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FEW OTHER IDENTITIES OF ROGERS-RAMANUJAN TYPE

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ABSTRACT

In this paper**,** few other identities of Rogers-Ramanujan type modulo (4p+2)s, (12p+6)s where p ≥ 3 and p ≥ 4 respectively and s is any finite positive integer, have been derived by using some transformations of basic hyper geometric series. We derive such identities modulo 18, 54, and 66. Finally, we conclude by a general result which gives identities modulo 42, 54, 66… and so on.

Key words: Basic Hypergeometric Series, q- analogue of Saalschutz Theorem, Jacobi's Triple Product Identity.

1. Introduction

For |q|<1, the q-shifted factorial is defined by

$$(a;q)_0 = 1$$

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \text{ for } n\geq 1$$

and $$(a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^k).$$

It follows that $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$

The multiple q-shifted factorial is defined by

$$(a_1,a_2,....,a_m;q)_n = (a_1;q)_n(a_2;q)_n....(a_m;q)_n$$

$$(a_1,a_2,....,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty....(a_m;q)_\infty.$$

The Basic Hyper geometric Series is

$$ {}_{p+1}\phi_{p+r}\left(\begin{matrix} a_1, a_2, \ldots\ldots, a_{p+1}; q; x \\ b_1, b_2, \ldots\ldots, b_{p+r} \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \ldots (a_{p+1};q)_n x^n (-1)^{nr} q^{\frac{n(n-1)r}{2}}}{(q;q)_n (b_1:q)_n (b_2;q)_n \ldots (b_{p+r};q)_n} $$

The series ${}_{p+1}\phi_{p+r}$ converges for all positive integers r and for all x. For r=0 it converges only when |x|<1.

1.1 The q-analogue of Saalschutz Theorem is

$$ {}_3\Phi_2 \left(\begin{matrix} e, f, q^{-n}; q \\ \frac{aq}{c}, \frac{cefq^{-n}}{a} \end{matrix}\right) = \frac{\left(\frac{aq}{ec}\right)_n \cdot \left(\frac{aq}{cf}\right)_n}{\left(\frac{aq}{c}\right)_n \cdot \left(\frac{aq}{cef}\right)_n}. $$

1.2 We require the following **Jacobi's Triple Product Identity** ***(see [3], 2.2.10 , 2.2.11)***

$(zq^{1/2}, z^{-1}q^{1/2}, q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n^2/2}$, and its corollary

$$ \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{(2k+1)n(n+1)}{2} - in} = \sum_{n=0}^{\infty} (-1)^n q^{\frac{(2k+1)n(n+1)}{2} - in} . (1 - q^{(2n+1)i}) $$

$$ = \prod_{n=0}^{\infty} \left(1 - q^{(2k+1)(n+1)}\right) \left(1 - q^{(2k+1)n+i}\right) \left(1 - q^{(2k+1)(n+1)-i}\right) $$

2. We begin by introducing the following transformations:

$$ {}_{10}\Phi_9 \left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, x, -x, y, -y, q^{-n}, -q^{-n}; q; -\frac{a^3 q^{3+2n}}{bx^2 y^2} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{x}, \frac{-aq}{x}, \frac{aq}{y}, \frac{-aq}{y}, -aq^{n+1}, aq^{n+1} \end{matrix}\right) $$

$$ = \frac{(a^2q^2; q^2)_n \left(\frac{a^2q^2}{x^2y^2}; q^2\right)_n}{\left(\frac{a^2q^2}{x^2}; q^2\right)_n \left(\frac{a^2q^2}{y^2}; q^2\right)_n} \times $$

$$ {}_5\Phi_4 \left(\begin{matrix} x^2, y^2, \frac{-aq}{b}, \frac{-aq^2}{b}, q^{-2n}; q^2; q^2 \\ -aq, -aq^2, \frac{a^2q^2}{b^2}, \frac{x^2y^2}{a^2} q^{-2n} \end{matrix}\right) \quad (2.1) $$

and,

$$ {}_{12}\Phi_{11} \left(\begin{matrix} a, q^3\sqrt{a}, -q^3\sqrt{a}, x, xq, xq^2, y, yq, yq^2, q^{-n}, q^{-n+1}, q^{-n+2}; q^3; \frac{a^4 q^{3+3n}}{x^3 y^3} \\ \sqrt{a}, -\sqrt{a}, \frac{aq^3}{x}, \frac{aq^2}{x}, \frac{aq}{x}, \frac{aq^3}{y}, \frac{aq^2}{y}, \frac{aq}{y}, -aq^{3+n}, aq^{2+n}, aq^{1+n} \end{matrix}\right) $$

