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HISTORICAL ORIGIN AND SCIENTIFIC DEVELOPMENT OF GRAPHS, HYPERGRAPHS AND DESIGN THEORY

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ABSTRACT

The origin and the various steps of the development of modern combinatorial theories, as Graph Theory, Hypergraphs and Designs Theory, are examined. The most important conjectures and results are pointed out.

1. Introduction

The modern combinatorial theories had their origin and development in the middle of the XIX century, overall for two important problems: *the Four Color Problem* for Graph Theory and *the Woolhouse Problem* for Steiner systems. The *Konigsberg Bridge Problem* (1735) had given a contribute, but after few things have been done, probably because it was soon solved by Euler.

The Four Color Problem, become soon afterwards *Four Color Conjecture* (4CC) was solved in 1976 and the many attempts to solve it have given a big contribute to the great development of Graph Theory.

The *Woolhouse Problem* is today still unsolved in its generality and it can be considered solved only for few particular cases.

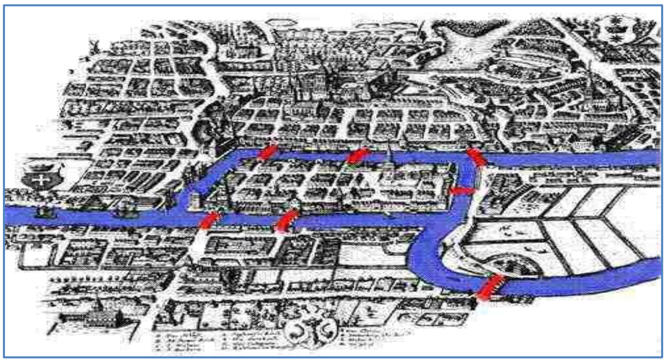
We see the various steps of the development of these problems and other historical combinatorial problems.

2. Konigsberg Bridge Problem - 1735

In 1735 the *seven bridges problem of Konigsberg* was solved by Euler. It is an historical problem in combinatorial mathematics.

The city of Konigsberg was in Prussia. Now it is called Kaliningrad and it is in Russia. It is situated on the banks of the Pregel river, including the two isles Kneiphof and Lomse. The various *zones of the city are connected to each other by seven bridges. The problem was to determine, if possible, a walk through the city that would cross each of bridges once and only once.*

Someone added the condition that the walk should be a cycle.



Euler proved that the problem has not any no solution. He developed a suitable technique to establish this assertion with mathematical rigor.

The year 1736 is considered the birth date of *Graph Theory*, because, it seemed that Euler used graphs for his proof. But today it seems that this is not true. On August 26, 1735, Euler presented a paper containing the solution to the Konigsberg bridge problem, having the title:

> "*Solutio problematis ad geometriam situs pertinentis*", and later he published in 1741 in *Commentarii Academiae Scientarum Imperialis Petropolitanae* 8 (1736), 128-140.

Precisely for this reason, O. Ore wrote in his book "*Graphs and their uses*"(1955):

> "*Graph Theory is one of the few fields of mathematics with a definite birth date*", even if this is questioned today.

Euler's paper is divided into twenty-one paragraphs. At first he introduced the Konigsberg Bridge problem. After Euler stated that he believed that this problem concerned geometry, that not involved measurements and calculations, which Leibniz referred to as Geometry of Position. Then Euler indicated the seven distinct bridges by the letters: a, b, c, d, e, f, g and gave his combinatorial proof (it seems) without use the term *graph*.

In Graph Theory, a cycle in a graph G which contains all the edges exactly once is called an *eulerian cycle*. A graph G which admits an eulerian cycle is said an *eulerian graph.*

Euler gave also a characterization of eulerian graphs:

"A connected graph is eulerian if and only if all its vertices have even degree".

3. Woolhouse Design Problem - 1844

In 1844, Woolhouse pointed out the following problem, where we use a modern mathematical language:

Let X={x_1, x_2, … , x_v} be a v-set, whose elements are called vertices, and let B={E_1, E_2, … , E_b} be a family of k-subsets of X, called blocks, such that every h-subset H of X is contained in exactly one block of F. For which integers h,k,v is it possible to construct a family B ?

