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IMPROVED SEKI FORMULA FOR ELLIPSE PERIMETER

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ABSTRACT
An exact formula is not yet discovered for the perimeter of the ellipse in terms of its semi-radii. However, approximation formulae are proposed by several eminent mathematicians, including Srinivasa Ramanujan. The quest for simple and more accurate formula goes on even in the 21st century. Some of the formulae are simple, but the simple formulae are found to be, in general, very deficient in accuracy. One of the well-known simple formulae for the perimeter of the standard ellipse is known after the 17th century Japanese Mathematician Takakazu Seki. Though simple, Seki formula overestimates the ellipse perimeter. A proof for his formula could not be traced in Mathematics literature. In this paper, the author derives Seki Formula using Plane Geometry method and introduces an 'Improved Seki Formula'. This improved formula is partly empirical and partly based on the author's discovery of a property of the Asteroid Curve. Moreover, it is simple to calculate and reduces the maximum absolute relative error to the order of 10^{-5} for ellipses with aspect ratio (b/a) ≥ 0.2 and to that of 10^{-4} for other ellipses. Subsequently, the author develops an error function, which, if used as a correction factor, reduces the maximum relative error further, to the order of 10^{-6} for all aspect ratios.
Keywords: Ellipse, Major/Minor Radii, Aspect Ratio, Seki Formula, Error, Correction Factor.

1. Introduction

From Integral Calculus, it is known that, the perimeter P (a, b) of an ellipse of semi-radii *a* and *b,* given by the Cartesian equation $(x/a)^2 + (y/b)^2 = 1$, $(a \geq b \geq 0, a \neq 0)$ is given by:

$$P(a,b) = \int_0^{2\pi} \sqrt{a^2 sin^2\theta + b^2 cos^2\theta}\, d\theta \quad (1)$$

Where ($a \cos\theta$, $b \sin\theta$), $0 \leq \theta < 2\pi$, is a parametric point on the ellipse. As the ellipse is symmetric about its axes, the Quarter Perimeter Q (*a, b*) is given by:

$$Q(a,b) = \int_0^{\pi/2} \sqrt{a^2 sin^2\theta + b^2 cos^2\theta}\, d\theta \quad (2)$$

P (*a, b*) and Q (*a, b*) both are symmetric w.r.t. *a* and *b*. Further, P (*a*, 0) = 4*a*; P (*a, a*) = 2π*a*, and, as the aspect ratio (***b/a***) increases from 0 to 1, P (*a, b*) increases from 4*a* to 2π*a*.

The definite integrals (1) and (2) could not be evaluated so far analytically. However, several eminent mathematicians of yester centuries/years had proposed formulae of varying types as approximations. A few of such formulae can be seen in [2] (Koshy 2020). Recently, based on the partial differential equation satisfied by Q (a, b), the author published an Empirical Formula, applicable for all aspect ratios, which is better in accuracy than many other formulae [3] (Koshy 2019). It is acknowledged that for comparing the accuracy of the formula the author relies up on Simpson's (1/3)-Rule [1] (Kreyszig 2010).

2. Terminology & Notations

- 'Ellipse' means the standard ellipse whose Cartesian equation is $(x/a)^2 + (y/b)^2 = 1$, having *a* and *b* as 'major' and 'minor' radii ($a \neq 0, a \geq b \geq 0$).
- The ratio *(b/a)* is called aspect ratio; obviously *(b/a)* ∈ [0, 1]. If *b/a*= 1, the ellipse is a circle of radius *a* and perimeter $2\pi a$; if *b/a* = 0, the ellipse degenerates to the line segment [-*a, a*] on the x-axis counted twice, and is of length 4*a*.
- An ellipse with aspect ratio less than 0.2 is called a flat ellipse in this paper.
- P (*a, b*) and Q (*a*, *b*) respectively denote the perimeter of the full ellipse and the 'quarter ellipse in the first quadrant'. Obviously, P (*a, b*) = 4* Q (*a*, *b*)
- Named estimates of P (*a, b*) and Q (*a*, *b*) are shown by suffixes. For example, Q_S (*a, b*) denotes the estimate of Q (a, b) by Simpson's one-third Rule, dividing [0, $\frac{\pi}{2}$] into 300 subintervals. We take Q_S (a, b) for comparison, as the deviation (error) in Simpson values is to the order of h^4 or 10^{-10}.
- Q_K and Q_I are respectively the estimates of Q by Seki and Improved Seki formulae.
- Correction Factor (CF) is an estimate of the error in an approximation. In this article, f (b) is introduced as the CF for Q_I (100, *b*) with a = 100 , whereas the C F for general Q_I (*a, b*) is (a/100)*f (b_t), where b_t =100* b/a. Thus, for arbitrary a and *b*, the Corrected estimate of Q_I (a, b) is Q_{CI} (a, b):= Q_I (a, b)$-\left(\frac{a}{100}\right) * f(b_t)$.