$$ = \frac{(aq; q)_n \left(\frac{aq}{xy}; q\right)_n}{\left(\frac{aq}{x}; q\right)_n \left(\frac{aq}{y}; q\right)_n} \times $$

$$ {}_6\Phi_5 \left(\begin{matrix} a^{1/3}, \omega a^{1/3}, \omega^2 a^{1/3}, x, y, q^{-n}; q; q \\ a^{1/2}, -a^{1/2}, (aq)^{1/2}, -(aq)^{1/2}, \frac{xy}{a} q^{-n} \end{matrix}\right) \quad (2.2) $$

(*For proof of the transformations (2.1) and (2.2), (see* [1], *(1.3) and (1.6) respectively*))

The multiple series generalisation of the transformation (2.1) is ***(see*** **[1]*, (4.1))*** given by

For $p \geq 3$,

$$ {}_{2p+4}\Phi_{2p+3}\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, x, -x, y, -y, (c_{p-3}).(d_{p-3}), -q^{-n}, q^{-n}; q; \frac{-a^p q^{p+2n}}{bx^2y^2c_1d_1\ldots c_{p-3}d_{p-3}} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{x}, -\frac{aq}{x}, \frac{aq}{y}, -\frac{aq}{y}, \frac{aq}{c_{p-3}}, \frac{aq}{d_{p-3}}, -aq^{1+n}, aq^{1-n} \end{matrix}\right) $$

$$ = \frac{(a^2q^2;q^2)_n\,(\frac{a^2q^2}{x^2y^2};q^2)_n}{(\frac{a^2q^2}{x^2};q^2)_n\,(\frac{a^2q^2}{y^2};q^2)_n}.\sum_{r_1,r_2,\ldots r_{p-3}\geq 0}\prod_{j=1}^{p-3}\left\{\frac{(\frac{aq}{c_jd_j};q)_{r_j}(c_j;q)_{M_{j-1}}\,(d_j;q)_{M_{j-1}}}{(q;q)_j\,(\frac{aq}{c_j};q)_{M_j}\,(\frac{aq}{d_j};q)_{M_j}}\right\}. $$

$$ \frac{(b;q)_{M_{p-3}}\,(x^2;q^2)_{M_{p-3}}\,(y^2;q^2)_{M_{p-3}}\,(q^{-2n};q^2)_{M_{p-3}}\,q^{M_{p-3}(M_{p-3}+1)/2}}{(-aq;q)_{2M_{p-3}}(\frac{aq}{b};q)_{M_{p-3}}(\frac{x^2y^2}{a^2}q^{-2n};q^2)_{M_{p-3}}} $$

$$ \times(-\frac{aq^2}{b})^{r_{p-3}}(-\frac{a^2q^3}{bc_{p-3}d_{p-3}})^{r_{p-4}}\ldots\ldots(-\frac{a^{p-3}q^{p-2}}{bc_{p-3}d_{p-3}\ldots c_2d_2})^{r_{p-4}}\times $$

$$ {}_5\Phi_4\left(\begin{matrix} x^2q^{2M_{p-3}}, y^2q^{2M_{p-3}}, \frac{-aq^{1+M_{p-3}}}{b}, \frac{-aq^{2+M_{p-3}}}{b}, q^{-2n+2M_{p-3}}; q^2; q^2 \\ -aq^{1+2M_{p-3}}, -aq^{2+2M_{p-3}}, \frac{a^2}{b^2}q^{2+2M_{p-3}}, \frac{x^2y^2}{a^2}q^{-2n+2M_{p-3}} \end{matrix}\right) \qquad (2.3) $$

where $M_i = r_1 + r_2 + \ldots . + r_i$, $M_{-1} = M_0 = 0$ and as usual $(a_{M,n})$ stands for $(n - M + 1)$ symbols $a_M, a_{M+1}, \ldots a_{n.}$. When $M = 1$, we would drop it and write (a_n) instead of writing $(a_{1,n})$. Similarly, the multiple series generalisation of the transformation (2.2) is ***(see*** **[1]*, (4.5))***

