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RESEARCH ARTICLE

SLASH EXPONENTIAL DISTRIBUTION: THEORY AND APPLICATIONS

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ABSTRACT

The present paper introduces heavy tailed generalization of exponential distribution called slash exponential distribution. Slash exponential distribution is the ratio of independent exponential and uniform power function distributions. We derived probability density function, reliability measures and studied various properties. The maximum likelihood estimation procedure is employed to estimate the parameters of the proposed distribution. An algorithm in R package was developed to carry out the estimation. Simulation studies for various choices of parameter values were performed to validate the algorithm. Finally the application of slash exponential distribution to real datasets were illustrated.

Keywords - Exponential distribution, slash distribution, Slash exponential distribution.

1 INTRODUCTION

Kafadar (1982, 1988) proposed the univariate slash distribution which is defined as the resulting distribution of the ratio of a standard normal random variable to an independent uniform random variable. Slash distribution have heavier tails than the normal distribution. Wang and Genton (2006) generalized the univariate slash normal distribution to multivariate slash and skew-slash distributions.

The standard slash normal distribution is obtained as the distribution of the ratio $X = \frac{Y}{U^{1/q}}$, where Y is a standard normal random variable, U is an independent uniform random variable over the interval (0,1) and q > 0. For q = 1, it has the probability density function (pdf),

$$f(x) = \begin{cases} \dfrac{\phi(0) - \phi(x)}{x^2}, & x \neq 0, \\ \dfrac{\phi(0)}{2}, & x = 0, \end{cases}$$

where $\phi(.)$ denotes the probability density function (pdf) of the standard normal distribution. As $q \to \infty$ the slash normal reduce to the normal distribution.

Recently there is lot of interest in developing new slash distributions using well known families of distributions. For example alternative skew-slash distribution in multivariate setting (Olcay Arslan, 2008), skew slash distribution generated by normal kernel (Punathumparambath, 2011), multivariate asymmetric slash Laplace (Punathumparambath, 2012a), skew slash logistic distribution (Punathumparambath and George, 2012b), skew slash distribution generated by Cauchy Kernel (Punathumparambath, 2013), multivariate skew-slash t and skew-slash Cauchy (Punathumparambath, 2012c), Modified slashed-Rayleigh distribution (Iriarte et al., 2018) and Generalized modified slash (Jimmy Reyes et al., 2020).

Many authors studied several generalized forms of exponential distribution for modeling lifetime datasets. In the present work we study slash exponential distribution which is the heavy tailed generalization of exponential distribution. This article is organized as follows. In section 2 standard slash exponential distribution is derived and various properties were explored. In section 3 we derived the two parameter slash exponential distribution, its reliability measures and maximum likelihood estimators for the parameters. Section 4 is devoted to simulation studies. Applications of the proposed distributions to microarray gene expression studies were illustrated in section 5. Finally some concluding remarks are given in section 6.

2 STANDARD SLASH EXPONENTIAL DISTRIBUTION

In this section we introduce standard slash exponential distribution and studied its properties. The standard slash exponential (SE) distribution can be defined as the distribution of the ratio $X = \frac{Y}{U^{1/q}}$, where Y is a standard exponential random variable and U is an independent uniform random variable over the interval (0,1) and q > 0. It is denoted by $X \sim SE(1;\, q)$ or SE (q). Now we define the standard slash exponential variable X.

Definition 2.1 A random variable X denoted by $X \sim SE(1;\, q)$ is said to have a standard slash exponential (SE) distribution if its probability density function is given by

$$g(x,q) = \int_0^1 u^{\frac{1}{q}} f\left(x u^{\frac{1}{q}}\right) du, x \geq 0, q > 0. \quad (2.1)$$

Where f (.) is the pdf of the standard exponential distribution with pdf given, $f(x) = e^{-x}, x \geq 0$.

