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RESEARCH ARTICLE

A DETAILED STUDY OF A NON-LINEAR MECHANICAL OSCILLATOR AND THE EXPLORATION OF CHAOTIC CHARACTERISTICS

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ABSTRACT

Many mechanical systems appear to be very simple and deterministic, but within certain ranges of parameters, systems can exhibit extremely complicated and unpredictable behavior. In this paper, we have considered a simple pendulum subject to damped and sinusoidal driven force as a mechanical system and discussed its oscillations under certain ranges of parameters. We have analyzed the nature of fixed and periodic points, and by means of period doubling phenomenon, we have proved that the oscillations become chaotic as the parameters are varied.
Keywords: Chaos, dynamical system, nonautonomous system, stability, critical points, almost linearity, period doubling.
Mathematics Subject Classification: 37, 37C, 37C05, 37C1.

1. Introduction

Most of the phenomenon taking place in nature is observed to be nonlinear, extremely complex and depending upon so many parameters which are difficult to control. The mathematical modelling of such phenomenon has been a tough challenge before scientists all over the world. In this regard, the theory of difference equations and differential equations are extremely useful. Getting exact solution of such nonlinear equations is quite a big challenge, and most of the times, we use approximations to the solutions using numerical techniques. However, very small errors in the initial conditions can lead to false or very strange conclusions, which are termed as 'Butterfly Effect'. The kind of strange property of nonlinear systems [4] is also termed as *chaos*[1, 5]. In the coming sections we will consider a damped driven pendulum as a mechanical system and study its oscillations for different values of the damping and driving forces.

2. Damped driven Pendulum as a Dynamical System

As an example of a dynamical system[3], we consider a simple pendulum of mass m and length L. With a small initial force, suppose that the pendulum is swinging back and forth under the action of three forces *viz*. its weight mg acting in the downward direction, a damping force and a periodic force. A force of damping can be acted upon the pendulum by immersing it in a medium like air or oil or any other fluid. Let $\theta(t)$ denote the angle made by the pendulum with the normal at time t. Let D denote the damping parameter. Then the damping force acting upon the pendulum, which opposes the motion of the pendulum, is$DL^2\frac{d\theta}{dt}$. The third force acting on the pendulum is the periodic driven force$A_D\sin\omega_D t$, where A_D is the amplitude and ω_D is the angular frequency of the periodic driven force. Then the differential equation of motion of the pendulum, using Newton's laws of motion, is given by

$$mL^2\frac{d^2\theta}{dt^2}+DL^2\frac{d\theta}{dt}+mgL\sin\theta=LA_D\sin\omega_D t \quad (1)$$

The equation (1) is nonlinear because of the term$\sin\theta$. This term has the speciality that it makes the system periodic but at the same time, unpredictable for certain ranges of parameters. This property is referred to as chaos. In order to have a proper analysis of the pendulum system, we reduce the number of parameters by choosing ${\omega_D}^{-1}$ as the unit of time. Also, using the notations$\omega^2=\frac{g}{L}$,$d=\frac{D}{m}$, $a=\frac{A_D}{mL}$, equation (1) can be written as

$$\frac{d^2\theta}{dt^2}+d\frac{d\theta}{dt}+\omega^2\sin\theta=a\sin t \quad (2)$$

As suggested by John R. Taylor[7], in search of chaos, we will choose $\omega=\frac{3}{2}$ and $d=\frac{3}{4}$ and let a vary. We will prove that the pendulum system undergoes the period doubling[8] phenomenon which is one of the main characteristics of chaotic systems. Kulkarni P. R. and Borkar V. C.[9] have proved that varying the amplitude of the driven periodic force $P(t)=A_D\cos\omega_D t$, the oscillations of a damped driven pendulum is chaotic.

3. Nature of Oscillations with Varying Periodic Force

In this section we will study the oscillations of the pendulum under the action of periodic force for different values of a. First suppose that $a=0$. Then in this case, the pendulum system has the differential equation

$$\frac{d^2\theta}{dt^2}+d\frac{d\theta}{dt}+\omega^2\sin\theta=0 \quad (3)$$

Taking

$\theta=x_1$ and $\frac{d\theta}{dt}={x_1}'=x_2$, $\frac{d^2\theta}{dt^2}={x_2}'$,

equation (3) can be written as a system of differential equations[10]

$${x_1}'=x_2, \quad (4)$$

$${x_2}'=-\frac{3}{4}x_2-\frac{9}{4}\sin x_1 \quad (5)$$

The pendulum system is not linear, but it is almost linear [6] at the origin. The auxiliary system[12] is

$${x_1}'=x_2, \quad (6)$$

$${x_2}'=-\frac{3}{4}x_2-\frac{9}{4}x_1 \quad (7)$$

and the associated matrix is $A = \begin{bmatrix} 0 & 1 \\ -2.25 & -0.75 \end{bmatrix}$.