given by

For $p \geq 4$,

$$ {}_{2p+4}\Phi_{2p+3}\left(\begin{matrix} a, q^2\sqrt{a}, -q^3\sqrt{a}, x, xq, xq^2, y, yq, yq^2, (c_{p-4}).(d_{p-4}), q^{-n}, q^{-n+1}, q^{-n+2}; q^3; \frac{a^pq^{3p-9+3n}}{bx^2y^2c_1d_1\ldots c_{p-3}d_{p-3}} \\ \sqrt{a}, -\sqrt{a}, \frac{aq^3}{x}, \frac{aq^2}{x}, \frac{aq}{x}, \frac{aq^3}{y}, \frac{aq^2}{y}, \frac{aq}{y}, \frac{aq^3}{c_{p-4}}, \frac{aq^3}{d_{p-4}}, aq^{3+n}, aq^{2+n}, aq^{1+n} \end{matrix}\right) $$

$$ = \frac{(aq;q)_n\,(\frac{aq}{xy};q)_n}{(\frac{aq}{x};q)_n\,(\frac{aq}{y};q)_n}.\sum_{r_1,r_2,\ldots r_{p-4}\geq 0}\prod_{j=1}^{p-4}\left\{\frac{(\frac{aq^3}{c_jd_j};q^3)_{r_j}(c_j;q^3)_{M_{j-1}}(d_j;q^3)_{M_{j-1}}}{(q^3;q^3)_{r_j}(\frac{aq^3}{c_j};q^3)_{M_j}(\frac{aq^3}{d_j};q^3)_{M_j}}\right\}. $$

$$ \frac{(x;q)_{3M_{p-4}}\,(y;q)_{3M_{p-4}}\,(q^{-n};q)_{3M_{p-4}}\,(aq^3;q^3)_{2M_{p-4}}\,q^{3M_{p-4}(M_{p-4}+1)}}{(-aq;q)_{6M_{p-4}}(\frac{xy}{a}q^{-n};q)_{3M_{p-4}}} $$

$$ (a)^{r_{p-4}}\,(\frac{a^2q^3}{c_{p-4}d_{p-4}})^{r_{p-5}}\ldots.(\frac{a^{p-4}q^{3p-15}}{c_{p-4}d_{p-4}\ldots\ldots c_2d_2})^{r_1}. $$

$$ {}_6\Phi_5\left(\begin{matrix} a^{1/3}, q^{2M_{p-4}}, \omega a^{1/3}q^{2M_{p-4}}, \omega^2a^{1/3}q^{2M_{p-4}}, xq^{3M_{p-4}}, yq^{3M_{p-4}}, q^{-n+3M_{p-4}}; q; q \\ a^{1/2}q^{3M_{p-4}}, -a^{1/2}q^{3M_{p-4}}, a^{1/2}q^{\frac{1}{2}+3M_{p-4}}, -a^{1/2}q^{\frac{1}{2}+3M_{p-4}}, \frac{xy}{a}q^{-n+3M_{p-4}} \end{matrix}\right) \qquad (2.4) $$

where ω is a cube root of unity and $M_i = r_1 + r_2 + \ldots . + r_i$ and $M_{-1} = M_0 = 0$

3. ROGERS-RAMANUJAN TYPE IDENTITIES MODULO (4p+2)s : (where $p \geq 3$ and s is any finite positive integer)

In (2.3), taking $n, b, x, y, c_1, d_1, \ldots . c_{p-3}, d_{p-3} \to \infty$ and then replacing q by q^{2s} , we get

$$(a^2q^{4s};q^{4s})_\infty \sum_{n=0}^{\infty} \sum_{r_1,r_2,\ldots r_{p-3}=0}^{\infty} \frac{q^{2s(M_1^2+\cdots+M_{p-4}^2+3M_{p-3}^2+4nM_{p-3}+2n^2)}}{(q^{4s};q^{4s})_n(q^{2s};q^{2s})_{r_1}\ldots(q^{2s};q^{2s})_{r_{p-3}}} \cdot \frac{a^{M_1+\cdots+M_{p-4}+3M_{p-3}+2n}}{(-aq^{2s};q^{2s})_{2n+2M_{p-3}}}$$

$$= \sum_{n=0}^{\infty} \frac{(a;q^{2s})_n\,(1-aq^{4ns})(-1)^n\, a^{pn}\, q^{(2p+1)n^2s-ns}}{(q^{2s};q^{2s})_n\,(1-a)} \quad (3.1)$$

