1838e-6	1.3730e-6	4.1378e-9	2.3924e-6	1.8788e-6
0.3	5.9094e-9	1.1400e-5	3.9234e-6	5.5215e-9	2.9126e-6	2.1591e-7
0.4	2.8070e-6	1.6388e-7	2.3401e-6	3.540e-10	2.9845e-8	1.5195e-6
0.5	3.9592e-7	1.1115e-5	1.5362e-6	5.4416e-9	2.8701e-6	1.3527e-6
0.6	2.5892e-6	8.0748e-7	3.4352e-6	1.7948e-9	1.0210e-6	7.4989e-8
0.7	1.2229e-6	9.8436e-6	1.1954e-6	4.7474e-9	2.6845e-6	8.6459e-7
0.8	2.0672e-6	3.4702e-7	2.1476e-6	3.8368e-9	2.4722e-6	1.1367e-6
0.9	5.3961e-7	6.4052e-6	1.6823e-6	4.127e-11	2.3806e-7	5.6458e-8
1.0	5.7945e-8	1.2244e-6	8.1310e-8	1.2011e-8	7.4220e-6	2.3713e-7

7. CONCLUSION

In this paper, we introduced the method of the fractional Chebyshev operational matrix based on shifted fractional-order Chebyshev functions of the first kind and second kind for solving linear fractional differential equations with the initial condition. Through three examples presented, the accuracy of numerical solutions was improved errors of about 10^{-3} as shown in the Tables. For solving the linear fractional differentialequation with the initial condition, we have converted the fractional differential equation into the system of algebraic equations. And we have obtained error bound of the numerical solution by using fractionalChebyshev operational matrix of the first kind and second kind. The numerical solutions of the above three Examples were obtained using MATLAB version R(2013A) and $N = 9$.

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