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ON RANDOM COEFFICIENT LEHMANN MINIFICATION PROCESSES

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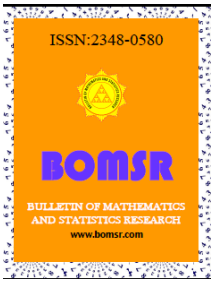

ABSTRACT
The Lehmann / proportional hazards family of distributions is a well-known family of distributions generated from a given distribution/survival function by raising it by a positive power and is found to be useful in lifetime studies. Several known distributions are generalized using this method. In this paper a new class of random coefficient first order autoregressive minification process is introduced which generates these types of generalized distributions as marginals. Several random coefficient minification models with distributions like generalized logistic, Burr Type XII, generalized exponential etc. as marginals are subclasses of this model. A random coefficient generalized semi-logistic process is also introduced. A special case is considered and the estimation of the parameters discussed.
Key words: Lehmann family of distributions, Minification processes, Random coefficient processes, Markov processes.

1. Introduction

The minification models, introduced by Tavares (1980) has led to the development of several non-Gaussian time series models. The minification models are developed and studied using the survival function of the underlying random variable. The autoregressive minification process introduced by Tavares (1980) is

$$X_n = \begin{cases} X_0 & n = 0 \\ k \min(X_{n-1}, \varepsilon_n) & n \geq 1 \end{cases}$$

where $k > 1$ is a constant and $\{\varepsilon_n\}$ is an innovation process of independent and identically (i.i.d.) random variables such that $\{X_n\}$ is a stationary Markov process. He considered a particular case, where $\{\varepsilon_n,\ n = 1, 2, \ldots\}$ is a sequence of i.i.d. exponential random variables

with mean $\theta(k-1)$ and X_0 is exponential with mean θ. This model generates first order autoregressive exponential process with mean θ and it finds useful in hydrological applications.

One may refer to Sim (1986), Yeh et al. (1988), Arnold (1989), Arnold and Robertson (1989), Arnold and Hallett (1989), Pillai (1991), Lewis and McKenzie (1991), Pillai et al. (1995), Jayakumar and Pillai (2002), Krishnarani and Jayakumar (2008a, 2008b, 2013), Ristic (2006,2008) etc. for an excellent exposition of minification models with different marginals.

Krishnarani and Jayakumar (2008b) introduced a new class of autoregressive minification model

$$X_n = \begin{cases} \phi^{-1}[\phi(X_{n-1}) - \ln\ p] & \text{w.p.} \quad p \\ \min\ [\phi^{-1}\ (\phi(X_{n-1}) - \ln\ p), \varepsilon_n] & \text{w.p.} \quad (1-p) \end{cases} \tag{1.1}$$

where $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables, ε_n is independent of $X_i's\ (i < n), 0 < p < 1.$

Extensions of the additive and minification models have been proposed by replacing the coefficients by random variables to get random coefficient autoregressive models. An example is the random coefficient model introduced and studied by Nicholls and Quinn (1982). Such models can be found in Gaver and Lewis (1980), Dewald and Lewis (1985), Lawrance and Lewis (1985), Sim (1986), Ristic (2008) and Han et al. (2018).

Estimation of parameters of the minification processes can be seen in Balakrishna (1998), Balakrishna and Jacob (2003) and Ristic (2008).

Now we consider some known distributions which are used in the following sessions.

It is well known that Lehmann family of distributions is generated from a given survival function in the following manner.

Let $\overline{F}(x)$ be an arbitrary known survival function. If γ is positive then,

$$\overline{S}(x) = \left(\overline{F}(x)\right)^{\gamma},$$

is also a survival function.

In particular if γ is a positive integer then, it represents the survival function of $\min(X_1, X_2, ..., X_n)$ where X_i's are i.i.d. random variables with F(x) as the common distribution function.