The cdf of the standard slash exponential variable X can be given by

$$G(x,q) = \int_0^1 F\left(x u^{\frac{1}{q}}\right) du, x \geq 0. \quad (2.2)$$

Where F (.) is the cdf of the standard exponential distribution with pdf given by,

$$F(x) = 1 - e^{-x},\ x \geq 0.$$

For the substitution $v = u^{\frac{1}{q}}$, the pdf and cdf of the standard slash exponential distribution are respectively given by,

$$f(x,q) = q\int_0^1 v^q\, e^{-vx} dv, x \geq 0, q > 0. \quad (2.3)$$

$$F(x,q) = q\int_0^1 v^{q-1}\,(1-e^{-vx})dv = 1 - q\int_0^1 v^{q-1}\, e^{-vx} dv, x \geq 0, q > 0. \quad (2.4)$$

Plot of the pdf and cdf of standard slash exponential distribution for various values of the parameter q is given below in Figure 1.

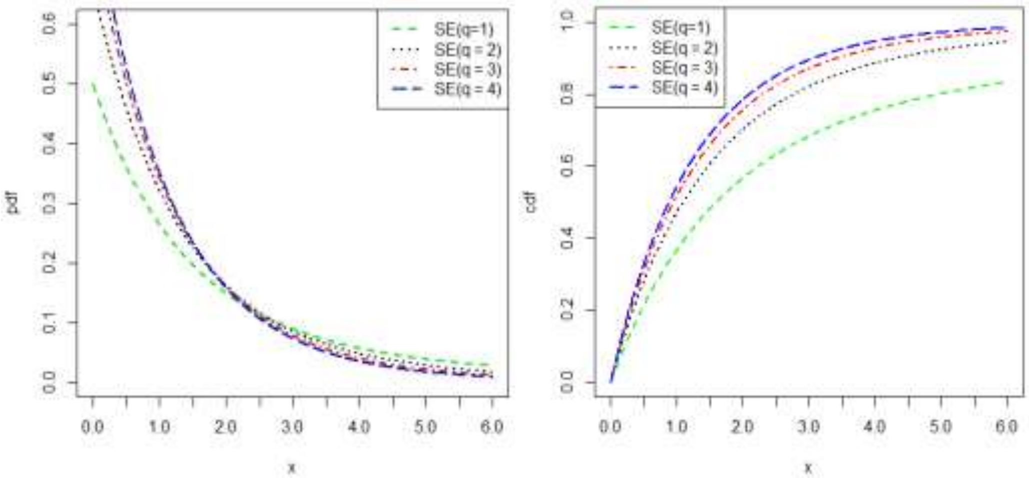

Figure 1. Plots of pdf (left panel) and cdf (right panel) of standard slash exponential for different values of q.

For $q = 1$, we get canonical standard slash exponential distribution and its pdf is given by,

$$f(x,1) = \int_0^1 v\, e^{-x} dv = \frac{1 - e^{-x}(1+x)}{x^2}, x > 0, q > 0. \quad (2.5)$$

The pdf of the univariate canonical slash exponential distribution has the same tail heaviness as the tail of the half Cauchy distribution.

Putting $q = 2$ in equation (2.3) we get the pdf of the standard slash exponential as given below,

$$f(x,2) = \begin{cases} \dfrac{4(1-e^{-x})}{x^3} - \dfrac{2e^{-x}(2-x)}{x^2} & , x \neq 0, q > 0. \\ \dfrac{2}{3}, \quad x = 0. & \end{cases} \quad (2.6)$$

In a similar way closed-form expressions for the pdf can be computed for different values of q.

Now we derive the closed form expression for the pdf in terms of incomplete gamma function. The lower incomplete gamma function and upper incomplete gamma function are respectively given by,

$$\gamma(\alpha, x) = \int_0^x z^{\alpha-1} e^{-z}\ dz, Re(\alpha) > 0,$$

$$\Gamma(\alpha, x) = \int_x^\infty z^{\alpha-1} e^{-z}\ dz, Re(\alpha) > 0.$$

For $\alpha \neq 0, -1, -2, \ldots$ we have $\gamma(\alpha, x) = \Gamma(x) - \Gamma(\alpha, x)$ and $\Gamma(\alpha + 1, x) = \alpha\Gamma(\alpha, x) + x^\alpha e^{-x}$.