Equation (3.1) for $a = 1, q^{2s}$ yeilds the following identities:

$$\frac{(q^{4s};q^{4s})_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r_1,r_2,\ldots r_{p-3}=0}^{\infty} \frac{q^{2s(M_1^2+\cdots+M_{p-4}^2+3M_{p-3}^2+4nM_{p-3}+2n^2)}}{(q^{4s};q^{4s})_n(q^{2s};q^{2s})_{r_1}\ldots(q^{2s};q^{2s})_{r_{p-3}}(-q^{2s};q^{2s})_{2n+2M_{p-3}}}.$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n\,(1-q^{2sn})q^{(2p+1)n^2s-ns}$$

$$= \prod_{n=1}^{\infty}(1-q^n)^{-1},\ where\ n \not\equiv 0, (2p+2)s, 2ps\ mod\ (4p+2)s \quad (3.2)$$

(on using the Jacobi's Triple Product Identity)

and,

$$\frac{(q^{4s};q^{4s})_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r_1,r_2,\ldots r_{p-3}=0}^{\infty} \frac{q^{2s(M_1^2+\cdots+M_{p-4}^2+3M_{p-3}^2+4nM_{p-3}+2n^2+M_1+\cdots+M_{p-4}+3M_{p-3}+2n)}}{(q^{4s};q^{4s})_n(q^{2s};q^{2s})_{r_1}\ldots(q^{2s};q^{2s})_{r_{p-3}}(-q^{2s};q^{2s})_{2n+2M_{p-3}+1}}.$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n\, q^{(2p+1)n^2s+(2p-1)ns}$$

$$= \prod_{n=1}^{\infty}(1-q^n)^{-1},\ where\ n \not\equiv 0, 2s, 4ps\ mod\ (4p+2)s \quad (3.3)$$

(on using the Jacobi's Triple Product Identity)

4. ROGERS-RAMANUJAN TYPE IDENTITIES MODULO (12p+6)s : (where $p \geq 4$ and s is any finite positive integer)

In (2.4), taking $n, b, x, y, c_1, c_2, \ldots . c_{p-4}, d_{p-4} \to \infty$ and then replacing q by q^{2s} , we get

$$(a^2q^{2s};q^{2s})_\infty \sum_{n=0}^{\infty} \sum_{r_1,r_2,\ldots r_{p-4}=0}^{\infty} \frac{(a;q^{6s})_{n+2M_{p-4}} q^{6s(M_1^2+\cdots+M_{p-5}^2)+24sM_{p-4}^2+12nsM_{p-5}+2n^2s}}{(q^{2s};q^{2s})_n(q^{6s};q^{6s})_{r_1}\ldots(q^{6s};q^{6s})_{r_{p-4}}}. \times$$

$$\frac{a^{n+4M_{p-4}+M_1+\cdots+M_{p-5}}}{(a;q^{2s})_{2n+6M_{p-4}+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(a;q^{6s})_n\,(1-aq^{12ns})(-1)^n\ q^{(6p+3)n^2s-3ns}a^{pn}}{(q^{6s};q^{6s})_n\,(1-a)} \quad (4.1)$$

which for $a = 1, q^{6s}$ yeilds the following identities:

$$\frac{(q^{2s};q^{2s})_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r_1,r_2,\ldots r_{p-4}=0}^{\infty} \frac{(q^{6s};q^{6s})_{n+2M_{p-4}-1}\, q^{6s(M_1^2+\cdots+M_{p-5}^2)+243M_{p-4}^2+2nsM_{p-4}+2n^2s)}}{(q^{2s};q^{2s})_n(q^{6s};q^{6s})_{r_1}\ldots(q^{6s};q^{6s})_{r_{p-4}}(q^{2s};q^{2s})_{2n+6M_{p-4}}}.$$

$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty}(-1)^n\, q^{(6p+3)n^2s-3ns}$

$= \prod_{n=1}^{\infty}(1-q^n)^{-1},\ where\ n\ \not\equiv 0,(6p+6)s,6ps\ mod\ (12p+6)s$ (4.2)