The eigenvalues of the matrix A are $\lambda_1 = -0.3750 + 1.4524i$ and $\lambda_2 = -0.3750 - 1.4524i$. Since the real part of the eigenvalues is -0.375 which is negative, it follows that the critical point $O = (0,0)$ is asymptotically stable and all the solutions tend to $O = (0,0)$ for the auxiliary pendulum system given by equations (6)-(7). Hence $O = (0,0)$ is an asymptotically stable critical point for the pendulum system given by equations (4)-(5).

As the system undergoes harmonic oscillations, the solutions for θ can be obtained by means of numerical methods. The graphs of some of the solutions obtained by using MATLAB for some of the initial conditions are as shown in the Figure 1 and Figure 2.

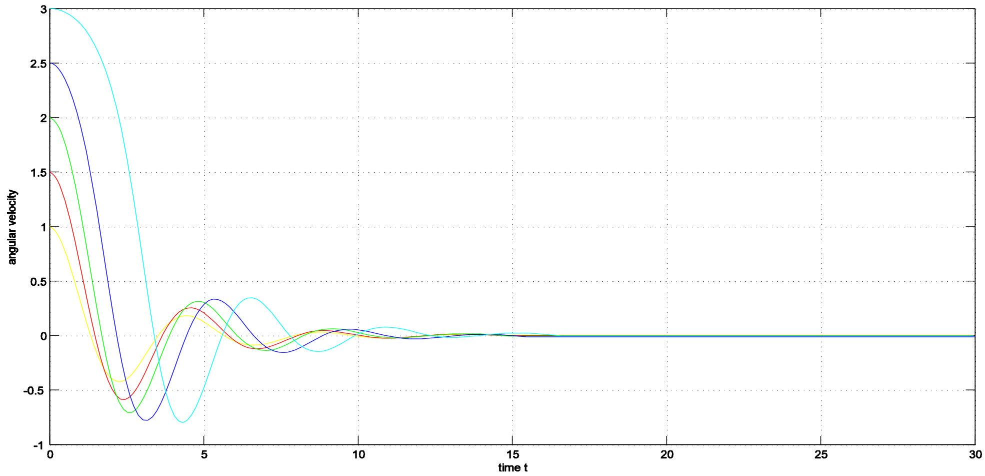

Figure 1: Trajectories in the absence of driven force

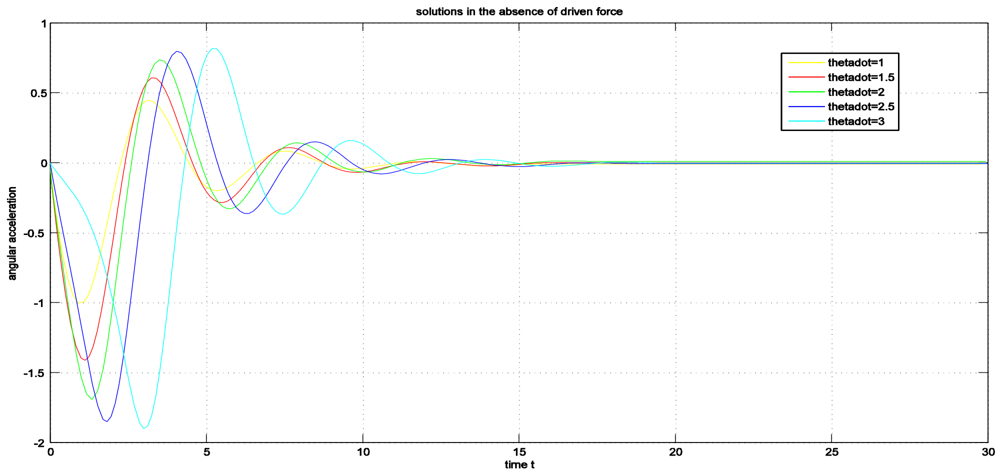

Figure 2: Some trajectories

The phase plain portrait is as shown in the Figure 3. It can be observed that the solution spirals in towards origin.