Definition 2.2 A random variable X denoted by $X \sim SE(1;\, q)$ is said to have a standard slash exponential (SE) distribution if its probability density function is defined as

$$f(x,1;q) = qx^{-(q+1)}\gamma(q+1,x), x > 0, q > 0. \tag{2.7}$$

Remark 2.1 When q = 1 we obtain the canonical standard slash exponential. The pdfof the SE for q = 1 is given by,

$$f(x,1;1) = x^{-2}\gamma(2,x), x > 0. \tag{2.8}$$

2.1. Properties

Figure 1 shows the density plots of standard SE distribution for various values of q. For standard SE distribution the tail heaviness is controlled by the parameter q and has heavier tail for q = 1. Plots survival and hazard function (sf) for various values of q is given in Figure 2. From Figure 2 we can see that standard SE has heavier tails for small values q. The hazard rate of standard SE is decreasing for lower values of q and remains constant for higher values of q. Hence standard SE can be used to model survival data which has decreasing and constant hazard rates.

Remark 2.2 Note that the SE random variable in (2.1) is a scale mixture of the exponential random variable as X| (U = u), $U \sim U(0,1)$ has standard exponential distribution.

Remark 2.3 The standard SE has heavier tails than the standard exponential distribution.

2.2 Reliability Measures

In this section we derive reliability measures of standard SE distribution. Here we derived expressions for survival function, hazard rate function, reverse hazard function, odds function and mean residual life function for the standard SE.

The survival function (sf) of standard SE is given by

$$S(t) = 1 - F(t) = q\int_0^1 v^{q-1}\, e^{-vx} dv, x \geq 0, q > 0. \tag{2.9}$$

The hazard function (hf) of standard SE is, $h(t) = \frac{f(t)}{S(t)}$ and is given by

$$h(t) = \frac{\int_0^1 v^q\, e^{-vx} dv}{\int_0^1 v^{q-1}\, e^{-vx} dv}, x \geq 0, q > 0. \tag{2.10}$$

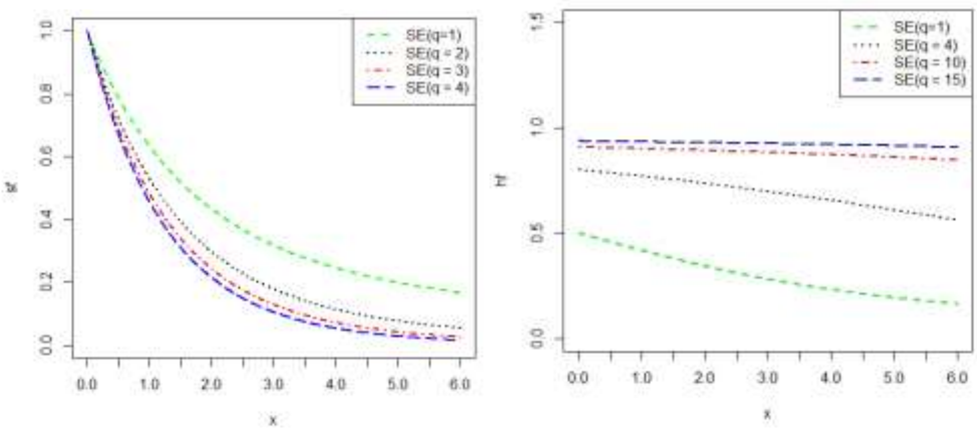

Figure 2. Plots of sf (left panel) and hf (right panel) of standard slash exponential for different values of q.

