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RESEARCH ARTICLE

# STIRLING'S NUMBERS AND POLYNOMIALS

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**ABSTRACT**

Among several class of interesting numbers that exist in mathematics, Stirling's numbers of first and second kind play significant role since they occur in variety of combinatorial situations and counting problems. In this paper, I formally introduce Stirling's numbers of both kinds and connect them with polynomials. In particular, in this paper, I will prove two important results providing the connection between Stirling's numbers and raising, falling factorial polynomials.

**Keywords**: Stirling's numbers of first and second kinds, Recurrence Relation, Falling Factorial Polynomial, Raising Factorial Polynomial, Mathematical Induction

## 1. Introduction

Stirling's numbers of first and second kinds were named after Scottish mathematician James Stirling. While Stirling's numbers of first kind were connected with number of disjoint cycle factorizations of permutations in symmetric groups, Stirling's numbers of second kind were related to counting number of partitions of a given finite set in to disjoint parts. In this paper, I connect Stirling's numbers with polynomials and prove two important theorems.

## 2. Definitions

**2.1 Stirling's Numbers of First Kind:** The number of permutations in $S_m$ whose disjoint cycle factorizations consists of exactly *n* cycles is defined to be the Stirling's Number of the First Kind denoted by $s(m,n)$ . In $s(m,n)$ we note that $1 \le n \le m$ .

The Stirling's numbers of first kind satisfies the recurrence relation

$$s(m+1,n) = s(m,n-1) + m\,s(m,n) \quad (2.1)$$

Further by definition of Stirling's numbers of first kind, we note that $s(m,1) = (m-1)!$ and $s(m,m) = 1$. For proof of recurrence relation (2.1) and these two results see **[4]** by the corresponding author.

Using (2.1) and other two results, we can form a tabular column containing Stirling's numbers of first kind as shown below:

| $m \backslash n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | |
| 2 | 1 | 1 | | | | | |
| 3 | 2 | 3 | 1 | | | | |
| 4 | 6 | 11 | 6 | 1 | | | |
| 5 | 24 | 50 | 35 | 10 | 1 | | |
| 6 | 120 | 274 | 225 | 85 | 15 | 1 | |
| 7 | 720 | 1764 | 1624 | 735 | 175 | 21 | 1 |

...

**Figure 1: Stirling's numbers of first kind**

**2.2 Stirling's Numbers of Second Kind :** The number of partitions of a set with *m* elements using *n* non-empty disjoint subsets is defined as the Stirling's numbers of second kind denoted by $S(m,n)$ or $\left\{ \begin{matrix} m \\ n \end{matrix} \right\}$. We notice that $1 \le n \le m$.

The Stirling's numbers of second kind satisfies the recurrence relation

$$S(m+1,n) = S(m,n-1) + n\,S(m,n) \quad (2.2)$$

For proof of (2.2),see **[5]** by the corresponding author. Using (2.2) and the fact that $S(m,1) = S(m,m) = 1$, we can form a tabular column containing Stirling's numbers of second kind as shown below:

| $m \backslash n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | |
| 2 | 1 | 1 | | | | | |
| 3 | 1 | 3 | 1 | | | | |
| 4 | 1 | 7 | 6 | 1 | | | |
| 5 | 1 | 15 | 25 | 10 | 1 | | |
| 6 | 1 | 31 | 90 | 65 | 15 | 1 | |
| 7 | 1 | 63 | 301 | 350 | 140 | 21 | 1 |

...

**Figure 2: Stirling's numbers of second kind**

**2.3 Falling Factorial Polynomial :** The Falling Factorial Polynomial is defined by

$$x^{(m)} = x(x-1)(x-2)\cdots(x-(m-2))(x-(m-1)),\ m \geq 0, x^{(0)} = 1 \ (2.3)$$

We notice from (2.3) that if $m \geq 1$ then $x^{(m)}$ is a polynomial in *x* of degree *m* whose roots are 0, 1, 2, . . . , *m* – 2, *m* – 1. For example, $7^{(3)} = 7 \times 6 \times 5 = 210$.