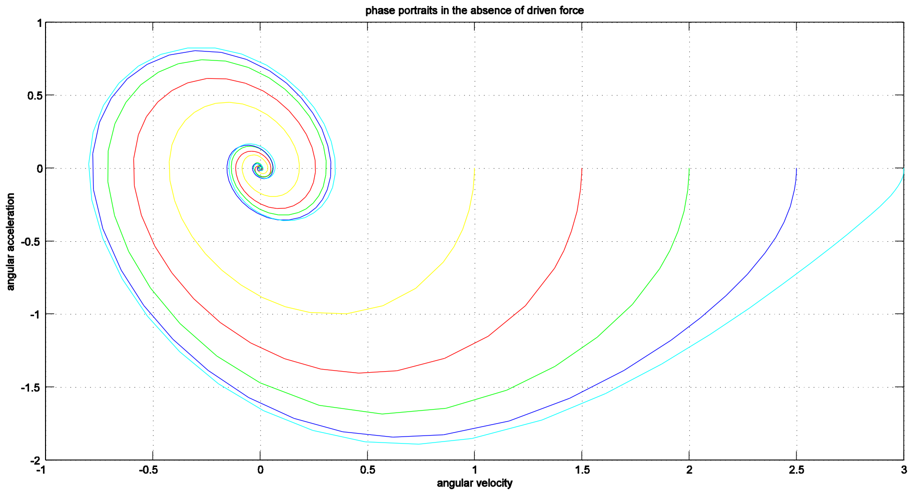

Figure 3: Phase Plain Portraits in the absence of driven force

It can be verified that there are other critical points of the system (6)-(7) which are given by $X_n = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}$, where n is an integer. In the next theorem, we will prove that the pendulum system (6)-(7) is almost linear at $X_n = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}$ for each integer n.

We will prove that the pendulum system

$$x_1{}' = x_2,$$

$$x_2{}' = -\frac{3}{4}x_2 - \frac{9}{4}\sin x_1$$

is almost linear at $X_n = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}$ for each integer n.

Changing the variables defined by $x_1 = n\pi + \mu$ and $x_2 = \nu$, the system $x_1{}' = x_2$, $x_2{}' = -\frac{3}{4}x_2 - \frac{9}{4}\sin x_1$ takes the form $\mu' = \nu$ and $\nu' = -\frac{9}{4}\sin(n\pi + \mu) - \frac{3}{4}\nu$. Of these equations, the first equation is linear and the second can be expressed as

$\nu' = -\frac{9}{4}\sin\mu - \frac{3}{4}\nu$ if n is even, and $\nu' = \frac{9}{4}\sin\mu - \frac{3}{4}\nu$ if n is odd.

In case n is even, the equation $\nu' = -\frac{9}{4}\sin\mu - \frac{3}{4}\nu$ is same as the equation (7), which we have shown that it is almost linear at the origin. By a similar argument, it can be easily verified that if n is odd, the equation $\nu' = \frac{9}{4}\sin\mu - \frac{3}{4}\nu$ is almost linear.

Now let us analyse the nature of the critical point $X_n = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}$ for each integer n. Whenever n is an even integer, the pendulum system (4)-(5) is asymptotically stable. In this case, if $x_1(0) = n\pi$ and $x_2(0) = 0$, the bob of the pendulum is at rest pointed in the downward direction and the pendulum is in the vertical position.

The behaviour of the pendulum system is a little bit different when n is odd. considering $n = 1$, and as shown in the theorem 1, the auxiliary system of the system (4)-(5) is given by

$$\mu' = \nu \qquad (8)$$

$$\nu' = \frac{9}{4}\mu - \frac{3}{4}\nu. \qquad (9)$$

The associated matrix $B = \begin{bmatrix} 0 & 1 \\ 2.25 & -0.75 \end{bmatrix}$ has eigenvalues 1.1712and -1.9212. Since the eigenvalues are real, distinct and have opposite signs, it follows that origin is an unstable critical point of the system (8)-(9), and hence the critical point $X_1 = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$ is an unstable critical point of the system (8)-(9). It is clear that when $x_1 = \theta = \pi$, the pendulum is standing upright with bob above the anchor in a vertical line, which is an unstable position. The same argument applies for all odd integer values of n.

Now suppose that $a = 0.4$. Then in this case, the pendulum system has the differential equation

$$\frac{d^2\theta}{dt^2} + d\frac{d\theta}{dt} + \omega^2 \sin\theta = 0.3 \sin t$$

Taking

$\theta = x_1$ and $\frac{d\theta}{dt} = x_1{}' = x_2$, $\frac{d^2\theta}{dt^2} = x_2{}'$,

equation (2) can be written as a system of differential equations

$$x_1{}' = x_2,$$
$$x_2' = -\frac{3}{4}x_2 - \frac{9}{4}\sin x_1 + 0.3\sin(t)$$

Solving these equations numerically with the initial conditions $\theta = x_1 = 1.6$ and $\frac{d\theta}{dt} = x_1' = x_2 = 0$ for the time limit $t = 0$ to $t = 100$, a solution curve is as shown by the Figure 4.