Today a pair ∑=(X,B), verifying the conditions introduced by Woolhouse, is called a *Steiner system S(h,k,v)*. In other words, using hypergraph terminology, a *Steiner systems S(h,k,v)* is an hypergraph of *order v*, *uniform of rank k*, in which every *h*-subset H of X has *degree one*.

Therefore in 1844 Woolhouse asked: *for which h,k,v does a Steiner System S(h,k,v) exist* ?

This problem remains unsolved in general until today. However several partial results were given. Observed that the first significant values of *h,k* were *h=2, k=3,* indicated by STS(v) any S(2,3,v), in 1847 Kirkman proved that.

"An STS(v) exists if and only if v ≡ 1 or 3, mod 6", and constructed systems S(*3,4,v*), also indicate by SQS(*v*), for $v=2^n$, for every *n=2,3,….* In 1853, J.Steiner asked for the existence of systems S(*h,h+1,v*) and probably for this reason these systems are called *Steiner systems*.

After many partial results, only in 1960 and using many complicated recursive constructions, H. Hanani proved that:

"An SQS(v) exists if and only if v ≡ 2 or 4, mod 6",

Soon after in 1962 H. Hanani proved also that:

"An S(2,4,v) exists if and only if v ≡ 1 or 4, mod 12".

After the cases *h=2* and *k=3,4*, *h=3* and *k=4*, the only known today solved case is for *h=2* and *k=5*:

"An S(2,5,v) exists if and only if v ≡ 1 or 5, mod 20".

For the other cases only some particular systems are known. Recently it is proved that any S(*4,5,17*) there exists, while it is open the problem to construct any S(*5,6,18*).

4. Kirkman 15 Schoolgirl problem - 1850

In 1850, the *Lady's and Gentleman's Diary* published on pg.48 the following problem proposed by K.T. Kirkman:

Query n.6: Fifteen young ladies in a college walk out three abreast for seven days in succession: it is required, if it is possible, to arrange them daily, so that no two shall walk twice abreast.

By this problem Kirkman introduced the theory of *resolvable* Steiner systems. Indeed, to solve that problem it is necessary to construct a *resolvable* STS(15).

Let Σ=(X,B) be Steiner system S(*h,k,v*). Two distinct blocks E',E" of B are said to be *paralleli* if they have no vertex in common. A *parallel class C* of Σ is a set of parallel blocks which cover all X. A system Σ is said to be *resolvable* if there exists a partition π of B in *parallel classes*.

Therefore, to solve the problem of Kirkman it is necessary to construct a resolvable STS(15).

Resolvable STS(15)

Set of schoolgirls X={1, 2, …, 15}:

Monday:	{1,6,11} - {2,7,12} - {3,8,13} - {4,9,14} - {5,10,1}
Tuesday:	{1,2,5} - {3,4,7} - {8,9,12} - {10,11,14} - {6,13,15}
Wednesday:	{1,7,14} - {2,3,6} - {4,5,8} - {9,10,13} - {11,12,15}
Thursday:	{1,12,13} - {2,4,10} - {3,14,15} - {5,6,9} - {7,8,11}

Friday: {1,8,10} - {2,13,14} - {3,5,11} - {4,6,12} - {7,9,15}

Saturday: {1,4,15} - {2,9,11} - {3,10,12} - {5,7,13} - {6,8,14}

Sunday: {1,3,9} - {2,8,15} - {4,11,13} - {5,12,14} - {6,7,10}

Resolvable systems of type S(2,3,v) were called *Kirkman Triple Systems* and they are today indicated by KTS(*v*). Kirkman proved that:

"A KTS(v) exists if and only if $v \equiv 3$, *mod 6".*

5. Four Colour Problem - 1852

The Four Colour problem was born in 1852. The student Francis Guthrie observed that any Geographical map in a plane can be coloured using 4 colours in such a way that regions sharing a common boundary (no a single point) do not share a same colour. Francis Guthrie was intrigued by the problem and passed the problem to his brother, who then asked to his professor De Morgan, who pointed out the question to the London Mathematical Society and in particular to Hamilton. So, the four colour problem soon became *Four Colour Conjecture.* There were many attempts of solution:

- A.Cayley in 1872 gave a proof in the paper "*On the colouring maps"*;
- A.Kempe in 1879 gave a proof of the conjecture which was considered valid for 11 years;
- P.Heawood in 1890 found a mistake in the proof of Kempe and proved the "*five colours theorem*".