3. Results

Result 1: Takakazu Seki's Formula

The perimeter P (a, b) = $4 * \sqrt{(a-b)^2 + \frac{\pi^2}{4}ab}$ (3)

Where *a* and *b* are the semi-radii of the ellipse, and, $0 \leq b \leq a$.

As the derivation of Seki Formula could not be traced in any Mathematics literature, we derive formula (3) before proceeding further.

Proof: Let arc AB be the circular arc of radius *a* in the 1st quadrant (Fig.1).

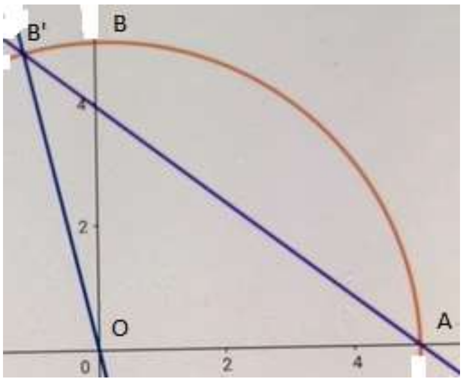

Fig. 1: Stretching the circular arc

Let the radius OB (on the y-axis) be rotated about O to the second quadrant till the arc AB, which is of length $\frac{\pi}{2}$ a, is stretched to the line- segment AB'. Let AÔB' = 2ϕ. Then, from the isosceles $\Delta AO\mathrm{B}'$, it follows that $\sin \varphi = \frac{\pi}{4}$ and cos (2ϕ) = 1- $\frac{\pi^2}{8}$.

Next consider the arc AC of the standard ellipse in the 1st quadrant, where C is on the y-axis and OC = *b* (Fig. 2). Now rotate OC about O in a circle of radius *b* to the second quadrant, stretching the elliptic arc AC to the straight line segment AC'

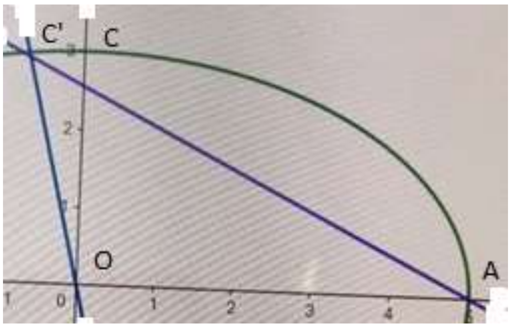

Figure 2: Stretching the elliptic arc

If we assume that C′ is on OB' (Fig. 3), then applying the Cosine Rule to $\Delta AO\mathrm{C}'$,

$(AC')^2 = a^2 + b^{\,2} - 2ab * cos2\varphi$

= $a^2 + b^{\,2} - 2ab * (1 - \frac{\pi^2}{8})$ = $a^2 + b^{\,2} - 2ab + \frac{\pi^2}{4}ab$ = $(a-b)^2 + \frac{\pi^2}{4}ab$.

Hence AC' = $\sqrt{(a-b)^2 + \frac{\pi^2}{4}ab}$. This is known as Takakazu Seki's Formula for

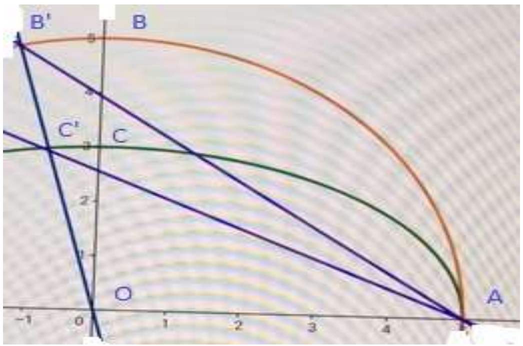

Figure 3: C' on OB'

Ellipse Perimeter. Obviously, the formula gives exact values for the perimeter for *b = a* (circle) and *b = 0*. But for other values of *b*, the Seki values are higher than the actual values as evident from Table 1, which gives values for the integral (2) by Simpson's one-third rule for Numerical Integration. Besides, the relative error grows up to 0.01354. (Table 1; Fig.4)