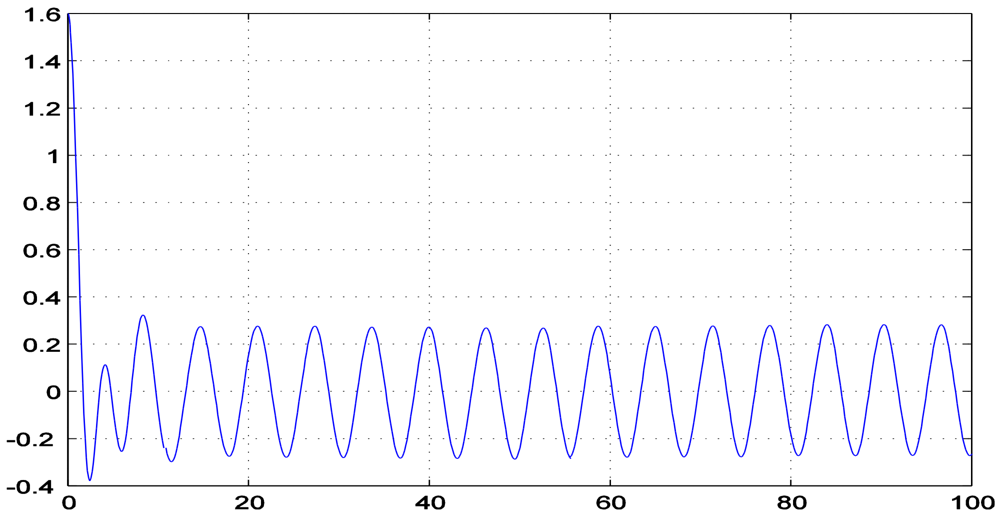

Figure 4: A trajectory for $a = 0.4$

It can be observed that after initial transient, the solution is harmonic with the period of oscillations 6.33 which is approximately equal to the period 2π of the driving force. An another way of observing the period of the solutions is by means of the phase space portrait in which $\theta = x_1$ is plotted against $\frac{d\theta}{dt} = x_1' = x_2$. Figure 5 shows the phase plane portrait[11] obtained by MATLAB.

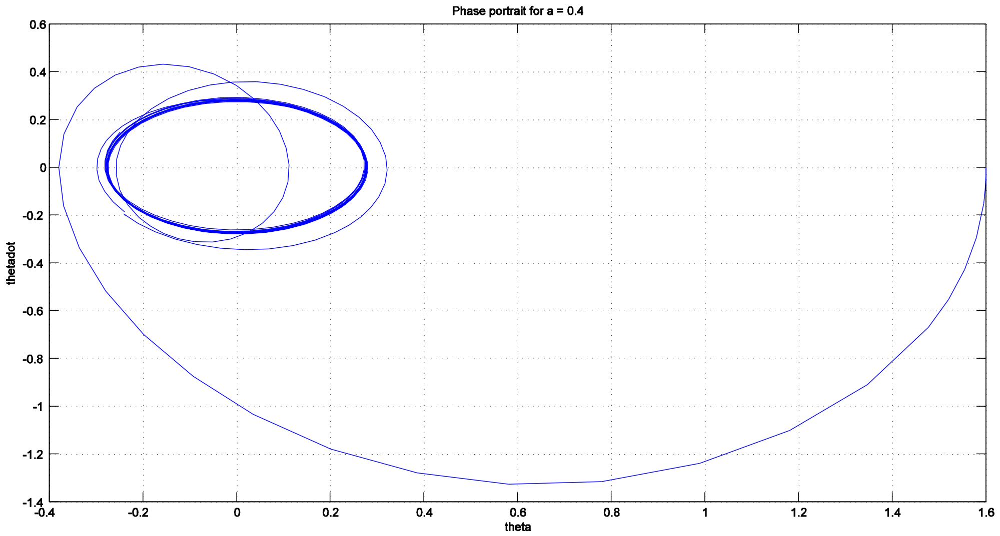

Figure 5: Phase Portrait for $a = 0.4$

After transient decay, there is a closed orbit with period 1 which is shown by means of the darkest lines. A much more clear insight on the phase plane portrait generated by mathematical software shown by the Figure 6, where a closed orbit is observed.

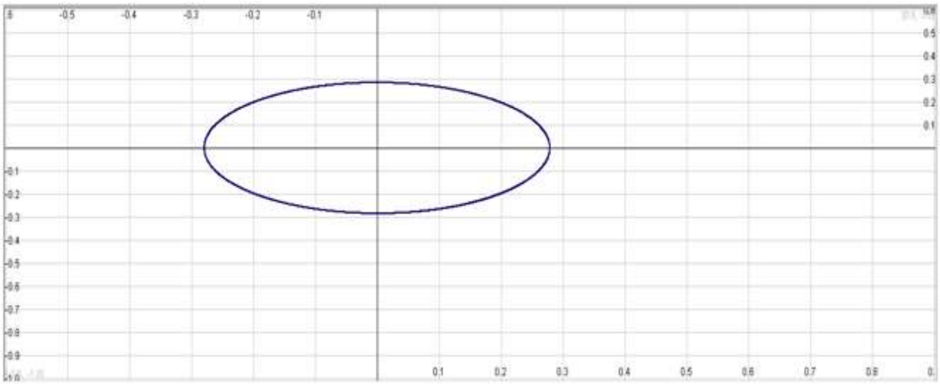

Figure 6: Another Phase Portrait for $a = 0.4$

As the phase portrait is a three dimensional figure, it becomes difficult to visulaize and understand completely when it is a little overcrouded. In such cases, Poincare sections[13] are generally used. In a Poincare section we plot $\theta(t)$ against $\frac{d\theta}{dt}$ whenever t is an integer multiple of 2π.The Poincare section for $a = 0.4$ is as shown in the Figure 7.

