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RESEARCH ARTICLE

SUMMING THROUGH PASCAL AND POWER MATRICES

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ABSTRACT

The process of summing powers of natural numbers has been carried out for many centuries by several eminent mathematicians. There are plenty of possible ways to sum powers of natural numbers. The most famous and well known Pascal's triangle contain exciting mathematical properties within it. In this paper, I use the entries of Pascal's triangle through square matrices and prove three theorems to determine the sum of powers of natural numbers. This concept will create a new dimension to the existing grand available literature regarding summing powers of natural numbers.

Keywords: Sum of powers of natural numbers, Hockey Stick Identity, Pascal Matrices, Power Matrices, Invertible Matrices.

1. Introduction

Ever since the Ancient mathematicians provided ways of summing natural numbers and sum of squares of natural numbers, several other mathematicians like Faulhaber, Euler, Bernoulli made great strides in determining wonderful ways of summing *m*th powers of first *n* natural numbers. In this paper, I just try to add one more feather to it by first proving a well known theorem and using it to determine sum of powers of first *n* natural numbers.

2. Definitions

2.1 The sum of *m*th powers of first *n* natural numbers is denoted by the expression

$$1^m + 2^m + \cdots + n^m \quad (2.1)$$

2.2 Using the binomial coefficients of the form $\binom{n}{r}$ where $0 \le r \le n$, we construct a triangular array of numbers for each value of n = 0, 1, 2, 3, 4, . . .

Such a triangle is called Pascal's triangle named after French mathematician Blaise Pascal. Though Pascal's triangle did not originate from Pascal himself, it was he who provided the significant mathematical aspects of the numbers involved in the triangle and applied them to probability. Hence, the triangle was named in his honor. See Figure 1 for Pascal's Triangle.

0 :	1							
1 :	1	1						
2 :	1	2	1					
3 :	1	3	3	1				
4 :	1	4	6	4	1			
5 :	1	5	10	10	5	1		
6 :	1	6	15	20	15	6	1	
7 :	1	7	21	35	35	21	7	1
...	...	...	...	...	...	...	...	...

Figure 1: The first eight rows of Pascal's Triangle written in Right Triangle form

In Figure 1, we notice that the entries above the leading 1's are all zero because $\binom{n}{r} = 0$ if *r*>*n*. The Pascal's triangle in Figure 1, posses so many mathematical properties, that a whole book can be written to brief them. In this paper, I will accomplish the task using the Pascal's triangle displayed in Figure 1. We now establish an interesting theorem called as "Hockey Stick Identity".

3. Theorem 1

If $0 \le r \le n$ then $\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$ (3.1)

Proof: To prove this we note that $\binom{r}{r} = \binom{r+1}{r+1}$ and shall make use of Pascal Identity $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ repeatedly.

$$\sum_{k=r}^{n}\binom{k}{r}=\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\cdots+\binom{n-1}{r}+\binom{n}{r}$$
$$=\binom{r+1}{r+1}+\binom{r+1}{r}+\binom{r+2}{r}+\cdots+\binom{n-1}{r}+\binom{n}{r}$$
$$=\binom{r+2}{r+1}+\binom{r+2}{r}+\cdots+\binom{n-1}{r}+\binom{n}{r}$$
$$=\binom{r+3}{r+1}+\cdots+\binom{n-1}{r}+\binom{n}{r}$$
$$=\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$=\binom{n-1}{r+1}+\binom{n-1}{r}+\binom{n}{r}=\binom{n}{r+1}+\binom{n}{r}$$
$$=\binom{n+1}{r+1}$$

This proves (3.1) and completes the proof.

4. Pascal Matrices

4.1 Definition

Let P_n be a $n\times n$ square matrix whose (i, j) th entry is the binomial coefficient $\binom{i}{j}$ where $1\le i, j\le n$. We call P_n as the Pascal matrix of order *n*.