The density function corresponding to $\overline{S}(x)$ is

$$f_\gamma(x) = \gamma\left(\overline{F}(x)\right)^{\gamma-1} f(x)$$

and the failure rate is

$$h_\gamma(x)=\frac{f_\gamma(x)}{\overline{S}(x)}=\frac{\gamma\left(\overline{F}(x)\right)^{\gamma-1}f(x}{\left(\overline{F}(x)\right)^{\gamma}}=\gamma\, h(x)$$

where $h(x)$ is the failure rate of $F(x)$. Thus hazards are proportional. Hence Lehmann family is also known as proportional hazards family.

A random variable X on $(0,\infty)$ is said to have semi- logistic distribution denoted by SL if it has the survival function

$$\overline{F}(x)=\frac{1}{1+\psi(x)}, \tag{1.2}$$

where $\psi(x)$ satisfies the functional equation

$$\psi(x)=\frac{1}{p}\,\psi\left(\frac{1}{\beta}\ln p+x\right),\quad ,\ \beta>0,\quad 0<p<1. \tag{1.3}$$

A random variable X on $(0,\infty)$ is said to have generalized semi- logistic distribution if it has the survival function

$$\overline{F}(x)=\left(\frac{1}{1+\psi(x)}\right)^{\gamma},\quad \gamma>0, \tag{1.4}$$

where $\psi(x)$ satisfies the functional equation (1.3).

We denote this distribution as GSL(γ) .

In the present study our aim is to construct random coefficient minification models, which can generate any marginal from the Lehmann family of distributions. In section 2, a generalized random coefficient minification model is constructed which can generate these types of generalized distributions as marginals. Some properties of the process are discussed. Several examples are given. In section 3, a random coefficient generalized semi-logistic process is introduced. Special cases are considered and estimation of the parameters is done in the last section.

2. A Generalized Random Coefficient Lehmann Minification Process

Here we introduce a new class of random coefficient process, which generates several known minification models. Let F(.) be a non-degenerate distribution function with $F(-\infty)=0$ and $F(\infty)=1.$

Consider the monotone transformation,

$$\phi(x)=\ln\frac{F(x)}{\overline{F}(x)} \tag{2.1}$$

where $\phi(-\infty)=-\infty$, $\phi(\infty)=\infty$ and $\overline{F}(x)=1-F(x).$

Let $\overline{G}(.)$ be a survival function with Lehmann structure given by

$$\overline{G}(x) = \frac{1}{[1+e^{\phi(x)}]^{r+1}}. \tag{2.2}$$

We call this distribution as generalized Lehmann logistic distribution and denote it by GLL(γ).

Consider the process with the structure

$$X_n = \min\left(\phi^{-1}(\phi(X_{n-1}) - \ln V_n), \varepsilon_n\right) \quad n = 1,2,\ldots \tag{2.3}$$

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of i.i.d. random variables such that V_n has the distribution $F_{V_n}(v) = v^{\gamma}, \quad 0 < v < 1, \quad \gamma > 0.$

The process (2.3) is called the generalized random coefficient Lehmann minification process.

Next we develop a result on the stationarity of the process (2.3).

Theorem 2.1

Let $\{X_n\}$ be defined by (2.3). Then if X_0 follows GLL(γ) and ε_n has distribution function F, then the process $\{X_n\}$ is stationary with GLL(γ) marginals.

Proof:

Assuming the structure of $\{X_n\}$ given by (2.3) the survival function is,

$$\overline{G}_{X_n}(x) = \overline{F}_{\varepsilon_n}(x)\int_0^1 \overline{G}_{X_{n-1}}\left(\phi^{-1}(\phi(x) + \ln v)\right) \gamma\, v^{\gamma-1} dv \tag{2.4}$$

Suppose X_0 follows GLL(γ) and ε_n has distribution F.

Then for n=1, (2.4) becomes,

$$\overline{G}_{X_1}(x) = \overline{F}_{\varepsilon_1}(x)\int_0^1 \overline{G}_{X_0}\left(\phi^{-1}(\phi(x) + \ln v)\right) \gamma\, v^{\gamma-1} dv$$

$$= \frac{1}{[1+e^{\phi(x)}]^{r+1}}. \tag{2.5}$$

That is, X_1 follows GLL(γ).

Similarly we can prove that if X_{n-1} follows GLL(γ), then X_n follows GLL(γ).