Table 1: Relative Error in Seki Formula

a	b	Q_S (a, b) by Simpson's one-third Rule	Q_K (a, b) by Seki Formula	Deviation from Simpson values	Relative Deviation /Error
100	100	157.079632679490	157.0796326795	0.0000000000	0.0000000000
100	99	156.295221198759	156.2954602434	0.0002390446	0.0000015294
100	98	155.512803035381	155.5137639653	0.0009609300	0.0000061791
100	97	154.732408602913	154.7345813729	0.0021727700	0.0000140421
100	96	153.954068977126	153.9579506314	0.0038816543	0.0000252131
100	95	153.177815915124	153.1839105539	0.0060946388	0.0000397880
100	94	152.403681875157	152.4125006112	0.0088187360	0.0000578643
100	93	151.631700037168	151.6437609417	0.0120609045	0.0000795408
100	92	150.861904324107	150.8777323614	0.0158280373	0.0001049174
100	91	150.094329424045	150.1144563741	0.0201269500	0.0001340953
100	90	149.329010813121	149.3539751813	0.0249643682	0.0001671769
100	89	148.565984779372	148.5963316924	0.0303469130	0.0002042656
100	88	147.805288447480	147.8415695344	0.0362810869	0.0002454654
100	87	147.046959804474	147.0897330624	0.0427732579	0.0002908816
100	86	146.291037726448	146.3408673691	0.0498296427	0.0003406199
100	85	145.537562006337	145.5950182950	0.0574562886	0.0003947867
100	84	144.786573382793	144.8522324381	0.0656590553	0.0004534886
100	83	144.038113570235	144.1125571637	0.0744435935	0.0005168326
100	82	143.292225290119	143.3760406143	0.0838153242	0.0005849258
100	81	142.548952303487	142.6427317188	0.0937794153	0.0006578752
100	80	141.808339444872	141.9126802022	0.1043407574	0.0007357872
100	79	141.070432657627	141.1859365948	0.1155039372	0.0008187679
100	78	140.335279030736	140.4625522412	0.1272732105	0.0009069224
100	77	139.602926837206	139.7425793096	0.1396524724	0.0010003549
100	76	138.873425574122	139.0260707999	0.1526452258	0.0010991680
100	75	138.146826004434	138.3130805529	0.1662545485	0.0012034627
100	74	137.423180200603	137.6036632580	0.1804830574	0.0013133378
100	73	136.702541590177	136.8978744612	0.1953328710	0.0014288898
100	72	135.984965003428	136.1957705730	0.2108055695	0.0015502123
100	71	135.270506723159	135.4974088753	0.2269021522	0.0016773956
100	70	134.559224536798	134.8028475289	0.2436229921	0.0018105261
100	69	133.851177790929	134.1121455793	0.2609677884	0.0019496862
100	68	133.146427448379	133.4253629632	0.2789355148	0.0020949531
100	67	132.445036148041	132.7425605140	0.2975243660	0.0022463988
100	66	131.747068267573	132.0637999673	0.3167316997	0.0024040892
100	65	131.052589989168	131.3891439647	0.3365539755	0.0025680834
100	64	130.361669368569	130.7186560585	0.3569866899	0.0027384329
100	63	129.674376407550	130.0524007149	0.3780243074	0.0029151812
100	62	128.990783130063	129.3904433167	0.3996601866	0.0030983624
100	61	128.310963662313	128.7328501652	0.4218865029	0.0032880004
100	60	127.634994316991	128.0796884820	0.4446941650	0.0034841085
a	b	Q_S (a, b) by Simpson's one-third Rule	Q_K (a, b) by Seki Formula	Deviation from Simpson values	Relative Deviation /Error

100	59	126.962953681968	127.4310264088	0.4680727269	0.0036866874
100	58	126.294922713731	126.7869330080	0.4920102943	0.0038957251
100	57	125.630984835900	126.1474782608	0.5164934249	0.0041111946
100	56	124.971226043167	125.5127330653	0.5415070221	0.0043330536
100	55	124.315735011060	124.8827692338	0.5670342227	0.0045612426
100	54	123.664603211928	124.2576594881	0.5930562762	0.0047956833
100	53	123.017925037629	123.6374774550	0.6195524174	0.0050362776
100	52	122.375797929393	123.0222976595	0.6464997301	0.0052829051
100	51	121.738322515427	122.4121955174	0.6738730020	0.0055354221
100	50	121.105602756846	121.8072473269	0.7016445701	0.0057936590
100	49	120.477746102595	121.2075302584	0.7297841558	0.0060574187
100	48	119.854863654071	120.6131223429	0.7582586888	0.0063264741
100	47	119.237070340246	120.0241024598	0.7870321196	0.0066005657
100	46	118.624485104170	119.4405503221	0.8160652180	0.0068793995
100	45	118.017231101804	118.8625464611	0.8453153593	0.0071626436
100	44	117.415435914284	118.2901722088	0.8747362946	0.0074499259
100	43	116.819231774770	117.7235096791	0.9042779044	0.0077408308
100	42	116.228755811240	117.1626417470	0.9338859358	0.0080348957
100	41	115.644150306677	116.6076520264	0.9635017198	0.0083316079
100	40	115.065562978324	116.0586248458	0.9930618674	0.0086304003
100	39	114.493147277843	115.5156452220	1.0224979442	0.0089306475
100	38	113.927062714469	114.9787988328	1.0517361183	0.0092316618
100	37	113.367475203505	114.4481719863	1.0806967828	0.0095326881
100	36	112.814557442816	113.9238515895	1.1092941467	0.0098328990
100	35	112.268489320333	113.4059251140	1.1374357936	0.0101313895
100	34	111.729458356001	112.8944805601	1.1650222041	0.0104271713
100	33	111.197660182088	112.3896064185	1.1919462364	0.0107191665
100	32	110.673299066356	111.8913916299	1.2180925635	0.0110062009
100	31	110.156588483269	111.3999255424	1.2433370591	0.0112869968
100	30	109.647751739223	110.9152978665	1.2675461273	0.0115601652
100	29	109.147022658760	110.4375986283	1.2905759696	0.0118241977
100	28	108.654646339879	109.9669181198	1.3122717799	0.0120774566
100	27	108.170879987965	109.5033468472	1.3324668592	0.0123181660
100	26	107.695993839586	109.0469754771	1.3509816375	0.0125444001
100	25	107.230272189460	108.5978947802	1.3676225907	0.0127540718
100	24	106.774014536549	108.1561955722	1.3821810357	0.0129449196
100	23	106.327536868368	107.7219686537	1.3944317853	0.0131144934
100	22	105.891173106713	107.2953047463	1.4041316396	0.0132601387
100	21	105.465276743059	106.8762944276	1.4110176845	0.0133789786
100	20	105.050222698445	106.4650280634	1.4148053650	0.0134678950
100	19	104.646409451062	106.0615957381	1.4151862871	0.0135235054
100	18	104.254261485833	105.6660871826	1.4118256968	0.0135421390
100	17	103.874232134835	105.2785917006	1.4043595658	0.0135198070
100	16	103.506806897050	104.8991980924	1.3923911954	0.0134521703
100	15	103.152507352688	104.5279945776	1.3754872249	0.0133345011
100	14	102.811895824480	104.1650687149	1.3531728904	0.0131616374
100	13	102.485580990886	103.8105073215	1.3249263306	0.0129279292
100	12	102.174224732294	103.4643963899	1.2901716576	0.0126271734

