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STIRLING'S NUMBERS AND SUMMATION

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ABSTRACT

Among several class of interesting numbers that exist in mathematics, Stirling's numbers play significant role since they occur in variety of combinatorial situations and counting problems. In this paper, I formally introduce Stirling's numbers of second kind and connect them with falling factorial polynomials. With this result as aid, I derived a nice closed expression for summing powers of natural numbers.

Keywords: Stirling's numbers of second kind, Recurrence Relation, Falling Factorial Polynomial, Mathematical Induction, Pascal's Identity

1. Introduction

The two kinds of Stirling's numbers were named after Scottish mathematician James Stirling. Stirling's numbers of second kind were related to counting number of partitions of a given finite set in to disjoint parts. In this paper, I connect Stirling's numbers of second kind with falling factorial polynomials and prove an important theorem leading us to connect Stirling's numbers with determining sum of *m*th powers of first *n* natural numbers.

2. Definitions

2.1 Stirling's Numbers of Second Kind

The number of partitions of a set with *m* elements using *n* non-empty disjoint subsets is defined as the Stirling's numbers of second kind denoted by $S(m,n)$ or $\left\{ \begin{matrix} m \\ n \end{matrix} \right\}$. We notice that $1 \le n \le m$.

The Stirling's numbers of second kind satisfies the recurrence relation

$S(m+1,n)=S(m,n-1)+nS(m,n)$ (2.1)

For proof of (2.1),see **[5]** by the corresponding author. Using (2.1) and the fact that $S(m,1)=S(m,m)=1$, we can form a tabular column containing Stirling's numbers of second kind as shown below:

m \ n	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	3	1				
4	1	7	6	1			
5	1	15	25	10	1		
6	1	31	90	65	15	1	
7	1	63	301	350	140	21	1
				...			

Figure 1: Stirling's numbers of second kind

2.2 Falling Factorial Polynomial

The Falling Factorial Polynomial is defined by

$$x^{(m)}=x(x-1)(x-2)\cdots(x-(m-2))(x-(m-1)),\ m\geq 0, x^{(0)}=1 \ (2.2)$$

We notice from (2.2) that if $m\geq 1$then $x^{(m)}$ is a polynomial in *x* of degree *m* whose roots are 0, 1, 2, . . . , *m* – 2, *m* – 1. For example, $6^{(4)}=6\times 5\times 4\times 3=360$.

We will now prove the following important theorem.

3. Theorem 1

For all integers, $m\geq 1$, $x^m=\sum_{n=1}^{m}S(m,n)x^{(n)}$ (3.1) where $x^{(n)}$ is the falling factorial polynomial defined in (2.2) and $S(m,n)$ are Stirling's numbers of second kind.

Proof: The proof is provided by Induction on *m*. If *m* = 1, then $x^m=x$ and $\sum_{n=1}^{1}S(1,n)x^{(n)}=S(1,1)x^{(1)}=1\times x=x$. Hence (4.1) is true if *m* = 1. We will assume that the result up to *m* and try to prove for *m* + 1.

Using the fact that the result is true up to *m*, we have

$$x^{m+1}=x^m\times x=\left(\sum_{n=1}^{m}S(m,n)x^{(n)}\right)\times x=\sum_{n=1}^{m}S(m,n)\left(x^{(n)}x\right) \ (3.2)$$

Since $x=(x-n)+n$, from (2.2) we have

$$x^{(n)}x=x^{(n)}\left((x-n)+n\right)=x^{(n)}(x-n)+nx^{(n)}=x^{(n+1)}+nx^{(n)}$$

Substituting this in (3.2), we get $x^{m+1} = \sum_{n=1}^{m} S(m,n) x^{(n+1)} + \sum_{n=1}^{m} S(m,n) n x^{(n)}$