Now we develop a necessary and sufficient condition for the generalized random coefficient Lehmann minification process.

Theorem 2.2

Let {X_n} be defined as (2.3), where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of i.i.d. random variables such that V_n has the distribution function $F_{V_n}(v) = v^{\gamma}, \quad 0 < v < 1, \quad \gamma > 0.$ Suppose that the process is stationary. Then X_n follows GLL(γ) if and only if ε_n follows F.

Proof:

Suppose that $\{X_n\}$ is stationary. Then from (2.4), we have

$$\overline{G}_X(x) = \overline{F}_{\varepsilon_n}(x)\int_0^1 \overline{G}_X\left(\phi^{-1}(\phi(x)+\ln v)\right)\gamma\, v^{\gamma-1}dv$$

$$= \overline{F}_{\varepsilon_n}(x)\int_{-\infty}^{x} \overline{G}_X(t)\,\gamma\, e^{\gamma\phi(t)}e^{-\gamma\phi(x)}\phi'(t)dt\,.$$

$$\text{Therefore, } \frac{e^{\gamma\phi(x)}\overline{G}_X(x)}{\overline{F}_{\varepsilon_n}(x)} = \int_{-\infty}^{x} \overline{G}_X(t)\,\gamma\, e^{\gamma\phi(t)}\phi'(t)dt\,. \tag{2.6}$$

Differentiating both sides with respect to x, and simplifying we get,

$$\frac{\overline{G}'_X(x)}{\overline{G}_X(x)} - \frac{\overline{F}'_{\varepsilon_n}(x)}{\overline{F}_{\varepsilon_n}(x)} = -\gamma\phi'(x)\left(1-\overline{F}_{\varepsilon_n}(x)\right). \tag{2.7}$$

Integrating the above now yields

$$\ln \overline{G}_X(x) - \ln \overline{F}_{\varepsilon_n}(x) \;=\; -\gamma\int \phi'(x)[1-\overline{F}_{\varepsilon_n}(x)]dx\,.$$

But since ε_n follows F, we have $\overline{F}_{\varepsilon_n}(x) = \dfrac{1}{1+e^{\phi(x)}}$.

Then the above equation becomes,

$$\ln \overline{G}_X(x) = \ln\left(\frac{1}{[1+e^{\phi(x)}]^{r+1}}\right).$$

That is, $\overline{G}_X(x) = \left(\dfrac{1}{1+e^{\phi(x)}}\right)^{\gamma+1}$ and so X_n has distribution GLL(γ).

Conversely assume $\{X_n\}$ is stationary and X_n has distribution GLL(γ). Then it follows that,

$$\left(\frac{1}{1+e^{\phi(x)}}\right)^{\gamma+1} \;=\; \overline{F}_{\varepsilon_n}(x)\int_0^1\left(\frac{1}{1+ve^{\phi(x)}}\right)^{\gamma+1}\gamma v^{\gamma-1}dv\,.$$

Therefore, $\overline{F}_{\varepsilon_n}(x) = \dfrac{1}{1+e^{\phi(x)}}$.

That is, ε_n has distribution function F. Hence the proof.

Also we derive the following expressions concerning the generalized random coefficient Lehmann minification process.

$$\overline{F}_{X_n,X_{n+1}}(x,y) = \frac{1}{1+e^{\phi(y)}}\int_0^1\left(\frac{1}{1+e^{\max(\phi(x),\phi(y)+\ln v)}}\right)^{\gamma+1}\gamma\, v^{\gamma-1}dv$$

$$P\left(X_{n+1} > X_n\right) = \frac{\gamma+1}{\gamma+2} \tag{2.8}$$

We can see that several minification models with marginals from the Lehmann family can be deduced as special cases from the process defined in (2.3). For instance we have the following generalized models already reported in the literature.

Example 2.1

On taking $\overline{F}(x) = \frac{1}{1+e^{x}}$, (2.3) takes the form,

$$X_n = \min\left(X_{n-1} - \ln V_n, \varepsilon_n\right) \tag{2.9}$$

which generates generalized Lehmann logistic marginals where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of i.i.d. random variables such that V_n has the distribution $F_{V_n}(v) = v^{\gamma}, \quad 0 < v < 1, \quad \gamma > 0$ and $\{\varepsilon_n\}$ follows logistic distribution.