The reversed hazard function (rhf) of the standard SE distribution $r(t) = \frac{f(t)}{F(t)}$ is given by,

$$r(t) = \frac{q\int_0^1 v^q\, e^{-vx} dv}{1 - q\int_0^1 v^{q-1}\, e^{-vx} dv}, x \geq 0, q > 0. \tag{2.11}$$

The odds function (Of) of the standard SE distribution, $O(t) = \frac{F(t)}{S(t)}$ is given by

$$O(t) = \frac{1 - q\int_0^1 v^{q-1}\, e^{-vx} dv}{q\int_0^1 v^{q-1}\, e^{-vx} dv} x \geq 0, q > 0. \tag{2.12}$$

2.3 Moments

If the random variable X has a standard slash exponential distribution then for $r > 0$, the r^{th} raw moment is given by

$$\mu_r' = E(X^r) = E(Y^r)E\left(U^{-\frac{r}{q}}\right) = \frac{q}{q-r}E(Y^r) = \frac{q}{q-r}r!, \qquad q > r. \tag{2.13}$$

For r=1 we get mean of the standard slash exponential distribution and is given by,

$$E(X) = E(Y)E\left(U^{-\frac{1}{q}}\right) = \frac{q}{q-1}E(Y) = \frac{q}{q-1}, q > 1,$$

and Variance is given by

$$V(X) = \mu_2 = \frac{2q}{q-2} - \left(\frac{q}{q-1}\right)^2, q > 2.$$

2.4 Skewness and Kurtosis

For q > 4, the Coefficient of skewness and Kurtosis of standard SE are given by,

$$\sqrt{\beta_1} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{\mu_2^{3/2}},$$

$$\beta_2 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}{\mu_2^2},$$

Where $\mu_2' = \frac{2q}{q-2}, q > 2$, $\mu_3' = \frac{6q}{q-3}, q > 3$ and $\mu_4' = \frac{24q}{q-4}, q > 4$.

2.5 Moments Generating Function

If the random variable X has a standard slash exponential distribution then the moment generating function is given by,

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}\, f(x)dx = \int_0^1 \phi\left(tu^{-\frac{1}{q}}\right) du,$$

where $\phi(t)$ is the characteristic function of standard exponential distribution and is given by,

$$\phi(t) = \frac{1}{1 - it}, t \in R.$$

2.6 Moment estimate

In this section we study the problem of estimating the unknown parameter, q of standard SE distribution. Let $x_1, x_2, \ldots, x_n$ be independent and identically distributed sample of size n from the standard SE distribution. The moment estimate of q can be obtained by solving the following normal equation,

$$\frac{q}{q-1} = \bar{x}.$$

Then the moment estimate of q is given by,

$$\hat{q} = \frac{\bar{x}}{\bar{x}-1}.$$

3 SLASH EXPONENTIAL DISTRIBUTION

In this section we introduce two parameter slash exponential distribution with parameters (λ, q). Now we define the density of the slash exponential distribution with scale parameter λ and tail parameter q, denoted by $X \sim SE(\lambda;\ q)$.

Definition 3.1. A random variable X denoted by $X \sim SE(\lambda;\ q)$ is said to have a slash exponential (SE) distribution if its probability density function is

$$f(x, \theta, \lambda, q) = \lambda q \int_0^1 v^q\, e^{-\lambda v x} dv, x \geq 0, \lambda, q > 0. \quad (3.1)$$

The cdf of the standard slash exponential variable X can be given by

$$F(t) = 1 - q\lambda \int_0^1 v^{q-1}\, e^{-\lambda v x} dv, x \geq 0, \lambda, q > 0. \quad (3.2)$$

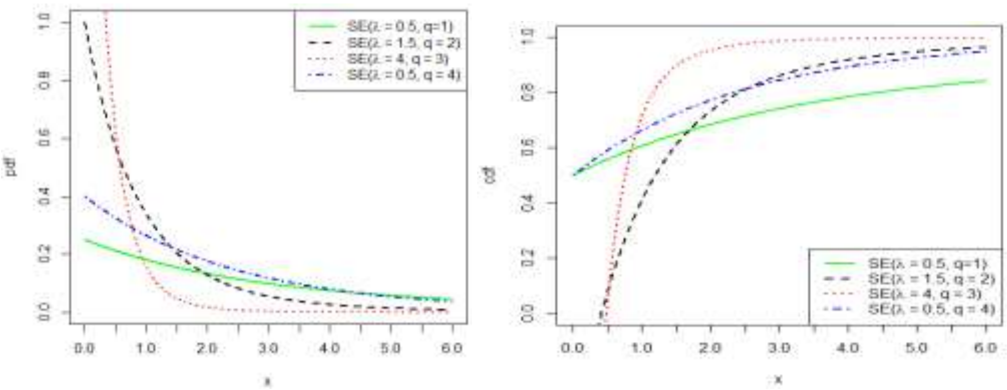

Figure 3. Plots of pdf (left panel) and cdf (right panel) of slash exponential for different values of λ and q.