**2.4 Raising Factorial Polynomial :** The Raising Factorial Polynomial is defined by

$$x^{[m]} = x(x+1)(x+2)\cdots(x+(m-2))(x+(m-1)),\ m \geq 0, x^{[0]} = 1 \ (2.4)$$

We observe from (2.4) that if $m \geq 1$ then $x^{[m]}$ is a polynomial in *x* of degree *m*. For example, $7^{[3]} = 7 \times 8 \times 9 = 504$.

**3.** In this section I prove the connection between the raising factorial polynomial and Stirling's numbers of first kind through the following theorem.

**3.1 Theorem 1**

For all integers, $m \geq 1$, $x^{[m]} = \sum_{n=1}^{m} s(m,n)x^n \ (3.1)$ where $x^{[m]}$ is the raising factorial polynomial defined in (2.4) and $s(m,n)$ are Sitrling's numbers of first kind.

**Proof**: The proof is provided using induction on *m*. First we note that if *m* = 1, then $x^{[1]} = x$ and $\sum_{n=1}^{1} s(1,n)x^n = s(1,1)x = x$. Thus (3.1) is true when *m* = 1.

We will assume that the result up to *m* and try to prove for *m* + 1.

Since the result is assumed to be true up to *m*, we have

$$x^{[m]} = s(m,1)x + s(m,2)x^2 + s(m,3)x^3 + \cdots + s(m,m)x^m \ (3.2)$$

From (3.2) we have

$$x \times x^{[m]} = s(m,1)x^2 + s(m,2)x^3 + \cdots + s(m,n-1)x^n + \cdots + s(m,m-1)x^m + s(m,m)x^{m+1} \ (3.3)$$

$$m \times x^{[m]} = ms(m,1)x + ms(m,2)x^2 + \cdots + ms(m,n)x^n + \cdots + ms(m,m)x^m \ (3.4)$$

Now adding (3.3) and (3.4), we get

$$\begin{aligned}(x+m)x^{[m]} &= ms(m,1)x + \left[s(m,1) + ms(m,2)\right]x^2 + \left[s(m,2) + ms(m,3)\right]x^3 + \cdots \\ &+ \left[s(m,n-1) + ms(m,n)\right]x^n + \cdots + \left[s(m,m-1) + ms(m,m)\right]x^m + s(m,m)x^{m+1} \ (3.5)\end{aligned}$$

Now by definition (2.4), we see that

$$(x+m)x^{[m]} = x(x+1)(x+2)\cdots(x+(m-1))(x+m) = x^{[m+1]}$$

Using (2.1), we get $ms(m,1) = m \times (m-1)! = m! = s(m+1,1)$ and

$$s(m,1) + ms(m,2) = s(m+1,2), s(m,2) + ms(m,3) = s(m+1,3), s(m,n-1) + ms(m,n) = s(m+1,n)$$
$$s(m,m-1) + ms(m,m) = s(m+1,m), s(m,m) = 1 = s(m+1,m+1).$$

Substituting these entries in (3.5), we get

$$x^{[m+1]} = s(m+1,1)x + s(m+1,2)x^2 + s(m+1,3)x^3 + \cdots + s(m+1,n)x^n$$

$$+\cdots + s(m+1,m)x^m + s(m+1,m+1)x^{m+1} = \sum_{n=1}^{m+1} s(m+1,n)x^n \quad (3.6)$$

Thus the result is true for *m* + 1 also. Hence by Induction Principle, the result is true for all integers *m* such that $m \geq 1$. This proves (3.1) and completes the proof.

**3.2 Illustrations**

If *m* = 3 then from (2.4) we have $x^{[3]} = x(x+1)(x+2) = 2x + 3x^2 + x^3$. We note that the coefficients of $x^{[3]}$ namely 2, 3, 1 are the third row entries in Figure 1 of Stirling's numbers of first kind for *m* = 3. Hence, $x^{[3]} = 2x + 3x^2 + x^3 = s(3,1)x + s(3,2)x^2 + s(3,3)x^3 = \sum_{n=1}^{3} s(3,n)x^n$

Similarly, if *m* = 4, we have $x^{[4]} = x(x+1)(x+2)(x+3) = 6x + 11x^2 + 6x^3 + x^4$. We note that the coefficients of $x^{[4]}$ namely 6, 11, 6, 1 are the fourth row entries in Figure 1 of Stirling's numbers of first kind for *m* = 4. Hence $x^{[4]} = \sum_{n=1}^{4} s(4,n)x^n$.

