Vol.8.Issue.2.2020 (April-June)

http://www.bomsr.com
Email:editorbomsr@gmail.com

RESEARCH ARTICLE

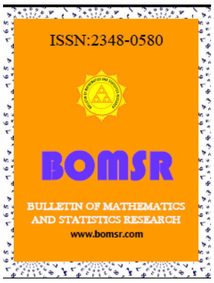
BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

PARAMETER ESTIMATION FOR THE CLASS OF LIFE-TIME DISTRIBUTIONS

ARUN KUMAR RAO[1], HIMANSHU PANDEY[2]
[1,2]Department of Mathematics & Statistics
DDU Gorakhpur University, Gorakhpur, INDIA
E-mail: himanshu_pandey62@yahoo.com
DOI: 10.33329/bomsr.8.2.42



ABSTRACT
A class of life-time distribution is considered. The classical maximum likelihood estimator has been obtained. Bayesian method of estimation is employed in order to estimate the scale parameter of the class of life-time distributions by using quasi and inverted gamma priors. In this paper, the Bayes estimators of the scale parameter have been obtained under squared error and precautionary loss functions.
Keywords: Class of life-time distributions, Bayesian method, quasi and inverted gamma priors, squared error and precautionary loss functions.

1. Introduction

The class of life-time distributions is very imperative concept when we study the reliability of the system. Let us consider the class of life-time distributions whose probability density function is given by

$$f(x;\theta)=\frac{k}{\Gamma(p)}\theta^{-p}x^{kp-1}e^{-\left(x^{k}\right)/\theta} \quad ; x\geq 0,\ \theta>0. \tag{1}$$

(Ahmad et al. [1]).

The following values of the constants k and p are of particular interest.

(i) For k=1, the density given in equation (1) reduces to Gamma distribution.

(ii) For p=1, the density given in equation (1) reduces to Weibull distribution.

(iii) For k=1, p=1, the density given in equation (1) reduces to Negative exponential distribution.

(iv) For k=2, p=1, the density given in equation (1) reduces to Rayleigh distribution.

(v) For k=2, p=3/2, the density given in equation (1) reduces to Maxwell distribution.

(vi) For k=1, with β (positive integer), the density given in equation (1) reduces to Erlang distribution.

2. Classical Method of Estimation

In classical approach, mostly we use the method of maximum likelihood. The alternative approach is the Bayesian approach which was first discovered by Rev. Thomas Bayes. In this approach, parameters are treated as random variables and data is treated as fixed. Recently Bayesian approach to estimation has received great attention by most researchers.

Theorem 1. Let $x_1, x_2, ..., x_n$ be a random sample of size n having probability density function (1), then the maximum likelihood estimator of parameter θ is given by

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^k}{np} \tag{2}$$

Proof. The joint density function or likelihood function of (1) is given by

$$f(\underline{x};\theta) = \left(\frac{k}{\Gamma(p)}\right)^n \theta^{-np} \left(\prod_{i=1}^{n} x_i^{kp-1}\right) e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^k} \tag{3}$$

The log likelihood function is given by

$$log\ f(\underline{x};\theta) = n\,log\,k - n\,log\,\Gamma(p) - np\,log\,\theta + log\prod_{i=1}^{n} x_i^{kp-1} - \frac{1}{\theta}\sum_{i=1}^{n} x_i^k \tag{4}$$

Differentiating (4) with respect to θ and equating to zero, we get

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^k}{np} \tag{5}$$

3. Bayesian Method of Estimation

In Bayesian analysis the fundamental problem are that of the choice of prior distribution g (θ) and a loss function $L\left(\hat{\theta},\theta\right)$. The squared error loss function for the scale parameter θ are defined as

$$L\left(\hat{\theta},\theta\right) = \left(\hat{\theta}-\theta\right)^2 \tag{6}$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, i.e,

$$\hat{\theta}_S = E(\theta) \tag{7}$$

This loss function is often used because it does not lead to extensive numerical computations but several authors (Zellner [2], Basu and Ebrahimi [3]) have recognized that the inappropriateness of using symmetric loss function. J.G.Norstrom [4] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions with quadratic loss function as a special case. A very useful and simple asymmetric precautionary loss function is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}} \quad . \tag{8}$$

The Bayes estimator under precautionary loss function is denoted by $\hat{\theta}_p$ and is obtained by solving the following equation.

$$\hat{\theta}_P=\left[E\left(\theta^2\right)\right]^{1/2} \tag{9}$$

Let us consider two prior distributions of θ to obtain the Bayes estimators.