Plots of pdf and cdf of slash exponential distribution for different values of λ and q is given in Figure 3. From the left panel of Figure 3 we can see that slash exponential distribution has heavier tails compared to exponential distribution. From Figure 4 we can see that the hazard function of the SE model is decreasing for smaller values of the parameter q and remains constant for higher values of q. Now we define the pdf of SE using incomplete gamma function.

Definition 3.2 A random variable X denoted by $X \sim SE(\lambda;\ q)$is said to have a standard slash exponential (SE) distribution if its probability density function is defined as

$$f(x, 1; q) = q\lambda^{-q} x^{-(q+1)} \gamma(q+1, x\lambda), x > 0, \lambda, q > 0. \quad (3.3)$$

The survival function (sf) of SE is given by

$$S(t) = 1 - F(t) = q\lambda \int_0^1 v^{q-1}\, e^{-\lambda v x} dv, x \geq 0, q > 0. \tag{3.4}$$

The hazard function (hf) of SE is, $h(t) = \frac{f(t)}{S(t)}$and is given by

$$h(t) = \frac{\int_0^1 v^q\, e^{-\lambda v x} dv}{\int_0^1 v^{q-1}\, e^{-\lambda v x} dv}, x \geq 0, q > 0. \tag{3.5}$$

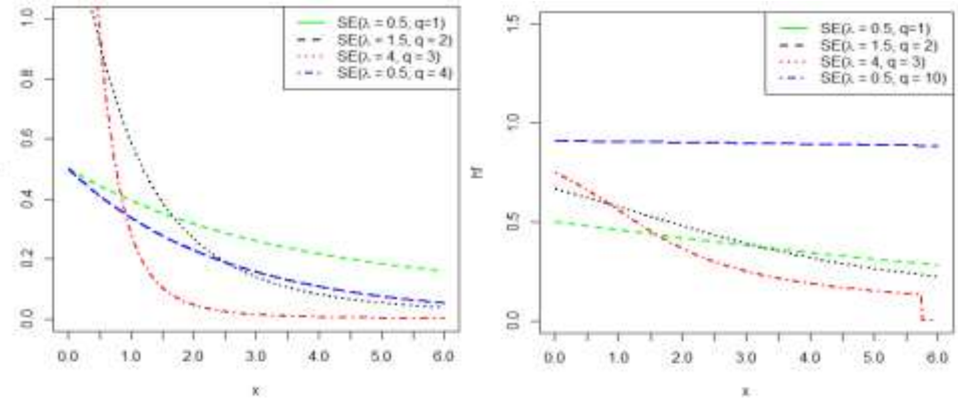

Figure 4. Plots of sf (left panel) and hf (right panel) of slash exponential for different values of λ and q.

The reversed hazard function (rhf) of the SE distribution $r(t) = \frac{f(t)}{F(t)}$ is given by,

$$r(t) = \frac{q\lambda \int_0^1 v^q\, e^{-\lambda v x} dv}{1 - q\lambda \int_0^1 v^{q-1}\, e^{-\lambda v x} dv}, x \geq 0, q > 0. \quad 3.6)$$

The odds function (Of) of the SE distribution, $O(t) = \frac{F(t)}{S(t)}$ is given by

$$O(t) = \frac{1 - q\lambda \int_0^1 v^{q-1}\, e^{-\lambda v x} dv}{q\lambda \int_0^1 v^{q-1}\, e^{-\lambda v x} dv} x \geq 0, q > 0. \tag{3.7}$$