(on using the Jacobi's Triple Product Identity)

and

$$\frac{(q^{2s};q^{2s})_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{r_1,r_2,\ldots r_{p-4}=0}^{\infty} \frac{(q^{6s};q^{6s})_{n+2M_{p-4}}\, q^{6s(M_1^2+\cdots+M_{p-5}^2)+243M_{p-4}^2+2n^2s)}}{(q^{2s};q^{2s})_n(q^{6s};q^{6s})_{r_1}\cdots(q^{6s};q^{6s})_{r_{p-4}}}.$$

$$\times \frac{q^{12nsM_{p-4}+6s(n+4M_{p-4}+M_1+\cdots+M_{p-5}}}{(q^{2s};q^{2s})_{2n+6M_{p-4}+3}}$$

$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty}(-1)^n\, q^{(6p+3)n^2s+(6p-3)ns}$

$= \prod_{n=1}^{\infty}(1-q^n)^{-1},\ where\ n\ \not\equiv 0,12ps,6s\ mod\ (12p+6)s$ (4.3)

(on using the Jacobi's Triple Product Identity)

Again taking $x,y,n \to \infty$, $c_1,d_1 \to 0$, $c_2,c_3,\ldots c_{p-4} \to \infty$, $d_2,d_3,\ldots d_{p-4} \to \infty$ in (2.4) we get (with $d_j = e_jq\ \forall_j = 1,2,\ldots p-4\ and\ d_1 = e_1q = c_1q$), *(see [4], (6), p-394 and its proof)* the following result:

$$(aq;q)_\infty \sum_{r_1=0}^{\infty} \cdots \sum_{r_{p-2}=0}^{\infty} \sum_{k=0}^{\infty} \frac{(aq^3;q^3)_{2M_{p-2}+k-1}(-1)^{3M_{p-2}-M_1}(a)^{M_1}}{(aq;q)_{2k+6M_{p-2}}(q;q)_k(q^3;q^3)_{r_1}\cdots(q^3;q^3)_{r_{p-2}}} \times$$

$$.\, q^{3(M_1^2+\cdots+M_{p-3}^2)-2(M_1+\cdots+M_{p-3})-\frac{3M_1^2+M_1}{2}+12M_{p-2}^2+6kM_{p-2}+3r_{p-3}+\cdots+(3p-9)r_1}$$

$= \sum_k^{\infty} \frac{(a;q^3)_k(aq^6;q^6)_k}{(q^3;q^3)_k(a;q^6)_k}\ (-1)^k a^{k(p-1)} q^{\frac{(6p-15)k^2-3k}{2}}$, for $p \geq 6$ (4.4)

Now, replacing $q\ by\ q^2$ and setting $a = 1$ in (4.4), we get,

$$(q^2;q^2)_\infty \sum_{r_1=0}^{\infty} \cdots \sum_{r_{p-2}=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^6;q^6)_{2M_{p-2}+k-1}(-1)^{3M_{p-2}-M_1}}{(q^2;q^2)_{2k+6M_{p-2}}(q^2;q^2)_k(q^6;q^6)_{r_1}\cdots(q^6;q^6)_{r_{p-2}}} \times$$

$$.\, q^{6(M_1^2+\cdots+M_{p-3}^2)-4(M_1+\cdots+M_{p-3})-3M_1^2-M_1+24M_{p-2}^2+12kM_{p-2}+6[r_{p-3}+\cdots+(p-3)r_1]}$$

$= \sum_k^{\infty} \frac{(q^6;q^6)_{k-1}(q^{12};q^{12})_k}{(q^6;q^6)_k(q^{12};q^{12})_{k-1}}\ (-1)^k q^{(6p-15)k^2-3k}$, for $p \geq 6$ (4.5)

The equation (4.5) is an interesting general result because upon setting $p = 6,7,8,\ldots\ldots$ in this equation, we get the following identities of Rogers-Ramanujan type modulo 42, 54, 66, ...