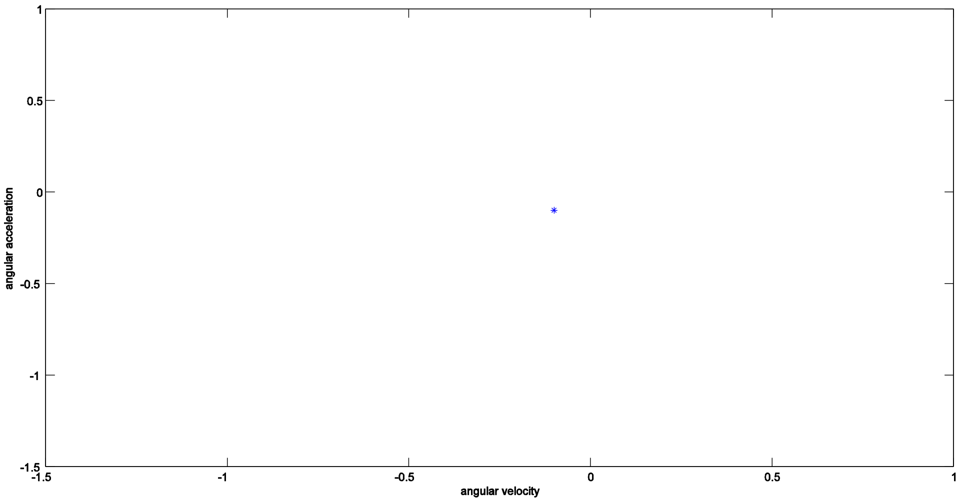

Figure 7: Poincare section for $a = 0.4$

From Figure 7, we can observe that there is a single dot which confirms the existence of a closed periodic orbit with period one.

We will keep varying the values of the parameter a in search of the chaos. For $a = 2.4$, solutions are observed to be periodic with period equal to 2.2π, which is two times the period 2π of the driven force Hence a period doubling of the cycles is noted. A solution curve in this case is as shown by the Figure 8, obtained by MATLAB. As it is a little difficult to observe the period doubling in this figure, we have taken a zoom in picture of this figure as shown by Figure 9.