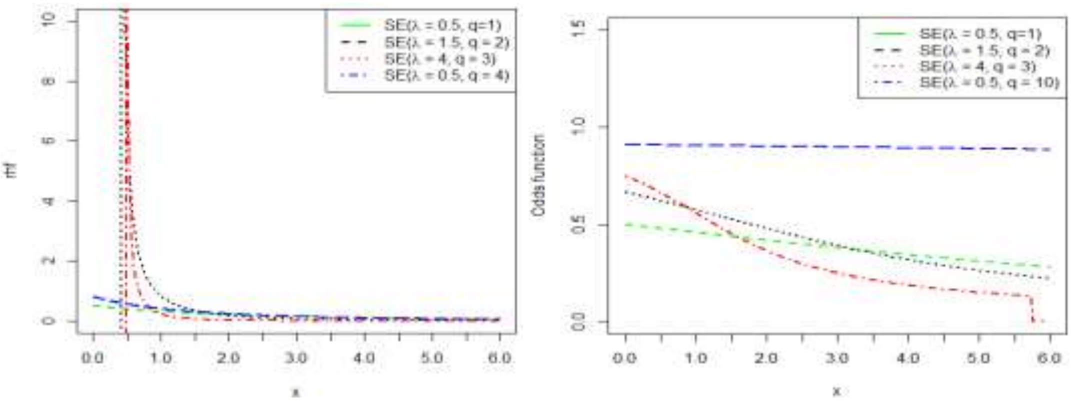

Figure 5.Plots of rhf (left panel) and Of (right panel) of slash exponential for different values of λ and q.

The rhf of the SE distribution is decreasing for smaller values of the parameter q and remains constant for higher values of q. Odds function is constant for larger values of q and is decreasing for different values of λ and smaller values of q.

3.1 Method Moments

In this section we study the problem of estimating two unknown parameters, $\Theta = (\lambda, q)$, of slash exponential distribution. The moment estimates of λ *and* $q$can be obtained by solving the following normal equations,

$$\frac{q}{\lambda(q-1)} = \bar{x}, \tag{3.8}$$

$$\frac{q}{\lambda^2(q-2)} = \frac{\sum_{i=1}^{n} {x_i}^2}{n}, \tag{3.9}$$

From equation (3.8) and (3.9) we get,

$$\hat{q} = \frac{\hat{\lambda}\bar{x}}{\hat{\lambda}\bar{x} - 1}, \tag{3.10}$$

$$\hat{\lambda} = \sqrt{\frac{n}{\sum_{i=1}^{n} {x_i}^2} \frac{\hat{q}}{\hat{q}-2}}, \qquad \hat{q} > 2\,. \tag{3.11}$$

3.2 Maximum Likelihood Estimation

In this section we derive the maximum likelihood estimate of the unknown parameters, λ and q of standard slash exponential distribution. Let $x_1, x_2, \dots, x_n$ be independent and identically distributed sample of size n from slash exponential distribution. The log likelihood function L is given by,

$$\text{L} = \log L(\Theta; X) = n \log\, \lambda + n \log q + \sum_{i=1}^{n} \log(H(x_i)), \tag{3.12}$$

where $H(x_i) = \int_0^1 v^q \, e^{-\lambda v x_i} dv$.

The score equations are given by

$$\frac{dL}{dq} = \frac{n}{q} + \sum_{i=1}^{n} \frac{H_1(x_i)}{H(x_i)} = 0. \tag{3.13}$$

$$\frac{dL}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} \frac{H_2(x_i)}{H(x_i)} = 0. \tag{3.14}$$

Where,

$$H_1(x_i) = \int_0^1 v^q \log_e v \, e^{-\lambda v x_i} dv.$$

$$H_2(x_i) = x_i \int_0^1 v^{q+1} \, e^{-\lambda v x_i} dv.$$

We estimate the unknown parameters using the R statistical software (R Core Team 2020) by maximizing the likelihood function (3.12).