Example 2.2

When $\overline{F}(x) = \frac{1}{1+x} \quad x > 0$, we have

$$X_n = \min\left(V_n^{-1} X_{n-1}, \varepsilon_n\right) \tag{2.10}$$

which generates generalized Lehmann Pareto marginals where $\{\varepsilon_n\}$ has the Pareto survival function and $\{V_n\}$ has the power function distribution with distribution function $F_{V_n}(v) = v^{\gamma}, \quad 0 < v < 1, \quad \gamma > 0$.

Example 2.3

If $\overline{F}(x) = (1-x), \ 0 < x < 1$, then (2.3) becomes,

$$X_n = \min\left(\frac{X_{n-1}}{V_n + (1-V_n)X_{n-1}}, \varepsilon_n\right) \tag{2.11}$$

which generates generalized Lehmann uniform marginals when $\{\varepsilon_n\}$ is a sequence of uniform random variables and $\{V_n\}$ has the power function distribution with distribution function $F_{V_n}(v) = v^{\gamma}, \quad 0 < v < 1, \quad \gamma > 0$.

Example 2.4

If $\overline{F}(x) = e^{-x}, \ 0 < x < \infty$, then the process is

$$X_n = \min\left[\ln\left(1 + V_n^{-1}\left(e^{X_{n-1}} - 1\right), \varepsilon_n\right)\right] \tag{2.12}$$

which generates generalized Lehmann exponential marginals when $\{\varepsilon_n\}$ is a sequence of exponential random variables and $\{V_n\}$ has the power function distribution with distribution function $F_{V_n}(v) = v^{\gamma}, \quad 0 < v < 1, \quad \gamma > 0$.

3. Random Coefficient Generalized Semi-Logistic Process

Consider the process with the structure

$$X_n = \min\left(X_{n-1} - \ln V_n, \varepsilon_n\right) \quad n = 1,2,\ldots \tag{3.1}$$

where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of i.i.d. random variables such that V_n has the distribution $F_{V_n}(v) = v^{\gamma}, \quad 0 < v < 1, \quad \gamma > 0.$

The minification process (3.1) can be used to generate generalized semi-logistic distributions as marginals. Proceeding on similar arguments as in the previous section it is easy to obtain conditions for stationarity, necessary and sufficient condition for the generalized semi- logistic distribution to be maginals of (3.1) and these results are stated in the following theorems.

Theorem 3.1

Let X_0 has GSL(γ) distribution with survival function (1.4) and ε_n has survival function (1.2), then the process $\{X_n\}$ is stationary with GSL(γ) as marginals.

Theorem 3.2

Let {X_n} be defined as (3.1), where $\{V_n\}$ and $\{\varepsilon_n\}$ are two independent sequences of i.i.d. random variables such that V_n has the distribution function $F_{V_n}(v) = v^{\gamma}, \quad 0 < v < 1, \quad \gamma > 0.$ Suppose that the process is stationary. Then X_n follows GSL(γ) if and only if ε_n follows SL.

4. A particular case

Now we consider a particular case where γ takes only positive integer values and $\{V_n\}$ is not random.

We define a model having the structure,

$$X_n = \min\left(\phi^{-1}\left(\phi(X_{n-1}) - \ln p\right), \eta_n\right) \tag{4.1}$$

$$\text{where } \eta_n = \min_{1 \le i \le \gamma} \left(\eta_{in}\right) \text{ and } \eta_{in} = \begin{cases} \infty & w.p. \quad p \\ \varepsilon_n & w.p. \quad (1-p) \end{cases} \tag{4.2}$$

where $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables with survival function $\frac{1}{1+e^{\phi(x)}}$ and ε_n is independent of $X_i's (i < n), \; 0 < p < 1.$

Note that when $\phi(x) = \ln \frac{F(x)}{\bar{F}(x)}$,

$$\bar{F}(x) = \frac{1}{1+e^{\phi(x)}}. \tag{4.3}$$

Next we seek a necessary and sufficient condition for the $\{X_n\}$ to be stationary.