$$(-q;q)_\infty \sum_{r_1=0}^{\infty} \cdots \sum_{r_4=0}^{\infty} \sum_{k=0}^{\infty}.$$

$$\frac{(q^6;q^6)_{2M_4+k-1}(-1)^{3M_4-M_1} q^{6(M_1^2+M_2^2+M_3^2)-4(M_1+M_2+M_3)-3M_1^2-M_1+24M_4^2+12kM_4+6[r_3+2r_2+3r_1]}}{(q^2;q^2)_{2k+6M_4}(q^2;q^2)_k(q^6;q^6)_{r_1}\cdots(q^6;q^6)_{r_4}}$$

$$= \frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k\,(1+q^{6k})q^{21k^2-3k}$$

$$= \prod_{k=1}^{\infty}(1-q^k)^{-1},\ where\ k\ \not\equiv 0,18,24\ (mod\ 42) \quad (4.6)$$

$$(-q;q)_\infty \sum_{r_1=0}^{\infty}\cdots\sum_{r_5=0}^{\infty}\sum_{k=0}^{\infty}.$$

$$\frac{(q^6;q^6)_{2M_5+k-1}(-1)^{3M_5-M_1}q^{6\left(M_1^2+\cdots+M_4^2\right)-4(M_1+\cdots+M_4)-3M_1^2-M_1+24M_5^2+12kM_5+6[r_4+\cdots+4r_1]}}{(q^2;q^2)_{2k+6M_5}(q^2;q^2)_k(q^6;q^6)_{r_1}\cdots(q^6;q^6)_{r_5}}$$

$$= \frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k\,(1+q^{6k})q^{27k^2-3k}$$

$$= \prod_{k=1}^{\infty}(1-q^k)^{-1},\ where\ k\ \not\equiv 0,24,30\ (mod\ 54) \quad (4.7)$$

$$(-q;q)_\infty \sum_{r_1=0}^{\infty}\cdots\sum_{r_6=0}^{\infty}\sum_{k=0}^{\infty}.$$

$$\frac{(q^6;q^6)_{2M_6+k-1}(-1)^{3M_6-M_1}q^{6\left(M_1^2+\cdots+M_5^2\right)-4(M_1+\cdots+M_5)-3M_1^2-M_1+24M_6^2+12kM_6+6[r_5+\cdots+5r_1]}}{(q^2;q^2)_{2k+6M_6}(q^2;q^2)_k(q^6;q^6)_{r_1}\cdots(q^6;q^6)_{r_6}}$$

$$= \frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k\,(1+q^{6k})q^{33k^2-3k}$$

$$= \prod_{k=1}^{\infty}(1-q^k)^{-1},\ where\ k\ \not\equiv 0,30,36\ (mod\ 66) \quad (4.8)$$

and so on.

5. Some particular cases:

Setting $p=4, s=1$ in (3.2) and (3.3), we get

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\ \sum_{r=0}^{\infty}\frac{q^{6r^2+8nr+4n^2}}{(q^4;q^4)_n(q^2;q^2)_r(-q^2;q^2)_{2n+2r}}. = \prod_{n=1}^{\infty}(1-q^n)^{-1},\quad n\ \not\equiv 0,\pm 10\ (mod\ 18)$$

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\ \sum_{r=0}^{\infty}\frac{q^{6r^2+8nr+4n^2+6r+4n}}{(q^4;q^4)_n(q^2;q^2)_r(-q^2;q^2)_{2n+2r+1}}. = \prod_{n=1}^{\infty}(1-q^n)^{-1},\quad n\ \not\equiv 0,\pm 2\ (mod\ 18)$$

Setting $p=4, s=1$ in (4.2) and (4.3), we get

$$(-q;q)_\infty\sum_{n=0}^{\infty}\ \sum_{r=0}^{\infty}\frac{(q^6;q^6)_{n-1}q^{2n^2}}{(q^2;q^2)_n(q^2;q^2)_{2n}}. = \prod_{n=1}^{\infty}(1-q^n)^{-1},\quad n\ \not\equiv 0,\pm 24\ (mod\ 54)$$

$$(-q;q)_\infty\sum_{n=0}^{\infty}\ \sum_{r=0}^{\infty}\frac{(q^6;q^6)_n q^{2n^2+6n}}{(q^2;q^2)_n(q^2;q^2)_{2n+3}}. = \prod_{n=1}^{\infty}(1-q^n)^{-1},\quad n\ \not\equiv 0,\pm 6\ (mod\ 54)$$

In the same way, many identities can be obtained from (3.2), (3.3),(4.2), (4.3) by choosing diferent values of $p\ and\ s$ ($where\ p \geq 3$ in case of (3.2), (3.3) and $p \geq 4$ in case of (4.2), (4.3) and $s = 1,2,3,\ldots\ldots$).

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