4 SIMULATION

In this section a simulation study is conducted for investigating the performance of the maximum likelihood estimation for parameters $(\lambda, q)$of SE distribution. Using algorithm given below we generated 1000 random samples of sizes *n* = 50, 100 and 250, under the SE model with different parameter values. The algorithm developed in R was used to obtain the MLEs of the parameters.

To generate samples from $X \sim SE(\lambda, q)$ we use the following algorithm:

Step 1. Input number of replications N=1000

Step 2. Give various values for sample size n and parameters λ *and* q.

Step 3. Generate $U \sim U(0,1)$

Step 4. Generate $Y = \frac{-\log(1-U)}{\lambda}, \lambda > 0.$

Step 5. Generate $X = \frac{Y}{U^{1/q}}, q > 0.$

Step 6. Compute MLE's parameters λ *and* q.

Step 7. Repeat the steps 3 to 5, N times.

Step 8. Compute the estimate of the MLE's, and sample standard deviations (SD) over the replications, of the parameters.

The results from 1000 replications are presented in Table 1. It is clear from Table 1 that the estimation algorithm works satisfactorily for various choices of parameters. Also from Table 1 we can see that as sample size increases, estimates become closer to the true parameter values. Further results indicate that estimated standard deviations become smaller as sample size increases.

Table 1: Simulation study - Maximum likelihood estimates of λ and q for various choices of parameters over 1000 replications of datasets of size n = 50, 100, 250. SD stands for the sample standard deviation over 1000 replications.

n	λ	q	$\hat{\lambda}\ (SD)$	$\hat{q}(SD)$
50	0.5	1	0.387(0.351)	1.137 (0.362)
	1.5	2	1.712(0.378))	2.247 (0.473)
	3	4	2.812(0.401)	3.752 (0.637)
100	0.5	1	0.512(0.277)	1.059 (0.149)
	1.5	2	1.401(0.281)	2.215 (0.357)
	3	4	3.127(0.274)	3.874 (0.582)
250	0.5	1	0.489(0.113)	1.011 (0.105)
	1.5	2	1.521(0.126)	2.024 (0.117)
	3	4	3.107(0.154)	4.031 (0.122)

5 APPLICATIONS

In this section we illustrate the application of SE distribution to green (control) and red (test) intensity measurements in cDNA dual dye microarray (Experiment id-38067) downloaded from the Stanford Microarray Database. Each array chip contains approximately 42000 human cDNA elements, representing over 30000 unique genes. Descriptive statistics of the microarray green and red intensity data sets are given below in Table 2.

The maximum likelihood estimates are calculated directly by using optim() function in R software (R Core Team, 2020) by maximizing the likelihood function (3.12). The Histogram and the density function of fitted SE and exponential distributions are presented in Figure 6. For the assessment the goodness of fit of the SE over exponential we have calculated the Akaike's Information Criterion (AIC) (Akaike 1973; Burnham and Anderson 1998), Corrected Akaike information criterion (AICC)(Hurvich and Tsai, 1989), Consistent Akaike information criterion (CAIC) (Bozdogan 1987), Hannan-Quinn information criterion (HQIC) (Hannan and Quinns 1979), and Bayesian Information Criterion (BIC) (Schwarz 1978) for green and red intensity measurements in microarray gene expression. The results are presented in Table 3 and Table 4 respectively. The best model is the one which yield smaller values for these statistic and are considered to provide better fit to the data.

Table 2: Descriptive statistics for green and red intensity measurements in microarray data

	n	Minimum	Median	Mean	Maximum	SD	Skewness	Kurtosis
Green Intensity	43104	0.333	8.319	13.711	244.584	15.735	3.239	29.935
Red intensity	43104	0.765	6.003	12.079	250.618	17.195	3.955	26.665