100	11	101.878550604060	103.1268210036	1.2482703995	0.0122525339
100	10	101.599354502522	102.7978652515	1.1985107490	0.0117964406
100	9	101.337518361821	102.4776121416	1.1400937798	0.0112504608
100	8	101.094028165077	102.1661435125	1.0721153474	0.0106051304
100	7	100.869998319400	101.8635399453	0.9935416259	0.0098497238
100	6	100.666705836685	101.5698806742	0.9031748375	0.0089719320
100	5	100.485640478649	101.2852434965	0.7996030178	0.0079573859
100	4	100.328582826687	101.0097046828	0.6811218561	0.0067889114
100	3	100.197736240671	100.7433388869	0.5456026462	0.0054452592
100	2	100.095979045020	100.4862190554	0.3902400104	0.0038986582
100	1	100.027463597804	100.2384163384	0.2109527406	0.0021089482
100	0	100.000000000000	100.0000000000	0.0000000000	0.0000000000

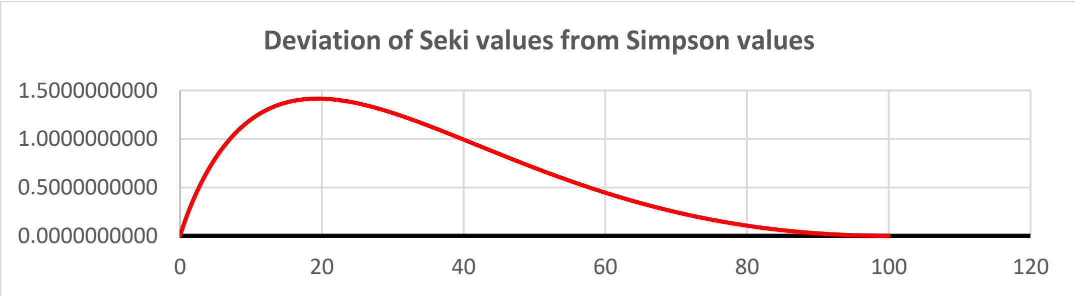

Figure 4: Error curve of Seki values.

It is the wrong assumption, that C′ is on OB', that leads to the inaccuracy of Seki formula. Actually, as the actual values are lower than the Seki Formula values, the elliptic arc AC, when rotated as above, will get stretched to a straight line segment even before reaching OB'.

Result 2: Quarter Perimeter of the Asteroid

As mentioned earlier, the objective of this paper is to improve the accuracy of Seki Formula. For this, we use, without proof, the following well-known result on asteroid. (It is a standard exercise in Integral Calculus).

Let $(x/a)^{2/3} + (y/b)^{2/3} = 1$, $(a \geq b \geq 0, a \neq 0)$ be the asteroid in the 1st quadrant.

Then its arc length is: L (a, b) = (a + b) – ab / (a + b).

In particular, for b = a, L (a, a) = 3a/2.

Result 3

If the 1st quadrant part of the asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ is stretched to a straight line as in Result 1, and 2φ is the angle between the x-axis and the new position of the rotated axis, then $\cos 2\varphi = -1/8$. (The simple proof is omitted.)