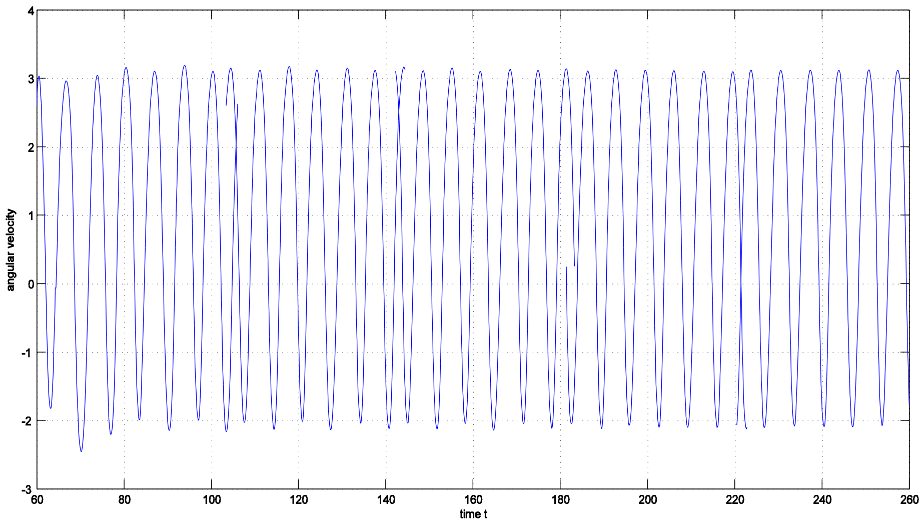

Figure 8: A solution curve for $a = 2.4$

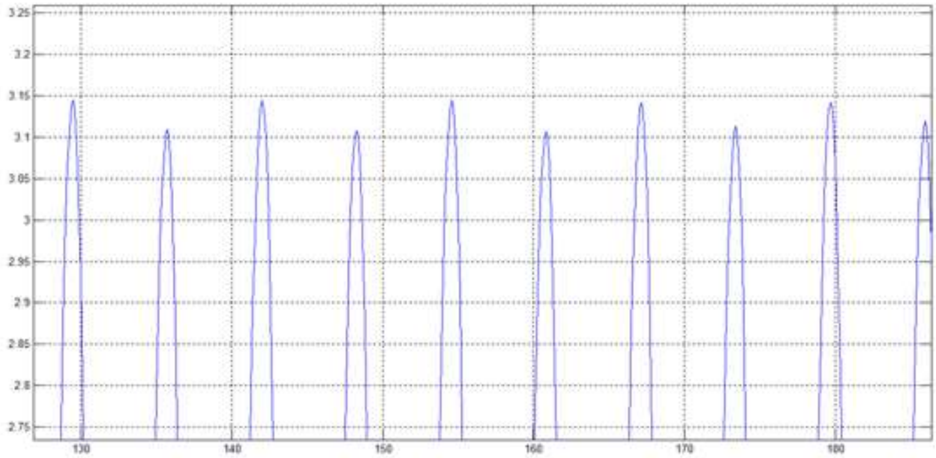

Figure 9: A zoom in on the solution curve for $a = 2.4$

Figure 10 is the phase portrait where periodic cycle of period two is observed. The same is viewed clearly in the Figure 11, where we can see a closed loop. In this case the trajectories can be seen to cross each other.