Altering the summation variable in the first summation and using the fact that $S(m,1) = S(m,m) = 1$ and using (2.1), we get

$$x^{m+1} = \sum_{n=2}^{m+1} S(m,n-1) x^{(n)} + \sum_{n=1}^{m} S(m,n) n x^{(n)}$$

$$= S(m,m) x^{(m+1)} + \sum_{n=2}^{m} S(m,n-1) x^{(n)} + \sum_{n=2}^{m} S(m,n) n x^{(n)} + S(m,1) x^{(1)}$$

$$= x^{(m+1)} + \sum_{n=2}^{m} \left[S(m,n-1) + nS(m,n) \right] x^{(n)} + x^{(1)}$$

$$= S(m+1,m+1) x^{(m+1)} + \sum_{n=2}^{m} \left[S(m+1,n) \right] x^{(n)} + S(m+1,1) x^{(1)}$$

$$= \sum_{n=1}^{m+1} S(m+1,n) x^{(n)}$$

Thus the result is also true for *m* + 1. Hence by induction principle, the result is true for all integers *m* such that $m \geq 1$. This proves (3.1) and completes the proof.

Using theorem 1, we now prove the following important theorem.

4. Theorem 2

For all positive integers *k* and *m*, we have $k^m = \sum_{n=1}^{m} \binom{k}{n} n! S(m,n)$ (4.1) where $\binom{k}{n}$ is the binomial coefficient and $S(m,n)$ are Stirling's numbers of second kind.

Proof: In (3.1), if we substitute *x* = *k* then we have

$$k^m = \sum_{n=1}^{m} S(m,n) k^{(n)} = \sum_{n=1}^{m} S(m,n) k(k-1) \times \cdots \times (k-(n-2)) \times (k-(n-1))$$

$$= \sum_{n=1}^{m} S(m,n) \times \frac{k!}{(k-n)!} = \sum_{n=1}^{m} n! S(m,n) \times \frac{k!}{n! \times (k-n)!} = \sum_{n=1}^{m} \binom{k}{n} n! S(m,n)$$

This completes the proof.

We now determine a closed expression for sum of *m*th powers of first *t* natural numbers.

5. Theorem 3

$$1^m + 2^m + \cdots + t^m = \sum_{n=1}^{m} \binom{t+1}{n+1} n! S(m,n) \quad (5.1)$$

Proof: From (4.1) of theorem 2, we have $k^m = \sum_{n=1}^{m} \binom{k}{n} n! S(m,n)$

Now considering k = 1, 2, . . . , t and summing we have

$$1^m + 2^m + \cdots + t^m = \sum_{n=1}^{m}\binom{1}{n} n!S(m,n) + \sum_{n=1}^{m}\binom{2}{n} n!S(m,n) + \cdots + \sum_{n=1}^{m}\binom{t}{n} n!S(m,n)$$

$$= \sum_{n=1}^{m} n!S(m,n)\begin{bmatrix}\binom{1}{n}+\binom{2}{n}+\cdots+\binom{n-1}{n}+\binom{n}{n}+\binom{n+1}{n}+\binom{n+2}{n}+\binom{n+3}{n} \\ +\cdots+\binom{t-1}{n}+\binom{t}{n}\end{bmatrix}$$

$$= \sum_{n=1}^{m} n!S(m,n)\left[0+0+\cdots+0+\binom{n+1}{n+1}+\binom{n+1}{n}+\binom{n+2}{n}+\binom{n+3}{n}+\cdots+\binom{t-1}{n}+\binom{t}{n}\right]$$

$$= \sum_{n=1}^{m} n!S(m,n)\left[\binom{n+2}{n+1}+\binom{n+2}{n}+\binom{n+3}{n}+\cdots+\binom{t-1}{n}+\binom{t}{n}\right]$$

$$= \sum_{n=1}^{m} n!S(m,n)\left[\binom{n+3}{n+1}+\binom{n+3}{n}+\cdots+\binom{t-1}{n}+\binom{t}{n}\right]$$

$$= \sum_{n=1}^{m} n!S(m,n)\left[\binom{n+4}{n+1}+\cdots+\binom{t-1}{n}+\binom{t}{n}\right] == \sum_{n=1}^{m} n!S(m,n)\left[\binom{t-1}{n+1}+\binom{t-1}{n}+\binom{t}{n}\right]$$

$$= \sum_{n=1}^{m} n!S(m,n)\left[\binom{t}{n+1}+\binom{t}{n}\right] = \sum_{n=1}^{m} n!S(m,n)\binom{t+1}{n+1}$$