Result 4 (New)

Let L (a, b) be the arc length of the asteroid: $(x/a)^{2/3} + (y/b)^{2/3} = 1$, $(a \geq b \geq 0, a \neq 0)$, in the first quadrant.

$$\text{Then L (a, b)} = \sqrt{(a^2 + b^{\,2} - 2ab * cos2\varphi * (GM/AM\,)^2} \quad (4)$$

where GM and AM are respectively the Geometric Mean and the Arithmetic Mean of *a* and *b,* and, 2φ is the angle between the x-axis and the new position of the rotated axis, when the 1st quadrant asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ is stretched to a straight line in the same way as the quarter circle is stretched in figure 1.

Proof

Note that the angle 2φ is related to the case when and b are equal. Let 2ψ be the angle when the arc of the quarter asteroid with $b \neq a$ is stretched. If L (a, b) is the corresponding arc length, then, from Fig. 3,

L^2 (a, b) = $(a^2 + b^2 - 2ab * cos2\psi)$, by cosine rule. But by Result 2,

$L^2(a, b) = [(a + b) - ab/(a + b)]^2 = a^2 + b^2 + (\frac{ab}{a+b})^2$.

Now, $(\frac{ab}{a+b})^2$ = -2ab* (-1/8)* (ab)/ $(\frac{a+b}{2})^2$ = -2ab* $cos2\varphi$ * $(\frac{GM}{AM})^2$

Hence, L^2(a, b) = $a^2 + b^2 - 2ab * cos2\varphi\ (GM/AM)^2$ and the result follows.

Result 5: Improved Seki Formula (New)

$$\text{The Formula: } \sqrt{(a^2 + b^2 + ab * \left(\frac{\pi^2}{4} - 2\right) * (GM/AM)^{2*k}} \quad (5)$$

with k = 0.6521, approximates the quarter perimeter Q (a, b) of the standard ellipse, reducing the maximum absolute relative error to the order of **10^{-5}** for ellipses with aspect ratio b/a ≥ 0.2 and to 10^{-4} for flat ellipses: (0 < b/a < 0.2).

Table 2 gives the Improved Seki Formula values and the relative error for a =100 and b varying from zero to 100.

Table 2: Relative Error in Improved Seki Formula values before and after correction

a	b	Q_S (a, b) by Simpson's 1/3-Rule	Index	Q_I (a, b) by Improved Seki Formula	Relative Error in Q_I (a, b)	Correction Factor f (b) for Q_I (a, b)	Rel. Error in Q_I (a, b) after Correction
100	100	157.07963268	0.6521	157.07963268	0.000000E+00	0.00000000E+00	0.000000E+00
100	99	156.29522120	0.6521	156.29521649	-3.015671E-08	-6.22852871E-05	3.683538E-07
100	98	155.51280304	0.6521	155.51278409	-1.218112E-07	-1.29001746E-04	7.077137E-07
100	97	154.73240860	0.6521	154.73236579	-2.767220E-07	-2.00326722E-04	1.017944E-06
100	96	153.95406898	0.6521	153.95399252	-4.966186E-07	-2.76438357E-04	1.298971E-06
100	95	153.17781592	0.6521	153.17769595	-7.831952E-07	-3.57514971E-04	1.550791E-06
100	94	152.40368188	0.6521	152.40350842	-1.138103E-06	-4.43734395E-04	1.773469E-06
100	93	151.63170004	0.6521	151.63146305	-1.562945E-06	-5.35273209E-04	1.967143E-06
100	92	150.86190432	0.6521	150.86159366	-2.059264E-06	-6.32305922E-04	2.132025E-06
100	91	150.09432942	0.6521	150.09393490	-2.628538E-06	-7.35004064E-04	2.268409E-06
100	90	149.32901081	0.6521	149.32852218	-3.272169E-06	-8.43535183E-04	2.376668E-06
100	89	148.56598478	0.6521	148.56539178	-3.991472E-06	-9.58061765E-04	2.457257E-06
100	88	147.80528845	0.6521	147.80458080	-4.787667E-06	-1.07874004E-03	2.510719E-06
100	87	147.04695980	0.6521	147.04612724	-5.661867E-06	-1.20571872E-03	2.537682E-06
100	86	146.29103773	0.6521	146.29007000	-6.615063E-06	-1.33913755E-03	2.538864E-06
100	85	145.53756201	0.6521	145.53644892	-7.648116E-06	-1.47912584E-03	2.515074E-06
100	84	144.78657338	0.6521	144.78530480	-8.761739E-06	-1.62580085E-03	2.467209E-06
100	83	144.03811357	0.6521	144.03667946	-9.956484E-06	-1.77926596E-03	2.396260E-06
100	82	143.29222529	0.6521	143.29061573	-1.123273E-05	-1.93960886E-03	2.303308E-06
100	81	142.54895230	0.6521	142.54715752	-1.259066E-05	-2.10689948E-03	2.189527E-06
100	80	141.80833944	0.6521	141.80634984	-1.403024E-05	-2.28118785E-03	2.056177E-06
100	79	141.07043266	0.6521	141.06823884	-1.555122E-05	-2.46250173E-03	1.904610E-06
100	78	140.33527903	0.6521	140.33287185	-1.715310E-05	-2.65084421E-03	1.736264E-06
100	77	139.60292684	0.6521	139.60029740	-1.883510E-05	-2.84619106E-03	1.552658E-06
100	76	138.87342557	0.6521	138.87056531	-2.059616E-05	-3.04848789E-03	1.355393E-06