**4.** In this section, I prove the connection between the falling factorial polynomial and Stirling's numbers of second kind through the following theorem.

**4.1 Theorem 2**

For all integers, $m \geq 1$, $x^m = \sum_{n=1}^{m} S(m,n)x^{(n)}$ (4.1) where $x^{(n)}$ is the falling factorial polynomial defined in (2.3) and $S(m,n)$ are Sitrling's numbers of second kind.

**Proof**: The proof is provided by Induction on *m*. If m = 1, then $x^m = x$ and $\sum_{n=1}^{1} S(1,n)x^{(n)} = S(1,1)x^{(1)} = 1 \times x = x$. Hence (4.1) is true if *m* = 1. We will assume that the result up to *m* and try to prove for *m* + 1.

Using the fact that the result is true up to *m*, we have

$$x^{m+1} = x^m \times x = \left( \sum_{n=1}^{m} S(m,n)x^{(n)} \right) \times x = \sum_{n=1}^{m} S(m,n)\left(x^{(n)}x\right) \quad (4.2)$$

Since $x = (x-n) + n$, from (2.3) we have

$$x^{(n)}x = x^{(n)}\left((x-n)+n\right) = x^{(n)}(x-n) + nx^{(n)} = x^{(n+1)} + nx^{(n)}$$

Substituting this in (4.2), we get $x^{m+1} = \sum_{n=1}^{m} S(m,n)x^{(n+1)} + \sum_{n=1}^{m} S(m,n)nx^{(n)}$

Altering the summation variable in the first summation and using the fact that $S(m,1) = S(m,m) = 1$ and using (2.2), we get

$$x^{m+1} = \sum_{n=2}^{m+1} S(m,n-1)x^{(n)} + \sum_{n=1}^{m} S(m,n)nx^{(n)}$$

$$= S(m,m)x^{(m+1)} + \sum_{n=2}^{m} S(m,n-1)x^{(n)} + \sum_{n=2}^{m} S(m,n)nx^{(n)} + S(m,1)x^{(1)}$$

$$= x^{(m+1)} + \sum_{n=2}^{m} \left[ S(m,n-1) + nS(m,n) \right] x^{(n)} + x^{(1)}$$

$$= S(m+1,m+1)x^{(m+1)} + \sum_{n=2}^{m} \left[ S(m+1,n) \right] x^{(n)} + S(m+1,1)x^{(1)}$$

$$= \sum_{n=1}^{m+1} S(m+1,n)x^{(n)}$$

Thus the result is also true for *m* + 1. Hence by induction principle, the result is true for all integers *m* such that $m \geq 1$. This proves (4.1) and completes the proof.

We now provide a small illustration justifying (4.1) obtained in theorem 2.

If we consider *m* = 4, then using fourth row of Figure 2 for Stirling's numbers of second kind, we have

$$\begin{aligned}\sum_{n=1}^{4} S(4,n)x^{(n)} &= S(4,1)x^{(1)} + S(4,2)x^{(2)} + S(4,3)x^{(3)} + S(4,4)x^{(4)} \\ &= x + 7x(x-1) + 6x(x-1)(x-2) + x(x-1)(x-2)(x-3) \\ &= x + \left(7x^2 - 7x\right) + \left(6x^3 - 18x^2 + 12x\right) + \left(x^4 - 6x^3 + 11x^2 - 6x\right) \\ &= x^4\end{aligned}$$

This verifies (4.1) when *m* = 4.

**5. Conclusion**

After introducing the Stirling's numbers of first and second kinds, I had proved two important theorems through (3.1) and (4.1). The major idea behind theorem (3.1) is to view the fact that raising factorial polynomial can be expressed as linear combination of ordinary polynomials in which the coefficients turns out to be Stirling's numbers of first kind. In theorem (4.1), we proved the fact that any polynomial is expressed as linear combination of falling factorial polynomials in which the coefficients turns out to be Stirling's numbers of second kind. Thus, the ideas presented in this paper provides the wonderful connection between polynomials, raising and falling factorial polynomials and Stirling's numbers of first and second kinds. These results can further be used to determine the sum of natural numbers and in various other counting problems. Connecting two completely unrelated areas is the key aspect in mathematics and this paper has met that objective by connecting Stirling's numbers with suitable polynomials.

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