This is (5.1). Notice that we have applied Pascal's identity of binomial coefficients repeatedly to obtain the above equations. In particular the sum $\binom{1}{n}+\binom{2}{n}+\cdots+\binom{t-1}{n}+\binom{t}{n}=\binom{t+1}{n+1}$ can be obtained through the famous Hockey Stick Identity of the binomial coefficients present in Pascal's triangle.

This completes the proof.

6. We can see few illustrations explaining the closed expression obtained in (5.1) of theorem 3. We can first rewrite the (5.1) as in the following way for convenience.

$$1^m + 2^m + \cdots + n^m = \sum_{r=1}^{m}\binom{n+1}{r+1} r!S(m,r) \quad (6.1)$$

From (6.1), we can determine the sum of mth powers of first n natural numbers through binomial coefficients, factorials and Stirling's numbers of second kind.

For example, the sum of first n natural numbers is obtained by taking m = 1 in (6.1)

$$1+2+\cdots+n = \sum_{r=1}^{1}\binom{n+1}{r+1} r!S(1,r) = \binom{n+1}{2} S(1,1) = \binom{n+1}{2} = \frac{n(n+1)}{2} \, (6.2)$$

Similarly, the sum of squares of first n natural numbers is obtained by substituting m = 2 in (6.1).

$$1^2 + 2^2 + \cdots + n^2 = \sum_{r=1}^{2} \binom{n+1}{r+1} r! S(2,r) = \binom{n+1}{2} S(2,1) + \binom{n+1}{3} \times 2! \times S(2,2)$$

$$= \binom{n+1}{2} + 2\binom{n+1}{3} = \frac{n(n+1)(2n+1)}{6} \quad (6.3)$$

For obtaining sum of cubes of first *n* natural numbers we consider *m* = 3 in (6.1) to get

$$1^3 + 2^3 + \cdots + n^3 = \sum_{r=1}^{3} \binom{n+1}{r+1} r! S(3,r) = \binom{n+1}{2} S(3,1) + \binom{n+1}{3} \times 2! \times S(3,2) + \binom{n+1}{4} \times 3! \times S(3,3)$$

$$= \binom{n+1}{2} + 6\binom{n+1}{3} + 6\binom{n+1}{4} = \left[\frac{n(n+1)}{2}\right]^2 \quad (6.4)$$

Equations (6.2), (6.3) and (6.4) provides the sum of first, second and third powers of first *n* natural numbers respectively. Similarly, we can determine the sum of higher powers of natural numbers as we wish.

Further as a bonus, from (5.1) of theorem 3, in the right hand side if we consider the coefficient of $\binom{t+1}{n+1}$ namely $n!S(m,n)$ provides the number of onto functions (surjectives) from a set with *m* elements to *n* elements.

7. Conclusion

By considering Stirling's numbers of second kind, in this paper, I had proved an interesting formula for finding sum of powers of natural numbers. Though there are several formulas that exist in determining the sum of powers of natural numbers, this paper introduces a new and not so well known formula in the form of closed expression established in (5.1) of theorem 3 in section 5. For this, I proved two theorems 1 and 2 in sections 3 and 4 respectively. As a bonus of proving theorem 3, we find the coefficient term of $\binom{t+1}{n+1}$ namely $n!S(m,n)$ provides the number of surjectives that one can obtain between a set with *m* elements to a set with *n* elements provided $m \geq n$. If $m < n$ then there would be no surjectives because from Figure 1 of Stirling's numbers of second kind, we find that $S(m,n) = 0$ if $m < n$.

Thus the expression derived in (5.1) of theorem 3 serves twofold purpose namely, it provides the sum of *m*th powers of natural numbers and it also provides the number of onto functions (surjectives) between a set with *m* elements to a set with *n* elements. These observations will add a new dimension to already known formulas for both the cases.

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