a	b	Q_S (a, b) by Simpson's 1/3-Rule	Index	Q_I (a, b) by Improved Seki Formula	Relative Error in Q_I (a, b)	Correction Factor f (b) for Q_I (a, b)	Rel. Error in Q_I (a, b) after Correction
100	75	138.14682600	0.6521	138.14372669	-2.243491E-05	-3.25764727E-03	1.146144E-06
100	74	137.42318020	0.6521	137.41983400	-2.434962E-05	-3.47354557E-03	9.266561E-07
100	73	136.70254159	0.6521	136.69894109	-2.633821E-05	-3.69601966E-03	6.987394E-07
100	72	135.98496500	0.6521	135.98110327	-2.839822E-05	-3.92486342E-03	4.642600E-07
100	71	135.27050672	0.6521	135.26637735	-3.052676E-05	-4.15982417E-03	2.251336E-07
100	70	134.55922454	0.6521	134.55482169	-3.272049E-05	-4.40059878E-03	-1.668450E-08
100	69	133.85117779	0.6521	133.84649627	-3.497560E-05	-4.64682971E-03	-2.592089E-07
100	68	133.14642745	0.6521	133.14146272	-3.728776E-05	-4.89810087E-03	-5.004352E-07
100	67	132.44503615	0.6521	132.43978442	-3.965210E-05	-5.15393332E-03	-7.383526E-07
100	66	131.74706827	0.6521	131.74152657	-4.206319E-05	-5.41378075E-03	-9.709568E-07
100	65	131.05258999	0.6521	131.04675619	-4.451494E-05	-5.67702495E-03	-1.196265E-06
100	62	128.99078313	0.6521	128.98406998	-5.204365E-05	-6.47977689E-03	-1.809229E-06
100	61	128.31096366	0.6521	128.30395995	-5.458388E-05	-6.74881963E-03	-1.986509E-06
100	60	127.63499432	0.6521	127.62770330	-5.712393E-05	-7.01692017E-03	-2.147474E-06
100	59	126.96295368	0.6521	126.95537993	-5.965324E-05	-7.28292639E-03	-2.290632E-06
100	58	126.29492271	0.6521	126.28707218	-6.216035E-05	-7.54557975E-03	-2.414639E-06
100	57	125.63098484	0.6521	125.62286495	-6.463286E-05	-7.80351039E-03	-2.518324E-06
100	56	124.97122604	0.6521	124.96284580	-6.705741E-05	-8.05523241E-03	-2.600713E-06
100	55	124.31573501	0.6521	124.30710506	-6.941961E-05	-8.29913933E-03	-2.661046E-06
100	54	123.66460321	0.6521	123.65573597	-7.170400E-05	-8.53349991E-03	-2.698802E-06
100	53	123.01792504	0.6521	123.00883475	-7.389403E-05	-8.75645421E-03	-2.713721E-06
100	52	122.37579793	0.6521	122.36650079	-7.597203E-05	-8.96601029E-03	-2.705821E-06
100	51	121.73832252	0.6521	121.72883677	-7.791912E-05	-9.16004130E-03	-2.675425E-06
100	50	121.10560276	0.6521	121.09594879	-7.971525E-05	-9.33628345E-03	-2.623168E-06
100	49	120.47774610	0.6521	120.46794655	-8.133914E-05	-9.49233466E-03	-2.550023E-06
100	48	119.85486365	0.6521	119.84494348	-8.276822E-05	-9.62565432E-03	-2.457305E-06
100	47	119.23707034	0.6521	119.22705696	-8.397872E-05	-9.73356418E-03	-2.346684E-06
100	46	118.62448510	0.6521	118.61440848	-8.494553E-05	-9.81325059E-03	-2.220191E-06
100	44	117.41543591	0.6521	117.40533332	-8.604148E-05	-9.87604648E-03	-1.929488E-06
100	43	116.81923177	0.6521	116.80917198	-8.611419E-05	-9.85289586E-03	-1.771094E-06
100	42	116.22875581	0.6521	116.21877985	-8.583043E-05	-9.78901963E-03	-1.608422E-06
100	41	115.64415031	0.6521	115.63430216	-8.515908E-05	-9.68102609E-03	-1.445152E-06
100	40	115.06556298	0.6521	115.05588965	-8.406796E-05	-9.52544491E-03	-1.285202E-06
100	39	114.49314728	0.6521	114.48369885	-8.252399E-05	-9.31874677E-03	-1.132685E-06
100	38	113.92706271	0.6521	113.91789235	-8.049329E-05	-9.05736721E-03	-9.918355E-07
100	37	113.36747520	0.6521	113.35863919	-7.794138E-05	-8.73773488E-03	-8.669345E-07
100	36	112.81455744	0.6521	112.80611515	-7.483337E-05	-8.35630507E-03	-7.622117E-07
100	35	112.26848932	0.6521	112.26050318	-7.113426E-05	-7.90959891E-03	-6.817356E-07
100	34	111.72945836	0.6521	111.72199380	-6.680923E-05	-7.39424914E-03	-6.292851E-07
100	33	111.19766018	0.6521	111.19078550	-6.182400E-05	-6.80705309E-03	-6.082027E-07
100	32	110.67329907	0.6521	110.66708528	-5.614532E-05	-6.14503378E-03	-6.212312E-07
100	31	110.15658848	0.6521	110.15110913	-4.974148E-05	-5.40551016E-03	-6.703312E-07
100	30	109.64775174	0.6521	109.64308262	-4.258294E-05	-4.58617739E-03	-7.564822E-07
100	29	109.14702266	0.6521	109.14324147	-3.464309E-05	-3.68519842E-03	-8.794689E-07
100	28	108.65464634	0.6521	108.65183229	-2.589907E-05	-2.70130807E-03	-1.037654E-06
100	27	108.17087999	0.6521	108.16911325	-1.633284E-05	-1.63393111E-03	-1.227745E-06
100	26	107.69599384	0.6521	107.69535495	-5.932335E-06	-4.83315668E-04	-1.444558E-06