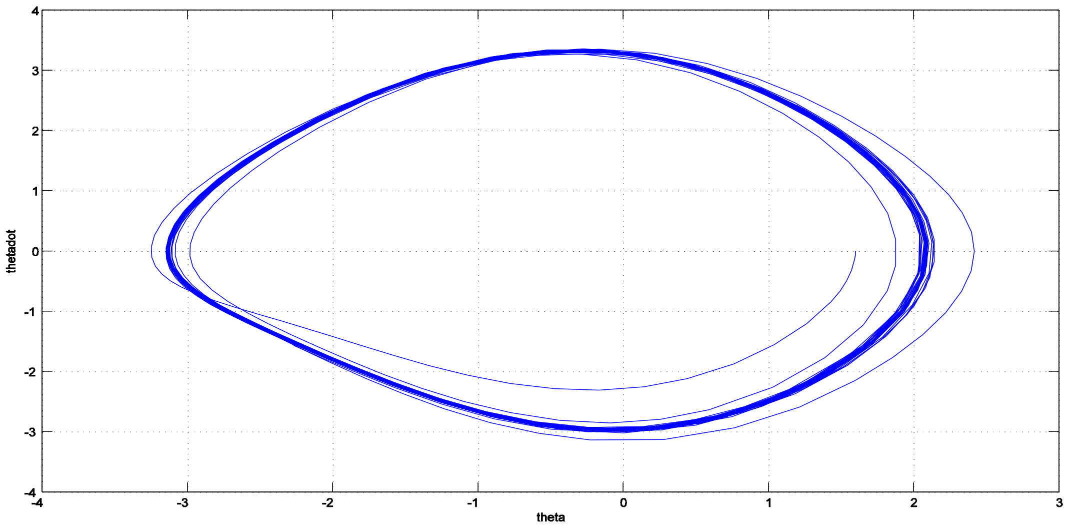

Figure 10: A phase portrait with darkest lines for $a = 2.4$

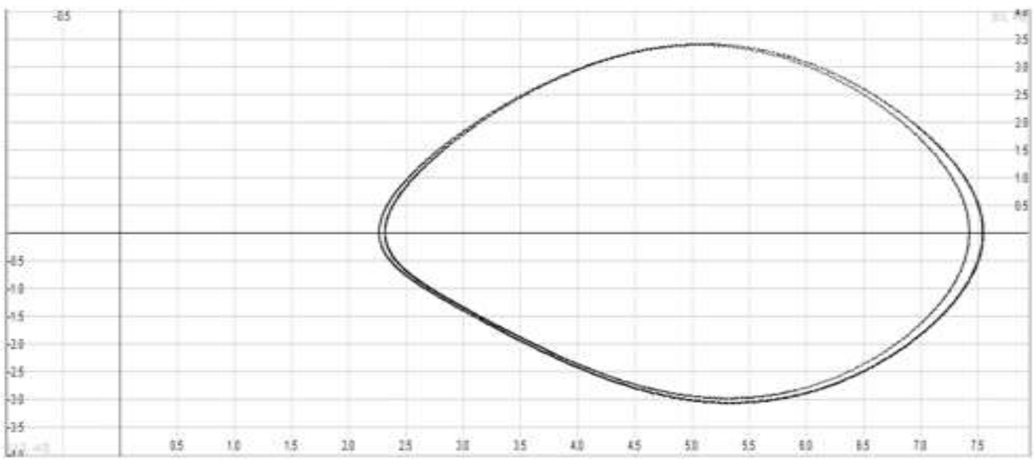

Figure 11: A phase portrait showing two closed loops for $a = 2.4$

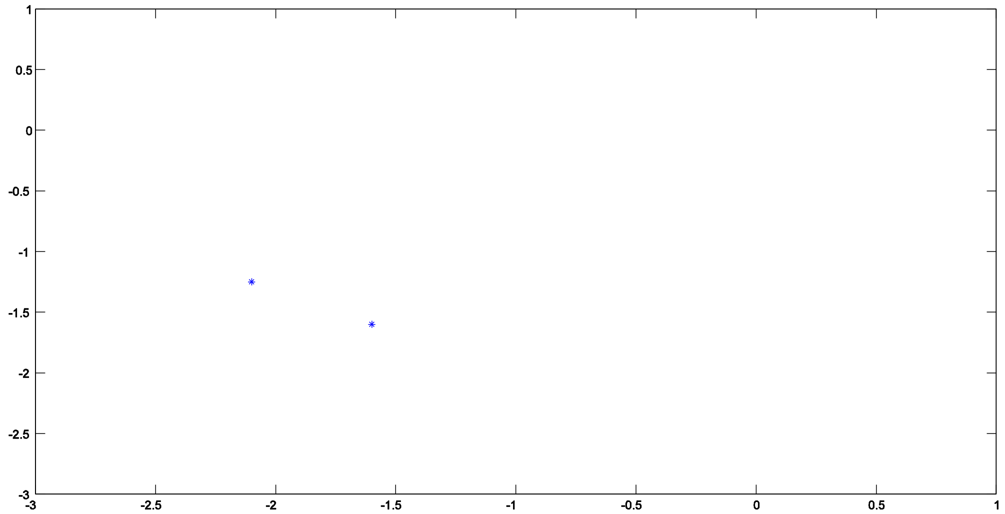

Figure 12: Poincare section showing two dots

Figure 12 is the Poincare section, where it can be observed that there are two points. With each period doubling, the number of points in the Poincare section is seen to be doubled. As we continue changing the value of the parameter a, it is observed that for $a = 3.3$, there is period four harmonic solution *i.e.* the solutions are periodic with period equal to four times the period of the driven force. In this way there is again a period doubling. As the trajectories become more and more messy with increasing cycles, we have obtained just the Poincare section in this case as shown in the Figure 13. We can observe four points which indicate the existence of a closed loop with period four.

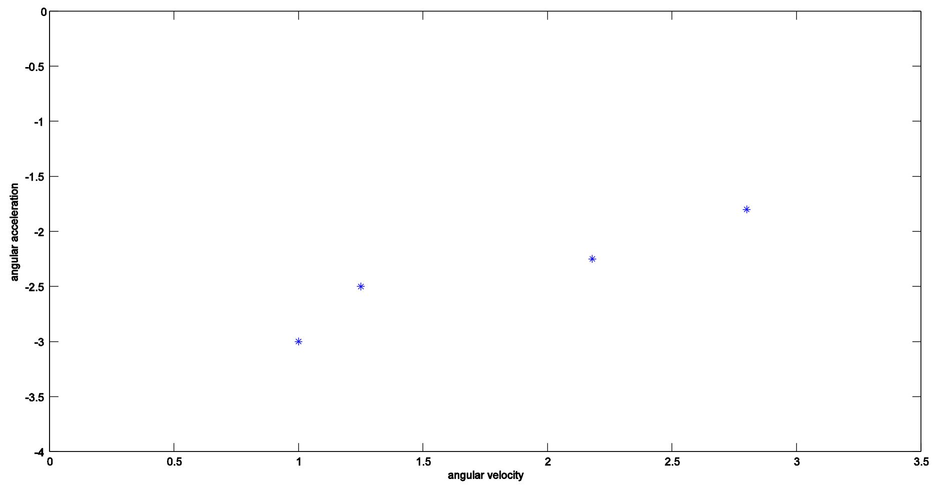

Figure 13: Poincare section for $a = 3.3$

3. CONCLISION

A lot of phenomenons which initially appear to be deterministic and regular transfer later in to a very strange one. It becomes very important to understand the exact state in which a system falls in to chaotic regime. However, this is a very difficult task. We can study some clues that point towards the transition from order to chaos. Of many indications, period doubling is one of the processes that point towards a chaotic system. We have observed the period doubling phenomenon for the values of the parameter $a = 0.4, 2.4, 3.3$ where there were periodic solutions of period 1, 2 and 4 respectively. It is very difficult to find the exact values of the parameter for which this period doubling process continues. We can just predict about the values of the parameter for which this period doubling process happens. Bifurcation diagrams are generally used to observe the points where there is bifurcations *i. e.* a doubling of the period of the periodic points. The bifurcation diagram for the pendulum system is as shown in Figure 14.