(i) Quasi-prior: For the situation where the experimenter has no prior information about the parameter θ, one may use the quasi density as given by

$$g_1\left(\theta\right)=\frac{1}{\theta^d}\ ;\theta>0,\ d\geq 0, \tag{10}$$

where *d = 0* leads to a diffuse prior and *d = 1*, a non-informative prior.

(ii) Inverted gamma prior: The most widely used prior distribution of θ is the gamma distribution with parameters α and $\beta\left(>0\right)$ with probability density function given by

$$g_2\left(\theta\right)=\frac{\beta^\alpha}{\Gamma\left(\alpha\right)}\theta^{-(\alpha+1)}e^{-\beta/\theta}\ ;\theta>0. \tag{11}$$

3.1 Bayes Estimators under $g_1\left(\theta\right)$

The posterior density of θ under $g_1\left(\theta\right)$, on using (3), is given by

$$f\left(\theta/\underline{x}\right)=\frac{\left(\frac{k}{\Gamma\left(p\right)}\right)^n\theta^{-np}\left(\prod_{i=1}^n x_i^{kp-1}\right)e^{-\frac{1}{\theta}\sum_{i=1}^n x_i^k}\theta^{-d}}{\int_0^\infty\left(\frac{k}{\Gamma\left(p\right)}\right)^n\theta^{-np}\left(\prod_{i=1}^n x_i^{kp-1}\right)e^{-\frac{1}{\theta}\sum_{i=1}^n x_i^k}\theta^{-d}d\theta}$$

$$= \frac{\theta^{-(np+d)}\, e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^k}}{\int_0^\infty \theta^{-(np+d)}\, e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^k}\, d\theta}$$

$$= \frac{\left(\sum_{i=1}^{n} x_i^k\right)^{np+d-1}}{\Gamma(np+d-1)}\, \theta^{-(np+d)}\, e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^k} \tag{12}$$

Theorem 2. Assuming the squared error loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_S = \frac{\sum_{i=1}^{n} x_i^k}{np+d-2} \tag{13}$$

Proof. From equation (7), on using (12),

$$\hat{\theta}_S = E(\theta) = \int \theta f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\sum_{i=1}^{n} x_i^k\right)^{np+d-1}}{\Gamma(np+d-1)} \int_0^\infty \theta^{-(np+d-1)}\, e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^k}\, d\theta$$

$$= \frac{\left(\sum_{i=1}^{n} x_i^k\right)^{np+d-1}}{\Gamma(np+d-1)} \frac{\Gamma(np+d-2)}{\left(\sum_{i=1}^{n} x_i^k\right)^{np+d-2}}$$

or, $$\hat{\theta}_S = \frac{\sum_{i=1}^{n} x_i^k}{(np+d-2)} .$$

Theorem 3. Assuming theprecautionary loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_P = \left[(np+d-2)(np+d-3)\right]^{-\frac{1}{2}} \sum_{i=1}^{n} x_i^k \tag{14}$$

Proof. From equation (9), on using (12),

$$\left(\hat{\theta}_P\right)^2 = E\left(\theta^2\right) = \int \theta^2 f(\theta/\underline{x})\, d\theta$$

$$=\frac{\left(\sum_{i=1}^{n}x_i^k\right)^{np+d-1}}{\Gamma(np+d-1)}\int_0^\infty \theta^{-(np+d-2)}\,e^{-\frac{1}{\theta}\sum_{i=1}^{n}x_i^k}\,d\theta$$

$$=\frac{\left(\sum_{i=1}^{n}x_i^k\right)^{np+d-1}}{\Gamma(np+d-1)}\frac{\Gamma(np+d-3)}{\left(\sum_{i=1}^{n}x_i^k\right)^{np+d-3}}$$

$$=\frac{\left(\sum_{i=1}^{n}x_i^k\right)^{2}}{(np+d-2)(np+d-3)}$$

$$\Rightarrow \qquad \hat{\theta}_P=\left[(np+d-2)(np+d-3)\right]^{-\frac{1}{2}}\sum_{i=1}^{n}x_i^k$$