a	b	Q_S (a, b) by Simpson's 1/3-Rule	Index	Q_I (a, b) by Improved Seki Formula	Relative Error in Q_I (a, b)	Correction Factor f (b) for Q_I (a, b)	Rel. Error in Q_I (a, b) after Correction
100	25	107.23027219	0.6521	107.23084127	5.307127E-06	7.49316113E-04	-1.680789E-06
100	24	106.77401454	0.6521	106.77587040	1.738125E-05	2.06159855E-03	-1.926805E-06
100	23	106.32753687	0.6521	106.33075592	3.027486E-05	3.44983185E-03	-2.170472E-06
100	22	105.89117311	0.6521	105.89582805	4.395969E-05	4.90876833E-03	-2.397036E-06
100	21	105.46527674	0.6521	105.47143506	5.839186E-05	6.43137309E-03	-2.589094E-06
100	20	105.05022270	0.6521	105.05794482	7.350882E-05	8.00855624E-03	-2.726675E-06
100	19	104.64640945	0.6521	104.65574662	8.922592E-05	9.62887414E-03	-2.787502E-06
100	18	104.25426149	0.6521	104.26525325	1.054322E-04	1.12781960E-02	-2.747468E-06
100	17	103.87423213	0.6521	103.88690333	1.219859E-04	1.29393329E-02	-2.581416E-06
100	16	103.50680690	0.6521	103.52116415	1.387083E-04	1.45916250E-02	-2.264326E-06
100	15	103.15250735	0.6521	103.16853494	1.553776E-04	1.62104825E-02	-1.773016E-06
100	14	102.81189582	0.6521	102.82955079	1.717210E-04	1.77668773E-02	-1.088528E-06
100	13	102.48558099	0.6521	102.50478733	1.874053E-04	1.92267779E-02	-1.993927E-07
100	12	102.17422473	0.6521	102.19486660	2.020262E-04	2.05505250E-02	8.939879E-07
100	11	101.87855060	0.6521	101.90046418	2.150951E-04	2.16921389E-02	2.173521E-06
100	10	101.59935450	0.6521	101.62231839	2.260240E-04	2.25985550E-02	3.595830E-06
100	9	101.33751836	0.6521	101.36124219	2.341070E-04	2.32087787E-02	5.082480E-06
100	**8**	**101.09402817**	**0.6521**	**101.11813901**	**2.384992E-04**	2.34529519E-02	6.507703E-06
100	7	100.86999832	0.6521	100.89402468	2.381914E-04	2.32513245E-02	7.683556E-06
100	6	100.66670584	0.6521	100.69005886	2.319836E-04	2.25131194E-02	**8.343437E-06**
100	5	100.48564048	0.6521	100.50759248	2.184591E-04	2.11352828E-02	8.127737E-06
100	4	100.32858283	0.6521	100.34824452	1.959730E-04	1.90011087E-02	6.584243E-06
100	3	100.19773624	0.6521	100.21403812	1.626971E-04	1.59787253E-02	3.225196E-06
100	2	100.09597905	0.6521	100.10767814	1.168787E-04	1.19194306E-02	-2.201289E-06
100	1	100.02746360	0.6521	100.03327106	5.80586839E-05	6.65586451E-03	-8.481687E-06

Result 6: Error Reduction in Improved Seki values by Correction Factor (CF)

A graphical description of the relative error in Improved Seki Formula and its correction is given in Figure 5.