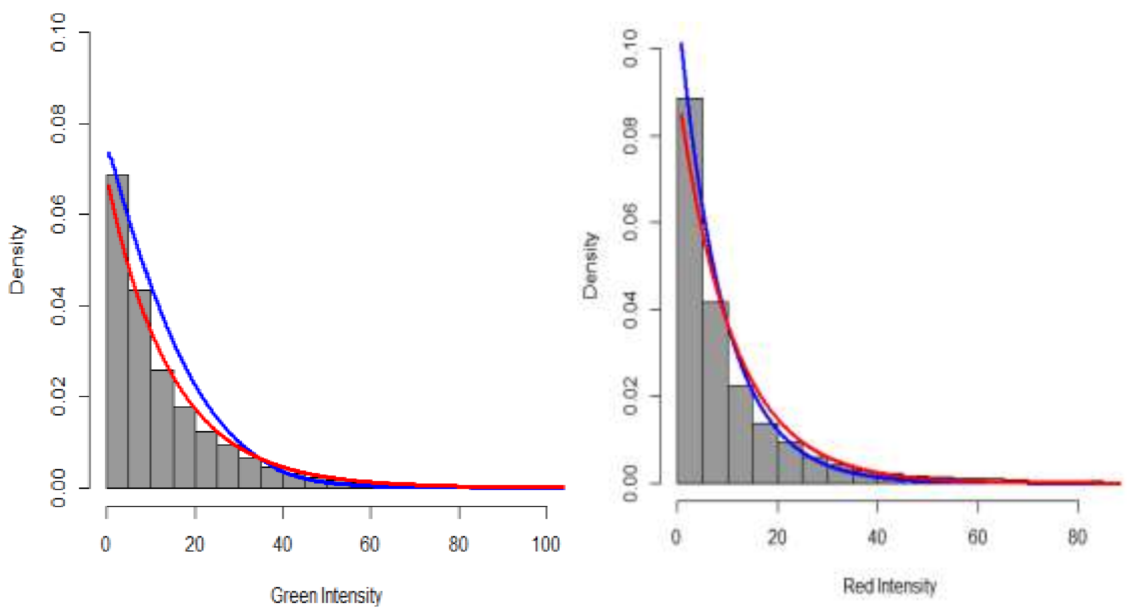

Figure 6. Fitted SE probability density function (red line) and Exponential density function (blue line) to the Green intensity left panel and red intensity right panel for microarray Experiment id 38607

Table 3: Application - maximum likelihood estimates, AIC, AICC, CAIC, HQIC and BIC for SE and exponential distributions for green intensity measurements in microarray dataset 38607

	$\hat{\lambda}$	$\hat{q}$	**AIC**	**AICC**	**CAIC**	**HQIC**	**BIC**
SE	0.078	8.512	3139.496	3139.501	3158.838	3140.231	3156.838
Exponential	0.082	-	3413.034	3413.033	3422.706	3413.402	3421.706

Table 4: Application - maximum likelihood estimates, AIC, AICC, CAIC, HQIC and BIC for SE and exponential distributions for red intensity measurements in microarray dataset 38607

	$\hat{\lambda}$	$\hat{q}$	AIC	AICC	CAIC	HQIC	BIC
SE	0.104	14.445	3509.049	3509.050	3528.392	3509.784	3526.392
Exponential	0.099	-	3867.381	3867.382	3877.052	3867.748	3876.052

From Table 3 and Table 4 we can see that the AIC, AICC, CAIC, HQIC and BIC for the SE had a lower value compared to exponential distribution. A smaller value indicates a better fit, and hence, SE fit the data better than exponential distribution. From Figure 6 it is clear that tail behaviour is better captured in SE than in exponential.

6 CONCLUSIONS

In this paper we introduced the two-parameter slash exponential (SE) distribution. We derived the pdf, cdf, sf, hf, rhf, odd function, moments and measures of skewness and kurtosis. From the plots of SE distribution we can see that SE distribution has heavier tails than exponential. The hazard function of the SE model is decreasing for smaller values of the parameter q and remains constant for higher values of q. For q=1 the slash exponential distribution has heavier tails like half Cauchy distribution. Finally we illustrated the application of slash exponential distribution using real datasets. The application illustrate that the SE distribution provides better fit than exponential distribution. We expect that the model presented in this paper will be usefull in the field of microarray data analysis.

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