3.2 Bayes Estimators under $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using equation (3), is obtained as

$$f(\theta/\underline{x})=\frac{\left(\frac{k}{\Gamma(p)}\right)^n\theta^{-np}\left(\prod_{i=1}^{n}x_i^{kp-1}\right)e^{-\frac{1}{\theta}\sum_{i=1}^{n}x_i^k}\,\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\beta/\theta}}{\int_0^\infty\left(\frac{k}{\Gamma(p)}\right)^n\theta^{-np}\left(\prod_{i=1}^{n}x_i^{kp-1}\right)e^{-\frac{1}{\theta}\sum_{i=1}^{n}x_i^k}\,\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\beta/\theta}d\theta}$$

$$=\frac{\theta^{-(np+\alpha+1)}\,e^{-\frac{1}{\theta}\left(\beta+\sum_{i=1}^{n}x_i^k\right)}}{\int_0^\infty\theta^{-(np+\alpha+1)}\,e^{-\frac{1}{\theta}\left(\beta+\sum_{i=1}^{n}x_i^k\right)}d\theta}$$

$$=\frac{\theta^{-(np+\alpha+1)}\,e^{-\frac{1}{\theta}\left(\beta+\sum_{i=1}^{n}x_i^k\right)}}{\Gamma(np+\alpha)\Big/\left(\beta+\sum_{i=1}^{n}x_i^k\right)^{np+\alpha}}$$

$$=\frac{\left(\beta+\sum_{i=1}^{n}x_i^k\right)^{np+\alpha}}{\Gamma(np+\alpha)}\theta^{-(np+\alpha+1)}\,e^{-\frac{1}{\theta}\left(\beta+\sum_{i=1}^{n}x_i^k\right)} \qquad (15)$$

Theorem 4. Assuming the squared error loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_S = \frac{\beta + \sum_{i=1}^{n} x_i^k}{np + \alpha - 1} \tag{16}$$

Proof. From equation (7), on using (15),

$$\hat{\theta}_S = E(\theta) = \int \theta f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} x_i^k\right)^{np+\alpha}}{\Gamma(np+\alpha)} \int_0^\infty \theta^{-(np+\alpha)}\, e^{-\frac{1}{\theta}\left(\beta + \sum_{i=1}^{n} x_i^k\right)} d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} x_i^k\right)^{np+\alpha}}{\Gamma(np+\alpha)} \frac{\Gamma(np+\alpha-1)}{\left(\beta + \sum_{i=1}^{n} x_i\right)^{np+\alpha-1}}$$

or, $$\hat{\theta}_S = \frac{\beta + \sum_{i=1}^{n} x_i^k}{np + \alpha - 1}.$$

Theorem 5. Assuming theprecautionary loss function, the Bayes estimate of the scale parameter θ, is of the form

$$\hat{\theta}_P = \left[(np+\alpha-1)(np+\alpha-2)\right]^{-\frac{1}{2}} \left(\beta + \sum_{i=1}^{n} x_i^k\right) \tag{17}$$

Proof. From equation (9), on using (15),

$$\left(\hat{\theta}_P\right)^2 = E\left(\theta^2\right) = \int \theta^2 f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} x_i^k\right)^{np+\alpha}}{\Gamma(np+\alpha)} \int_0^\infty \theta^{-(np+\alpha-1)}\, e^{-\frac{1}{\theta}\left(\beta + \sum_{i=1}^{n} x_i^k\right)} d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} x_i^k\right)^{np+\alpha}}{\Gamma(np+\alpha)} \frac{\Gamma(np+\alpha-2)}{\left(\beta + \sum_{i=1}^{n} x_i^k\right)^{np+\alpha-2}}$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} x_i^k\right)^2}{(np+\alpha-1)(np+\alpha-2)}$$

or, $$\hat{\theta}_P = \left[(np+\alpha-1)(np+\alpha-2)\right]^{-\frac{1}{2}} \left(\beta + \sum_{i=1}^{n} x_i^k\right).$$

Conclusion

In this paper, we have obtained a number of estimators of parameter. In equation (6) we have obtained the maximum likelihood estimator of the parameter. In equation (13) and (14) we have obtained the Bayes estimators under squared error and precautionary loss function using quasi prior. In equation (16) and (17) we have obtained the Bayes estimators under squared error and precautionary loss function using inverted gamma prior. In the above equation, it is clear that the Bayes estimators depend upon the parameters of the prior distribution.

References

[1]. Ahmad, S. P., et al., (2016).“Bayesian Estimation for the Class of Life-Time Distributions Under Different Loss Functions,” Statistics and Applications, Vol. 14, Nos. 1&2, pp. 75-91.

[2]. Zellner, A., (1986). “Bayesian estimation and prediction using asymmetric loss functions,” Jour. Amer. Stat. Assoc., 91, 446-451.

[3]. Basu, A. P. and Ebrahimi, N., (1991). “Bayesian approach to life testing and reliability estimation using asymmetric loss function,” Jour. Stat. Plann. Infer., 29, 21-31.

[4]. Norstrom, J. G., (1996). “The use of precautionary loss functions in Risk Analysis,” IEEE Trans. Reliab., 45(3), 400-403.