Theorem 5.1

Let X_0 has survival function $\overline{F}^{\gamma}(x)$. The process $\{X_n\}$ in (4.1) is a strictly stationary markov process if and only if $\varepsilon_n's$ in (4.2) are i.i.d. with survival function $\overline{F}(x)$.

For the stationary AR(1) Lehmann process,

$$P(X_n > x, X_{n+1} > y) = \begin{cases} \left(\frac{1}{1+e^{\phi(y)}}\right)^{\gamma} & if \quad x > \phi^{-1}(\phi(y) + \ln p) \\ \left(\frac{1+pe^{\phi(y)}}{(1+e^{\phi(x)})(1+e^{\phi(y)})}\right)^{\gamma} & if \quad x < \phi^{-1}(\phi(y) + \ln p) \end{cases}$$

and

$$P(X_{n+1} > X_n) = \left(\frac{\gamma + p}{\gamma + 1}\right)^{\gamma}. \tag{4.4}$$

Again if $\gamma = 1$ in (4.1) we have the autoregressive model

$$X_n = \begin{cases} \phi^{-1}[\phi(X_{n-1}) - \ln p] & \text{w.p.} \quad p \\ \min[\phi^{-1}(\phi(X_{n-1}) - \ln p), \varepsilon_n] & \text{w.p.} \quad (1\text{-}p) \end{cases}$$

where $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables, ε_n is independent of $X_i's$ $(i < n)$, $0 < p < 1$. This model is given in (1.1)

Now we estimate the parameters p and ϕ of this model where $\phi(x)$ is as defined in (2.1), $\phi(-\infty) = -\infty, \phi(\infty) = \infty$ and $\overline{F}(x) = 1 - F(x)$. Suppose we know a realization $(X_0, X_1, ..., X_N)$ from the above-defined process.

Consider the process $\{U_n\}$ defined by,

$$U_n = \begin{cases} 1 & if \quad X_{n+1} > X_n \\ 0 & if \quad X_{n+1} < X_n \end{cases} \quad , n = 0,1,2,\cdots,N-1. \tag{4.5}$$

We know that $P(X_{n+1} > X_n) = \frac{p+1}{2}$.

Therefore $E(U_n) = P(X_{n+1} > X_n) = \frac{p+1}{2}$.

$$E(U_n^{\,2}) = P(X_{n+1} > X_n) = \frac{p+1}{2}$$

$$V(U_n) = \frac{1-p^2}{4}.$$

Hence an estimator of p is obtained by solving,

$\overline{U}_N = \frac{1}{N}\sum_{i=0}^{N-1} U_i = \frac{\hat{p}+1}{2}$, which yields

$\hat{p} = 2\overline{U}_N - 1.$

Now let us check some of the properties of $\hat{p}$.

$E(\hat{p}) = E\left(2\overline{U}_N - 1\right) = p$ and

$V(\hat{p}) = 4V\left(\overline{U}_N\right) = \frac{1-p^2}{n} \to 0 \quad as \quad n \to \infty.$

Therefore $\hat{p}$ is an unbiased and consistent estimator.

To estimate ϕ, we define the level crossing process $\{Z_n(t)\}$ associated with $\{X_n\}$ by,

$$Z_n(t) = \begin{cases} 1 & if \quad X_n > t \\ 0 & if \quad X_n \le t \end{cases} \quad , n = 0,1,2,\cdots,N$$

Therefore $E\left(Z_n(t)\right) = P\left(X_n > t\right)$

$$= \frac{1}{1+e^{\phi(t)}}$$

Now $\phi(t)$ can be estimated by solving the equation,

$$\overline{Z}_{N+1}(t) = \frac{1}{N+1}\sum_{i=0}^{N} Z_i(t) = \frac{1}{1+e^{\hat{\phi}(t)}}.$$

Hence, the desired estimator is $\hat{\phi}(t) = \ln\left(\frac{1-\overline{Z}_{N+1}(t)}{\overline{Z}_{N+1}(t)}\right)$.

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