Since that years, the attempts to prove that "*every map is 4-colorable"* or that "*there exist maps 5-colorable but not 4-colorable"* were multiple, but the demonstrations were all wrong.

These gave a great contribute to the knowledge of the problem and at the same time gave a great contribute to the development of *Graph Theory.* Associating with every geographical map a *planar graph* and colouring its vertices, the problem of colouring a map became a problem of vertex-colouring of graph theory as follows:

"Every planar graph is 4 colourable".

The four colour problem was solved in 1976 by Appel and Heken, using big and powerful computers. The proof of 4CC is published in:

K.Appel, W.Haken, Illinois Journal of Mathematics, 84 (1977)-I, 429-490

K.Appel, W.Haken, J.Koch, Illinois Journal of Mathematics, 84 (1977)-II, 491-567

K.Appel, W.Haken, Scientific American, 237 (1977), 108-121

The authors arrived to the result, by 3 steps.

1) *Definition and determination of main maps, which it was proved were 1476*.
2) *Demonstration that the colouring of any map can be traced back reduced to the colouring of a main map.*
3) *Check by computer that the main maps are all 4-colorable.*

The idea used by the authors was the same of Kempe. But Kempe had proved that the number of main maps was exactly 4.

We can observe that step 1 and step 2 are of mathematical character, while in step 3 there is the use of computers. Therefore, the proof is only in part mathematical and there are many mathematicians who ask for a proof completely mathematical.

6. Hamilton Dodecahedron problem - 1859

In 1859, the Irish mathematician W. R. Hamilton devised a wood puzzle with a regular dodecahedron. He labelled the vertices with the names of important cities and he formulated the following problem:

Determine, if it is possible, a cycle along the edges of the dodecahedron which visites every city exactly once and returnes to the start.

The problem has solutions and it gave origin to the so-called *hamiltonian graphs*. It is easy to see that it is possible to represent the dodecahedron by a graph. A cycle on the dodecahedron , as requested by Hamilton, is a cycle which contains every vertex exactly once. In a graph, a cycle of this kind is called an hamiltonian cycle and a graph which admits an hamiltonian cycle is said an *hamiltonian graph*. From Hamilton Dodecahedron problem it is born one of the most important open problem in the modern graph theory: *the characterization of hamiltonian graphs.*

The problem of Hamilton is very similar to the Konigsberg bridges problem. In the eulerian graphs, the condition regards the edges of the graph (determination of a cycle which contains all the edges exactly once), in the hamiltonian graphs the condition regards the vertices (determination of a cycle which contains all the vertices exactly once). However, while eulerian graphs have been characterize by Euler (the necessary and sufficient condition is that in the connected graph every vertex has even degree), we do not know until today a necessary and sufficient condition for hamiltonian graphs.

7. Nine Prisoners problem – 1917

In 1917, the mathematician H.E.Dudeney formulated the following puzzle (called by himself):

In a jail there were 9 prisoners, indicated by integer numbers 1, 2, …, 9, having a particularly character. Each morning they were allowed to walk handcuffed in the prison yard. On Monday they walked as follows:

$$1 — 2 — 3$$

$$4 — 5 — 6$$

$$7 — 8 — 9$$

Could you arrange them for Tuesday — Saturday so that no pair of prisoners is handcuffed together twice ?

Note that the prisoners 1 and 3 are not handcuffed together on Monday, so that this puzzle is, as Dudeney remarks, an interesting puzzle similar to the problem of 15 schoolgirls of Kirkman.

We observe that this problem can be considered the primitive question from which the entire G-design theory had its origin.