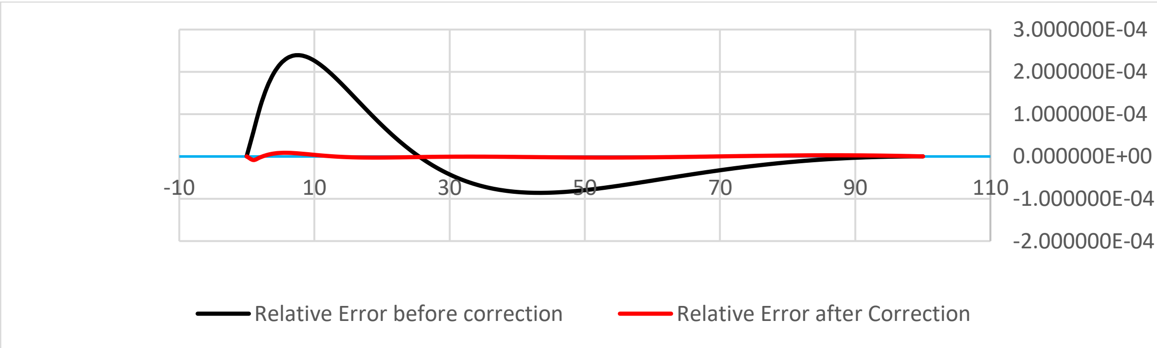

Figure 5: Relative Errors in Improved Seki formula before and after applying the Correction Factor f (b).

For a = 100, the error curve f (b) is a smooth curve, having the value 0 at b = 0 and at b = 100, and, crosses the b-axis between b = 25 and b = 26. It is found by trial that

$$f(b) := b*(b-25.6)*(b-100)*\exp(-b/17)/(3.45*10^5). \quad \textbf{(6)}$$

is a very good estimate of the error in Improved Seki values as the correlation coefficient between them is 0.99963. Hence, if *f (b)* is used as a Correction Factor for Q_I (100, b), then

$$Q_{CI} := Q_I(100, b) - f(b) \quad (7)$$

is a more improved estimate of Q_I (100, b) with maximum absolute relative error 2.787502*10^{-6}, if b/a ≥ 0.11 and 8.481687*10^{-6} if 0 < (b/a) < 0.11. (Table 2).

For arbitrary *a* and *b*, we may take (a/100)*f (b_t) as the Correction Factor, where b_t = 100* b/a.

Then the 'Corrected and Improved Seki Formula' takes the form:

$$Q_{CI}\ (a, b) := Q_I\ (a, b) - (a/100)*f\ (b_t) \quad (8)$$

It can be verified that the maximum relative error, then, is to the order of 10^{-6} *for all* (b/a). **(**Table 3).

Table 3: Illustrative Examples to verify that the Maximum Relative Error in Improved Seki Formula after applying Correction factor is of the order 10^{-6}. (See Introduction for Notations)

a	b	Q_S (a, b) by Simpson's one-third Rule	Q_I (a, b)	Rel. Error in Q_I (a, b)	CF = (a/100)*f (b_t)	Rel. Error in Q_I (a, b) after Correction	b_t = 100*b/a
80	60	110.5174608035	110.514981354	-0.0000224349	-0.0026061178	0.0000011461	75
80	40	96.8844822055	96.8767590344	-0.0000797153	-0.0074690268	-0.0000026232	50
80	30	90.9171569579	90.9099486222	-0.0000792847	-0.0071240442	-0.0000009271	37.5
80	20	85.7842177516	85.7846730193	0.0000053071	0.0005994529	-0.0000016808	25
80	15	83.6378093184	83.6456074240	0.0000932366	0.0080312813	-0.0000027879	18.75
80	10	81.8623915005	81.8783446034	0.0001948771	0.0159267101	0.0000003224	12.5
80	5	80.5724060773	80.5912730934	0.0002341623	0.0182027107	0.0000082448	6.25
80	3	80.2346525883	80.2497960027	0.0001887391	0.0146674321	0.0000059324	3.75
80	2	80.1144027442	80.1257344196	0.0001414437	0.0112709536	0.0000007579	2.5
80	1	80.0329290166	80.0388408920	0.0000738680	0.0064756528	-0.0000070443	1.25

4. Discussion

A formula for the arc length Q (a, b) of the quarter ellipse exactly same as equation (4) is not true. Hence, other possibilities are tried as the power of (GM/AM). It is empirically found that k = 0.6521 is the best suitable value, uniformly applicable across all aspect ratios (b/a), minimizing the error. It is conjectured that '**k'** depends upon *a* and *b* and the problem of finding that function in formula (5) is still open. Such a function will, obviously, provide an exact formula for Ellipse Perimeter. In that sense, Seki Formula is still important.

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