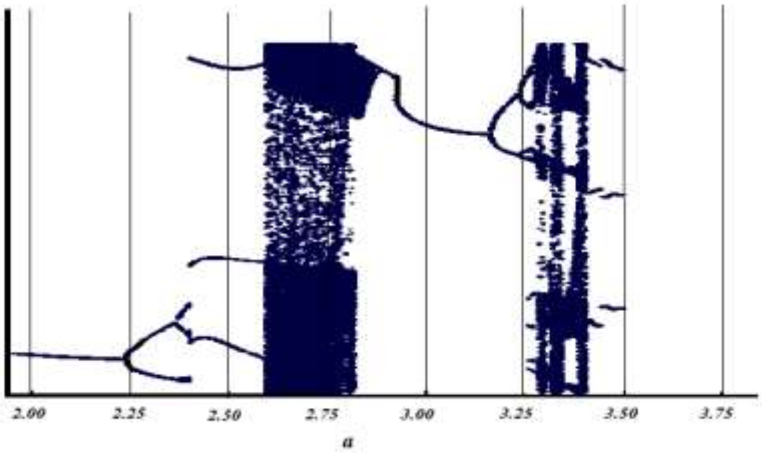

Figure 14: Bifurcation diagram

In a bifurcation diagram, there are regions where there are periodic points of other periods also. Such regions are called as windows. Such kind of window can be observed between the approximate parameter range $2.6 < \gamma < 2.8$. As we zoom on the bifurcation diagram, a similar structure *i. e.* a fractal [2] structure is obtained.

An indication of transition from predictability to chaos is intermittency, which consists of periodic or predictable motion where there is no period doubling but some sort of noise or intervals with irregular behaviour. As we go on increasing the values of the parameter, the irregularity in the periodic solutions is observed more frequently and lasting for a comparatively longer period. For the parameter value $a = 2.7$, the order in the motion of the pendulum is lost and the pendulum system becomes chaotic. A trajectory of the system is as shown in the Figure 15, where periodic behaviour is seen to be lost.

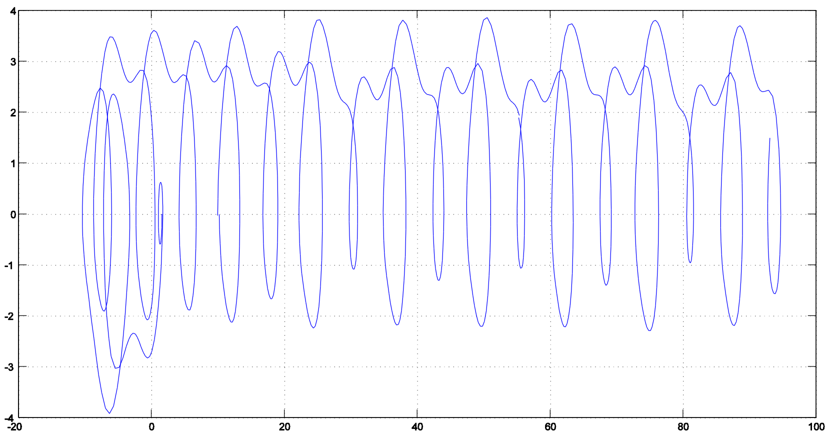

Figure 15: A solution curve for $a = 2.7$

Poincare section of the phase space for $a = 2.7$ is as shown in the Figure 16. The Poincare section is a strange attractor, which is a set of points which is an infinite set of points in the phase space. This set can be reproduced, has a complex and many layered structure. It is not just a set of points on a line, but it has fractal dimension which repeats itself as we keep zooming on its structure.

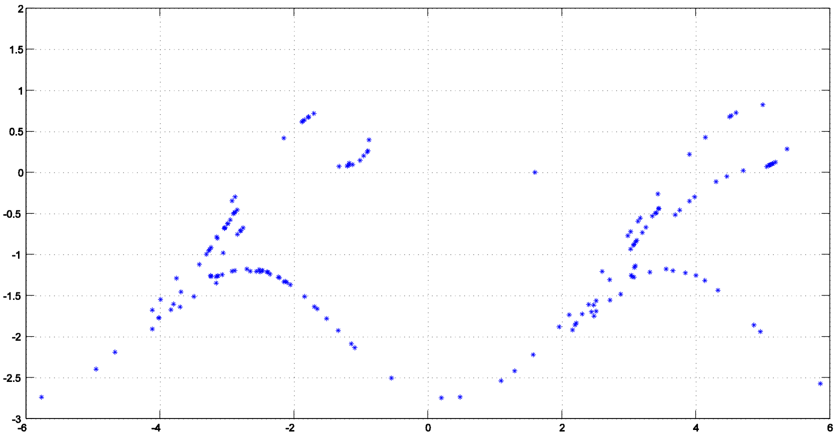

Figure 16: Poincare section for $a = 2.7$

We have observed that the pendulum system is deterministic in the absence of the external driven periodic force, but as we apply the driven force and keep changing the driven parameter value, a strange scenario comes in to focus. We conclude that however carefully we take into consideration the parameters involved in a nonlinear system, there will always be a kind of uncertainty and unpredictability in the system as we keep changing the parameter values. This uncertainty is because of the sensitive dependence on the initial conditions.

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