Further, 15 schoolgirls problem of Kirkman and 9 prisoners problem of Dudeney point out the considerable difference between *sets* and *graphs* and their use. In Kirkman's problem the triples of schoolgirls are all sets and as such, if {A,B,C} is one of them, it contains le pairs of schoolgirls {A,B},{B,C},{A,C} (which cannot appear in other triples). In Dudeney problem, the triples are not sets, indeed if A,B,C are the prisoners, in this order, the triples is made up of pairs {A,B},{B,C}. Therefore, we can say that it is a *path* P_3, that is the graph having vertices A,B,C and edges {A,B},{B,C}. From this point of view, a set as {A,B,C}can be seen as the complete graph K_3.

In other words, *this show how graphs can and will replace sets*.

Well, following design theory terminology, we observe that to solve Dudeney's problem it is necessary to construct a *resolvable P_3-designs*.

***Resolvable P_3-design of order* 15**

Set of prisoners X={1, 2, …, 15}:

Monday:	1 — 2 — 3	*Tuesday*:	5 — 1 — 6
	4 — 5 — 6		7 — 2 — 8
	7 — 8 — 9		9 — 3 — 4
Wednesday:	2 — 5 — 8	*Thursday*:	2 — 4 — 1
	9 — 1 — 3		5 — 3 — 8
	4 — 6 — 7		7 — 9 — 6
Friday:	1 — 7 — 5	*Saturday*:	5 — 9 — 2
	2 — 6 — 3		1 — 8 — 6
	9 — 4 — 8		4 — 7 — 3

It is well-known that:

(1) *P_3-designs exists if and only if $v \equiv 0$ or 1, mod 4.*

(2) *Resolvable P_3-designs exists if and only if $v \equiv 9$, mod 12.*

8. Edge-colouring - 1880 and Classification Problem - 1960

In the past, problems involving vertex-colorings and chromatic number received considerable attention mainly because the 4CC. After some studies about vertex-colorings, it was naturally to define edge-coloring and chromatics index. Although the origins of chromatic theory may be traced back to 4CC (1852) and the first papers on edge-colorings appeared in 1880, when P. G. Tait published two brief abstracts in the *Proceedings of the Royal Society of Edimburg.*

In this paper Tait proved that:

"If the 4CC is true, then the edges of every trivalent planar graph can be properly coloured using only 3 colours"

After this, little was done until D. Konig, who in 1916 proved that:

"*If G is a bipartite graph having maximum degree Δ, then its edges can be properly coloured using exactly Δ colours"*

It was soon evident that it was not possible to colour edges of a graph by Δ-1 colours and a curious fact was that every graph was always colourable by Δ or Δ +1 colours.

So, the great news arrived in 1964, when V.G.Vizing proved that:

"In every graph G, the chromatic index is always Δ or Δ+1"

Observe that this fact is completely different from what happens with the vertex- colourings.

The Theorem of Vizing gave a criterion to classify graphs. *A graph G is of class 1 if its chromatic index is Δ, it is of class 2 if its chromatic index is Δ+1.*

The problem to characterize if a graph is of class 1 or of class 2 is the *Classification Problem of Graphs* and it is one of the most important *open problem* of the modern Graph Theory.

9. Vizing's Conjecture for classification of planar graphs – 1965

Although the classification problem is far from solved in general, appreciable progress has been done in the particular case of planar graphs.

Even cycles are planar graph of class 1. Odd cycles are planar graphs of class 2. In 1965 Vizing proved the following surprising result:

"Every planar graph with $\Delta \geq 8$ *is necessary of class 1"*

The problem of determining what happens when the maximum degree is either 6 or 7 remains open and, in this connection, Vizing formulated the following conjecture:

"Every planar graph with Δ=6 or Δ=7 is of class 1"

We can see a different situation between vertex-colorings and edge-colorings. Given a graph G, the minimum possible value for chromatic index is Δ (maximum degree), while the minimum possible value for chromatic number is the density ω (the maximum number of vertices generating a complete subgraph), For edge colorings the possible values for the chromatic index χ' are only two (Theorem of Vizing): χ'= Δ or χ'= Δ+1. For vertex-colorings, there is a Mycielski construction for which it happens that:

"*For every positive integer h, there exists graphs G such that χ= ω+h*"

10. Berge's Conjecture for Linear Hypergraphs - 1975

Let $X=\{x_1,x_2,...,x_n\}$ be a finite set of cardinality n. Let Π be a family of non-empty subset of X, such that two different subsets of Π are disjoint or have only one element in common. Let Π* be the family of all the subsets of X, containing Π and containing all the non-empty subsets of the subsets of Π. In other words, Π* is obtained from Π, adding to every E that belongs to Π all its non-empty subsets.

Prove that it is always possible to define a partition $\Omega=\{E_1,E_2,...,E_n\}$ of Π* so that:

1) For every i=1,2,…,n, if F', F" belong to E_i then F' and F" are disjoint;

2) There exists at least an element x of X such that for every i=1,2,…,n there exists an E_i of Π* containing x;

The problem so formulated is an open problem. It is due to Berge, who formulated it using hypergraph terminology.

Let H=(X,B) be an hypergraph. H is said *linear* if two different edges of B are always disjoint or have only one vertex in common. The *closure* of H is the hypergraph H*=(X,B*), having the same vertices of H and any F is an edge of H* if and only if it is a non-empty subset of an edge of H. Given H, the symbol Δ(H) indicates the maximum degree of the vertices of H.

Conjecture of Berge:

If H is a linear hypergraph then the chromatic index of H is equal to* Δ(H*)

Observe that a Steiner system of type S(2,k,v) is a linear hypergraph. Therefore, Berge's conjecture can be formulated for S(2,k,v) as follows:

"*The closure of every S)(2,k,v) is resolvable*"

<u>Example:</u> Let H=(X,B) be an hypergraph, where X={1,2,…,7} and B is the family of edges so defined:

E_1={1,2,3}, E_2={1,4,5}, E_3={1,6,7},

E_4={2,4,6}, E_5={2,5,7}, E_6={3,4,7}

The hypergraph H*, *closure* of H, is always defined in X and has for edges all the edges of H with also all their non-empty subsets:

1,2 1,3 1,4 1,5 1,6 1,7

2,3 2,4 2,5 2,6 2,7 3,4

3,7 4,5 4,6 4,7 5,7 6,7

1, 2, 3, 4, 5, 6, 7

A partition of the closure H* of H in parallel classes, verifying the conjectures (classes must be red horizontally), is:

1 2 3	4 5	6 7		
1 4 5	2 6	3 7		
1 6 7	2 4	3 5		
2 4 6	1 3	5 7		
2 5 7	1 6	3 4		
3 4 7	1 2	5 6		
3 5 6	1 4	2 7		
1 5	2 3	4 6	7	
4 7	1	2	5	
1 7	2 5	3	4	6

11. The conjecture of Erdos-Strauss - 1948

There is a famous conjecture due to Erdos-Strauss regarding Number Theory.

For every positive integer $n \geq 2$ there exist positive integers x, y, z such that:

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

It is not clear if there is a demonstration of the truth of conjecture. However, it is easy to see that the conjecture is true for n=4k, n=4k+2, n=4k+3. Indeed:

- for *n=4k,* it is sufficient to take: *x=2k, y=2k, z=4k*
- for *n=4k+2*, it is sufficient to take: *x=2k+1, y=2k+1, z=4k+2*
- for *n=4k+3*, it is sufficient to take: *x=2k+2, y=2k+2, z=(k+1)(4k+3).*

For *n=4k+1*, again, it is not clear if there is a demonstration or not, however there are cases which confirm that the conjecture is true. Indeed, for n=5, n=45, n=81, we have respectively:

$\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}$; $\quad \frac{4}{45} = \frac{1}{18} + \frac{1}{45} + \frac{1}{90}$; $\quad \frac{4}{81} = \frac{1}{27} + \frac{1}{162} + \frac{1}{162}$.

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