For example, $P_5=\begin{pmatrix}1&0&0&0&0\\2&1&0&0&0\\3&3&1&0&0\\4&6&4&1&0\\5&10&10&5&1\end{pmatrix}$ (4.1)

We notice from the matrix P_5 in (4.1), that we do not consider the binomial coefficients for *n* = 0. Hence the matrix in (4.1) contain elements of Pascal's triangle in Figure 1 from *n* = 1 to 5 without the first column comprising of 1's. Moreover, the entries above the main diagonal are all zero making it a lower triangular matrix. But we need to know if P_5 is invertible? If so, what would be its inverse? The following theorem answers this question in general.

4.2 Theorem 2

The Pascal matrix P_n is invertible. In fact, the (i, j) th entry of P_n^{-1} is $(-1)^{i+j}\binom{i}{j}$ (4.2)

Proof: Since P_n is a $n\times n$ lower triangular matrix whose diagonal entries are all 1, it follows that $|P_n|=1$ for all *n*. Hence P_n is non-singular and so the Pascal matrix P_n is invertible for each *n*. To prove the remaining part, let us assume that $Q_n = \left(q_{i,j}\right)$ where $q_{i,j} = (-1)^{i+j}\binom{i}{j}, 1 \le i, j \le n$. We see that Q_n is inverse of P_n if and only if $P_nQ_n = Q_nP_n = I_n$ (4.3), where I_n is a unit matrix of order *n*. Now if we multiply the (i,k)th entry of P_n with the (k,j)th entry of Q_n then we get the following expression

$\sum_{k=1}^{n}\binom{i}{k}(-1)^{k+j}\binom{k}{j}$. Since $\binom{k}{j} = 0$ for $j > k$ we have

$$\sum_{k=1}^{n}\binom{i}{k}(-1)^{k+j}\binom{k}{j} = \sum_{k=j}^{n}(-1)^{k+j}\binom{i}{k}\binom{k}{j} \quad (4.4)$$

If $j > i$ then the right hand side of (4.4) becomes 0. Thus, we should have $i \ge k \ge j$. In this case, we get $\binom{i}{k}\binom{k}{j} = \binom{i}{j}\binom{i-j}{k-j}$. Thus (4.4) becomes

$$\sum_{k=j}^{n}(-1)^{k+j}\binom{i}{k}\binom{k}{j} = \sum_{k=j}^{n}(-1)^{k+j}\binom{i}{j}\binom{i-j}{k-j} = \binom{i}{j}\sum_{k=j}^{n}(-1)^{k+j}\binom{i-j}{k-j} = \binom{i}{j}\sum_{k=j}^{n}(-1)^{k-j}\binom{i-j}{k-j}$$
$$= \binom{i}{j}\sum_{r=0}^{i-j}(-1)^{r}\binom{i-j}{r} \quad (4.5)$$

Now we notice that if $i = j$ then the Right Hand Side of (4.5) becomes $\binom{i}{i}\times(-1)^0\times\binom{0}{0} = 1$ (4.6)

Also by binomial expansion $(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}$ for *n*> 0 if we consider $a = -1, b = 1$ then we get $\sum_{r=0}^{n}\binom{n}{r}(-1)^r = (-1+1)^n = 0$. Thus taking $i - j = m$ and noting that if $i \ne j$ then $m = i - j > 0$, for $i \ne j$ equation (4.5) becomes

$$\sum_{k=j}^{n}(-1)^{k+j}\binom{i}{k}\binom{k}{j} = \binom{i}{j}\sum_{r=0}^{i-j}(-1)^{r}\binom{i-j}{r} = \binom{i}{j}\sum_{r=0}^{m}(-1)^{r}\binom{m}{r} = 0 \quad (4.7)$$

Hence from (4.6) we see that the leading diagonal entries of the product matrix P_nQ_n are 1 and from (4.7) we see that the non-diagonal entries of P_nQ_n are 0. Hence $P_nQ_n = I_n$. By the same reasoning we can show that $Q_nP_n = I_n$. Therefore, $Q_n = P_n^{-1}$ and this completes the proof.

4.3 Illustration

Using the idea of theorem 2 we established, it will be fairly straightforward for us to determine the inverse of the Pascal matrix P_n for any given natural number *n*. In particular from (4.2) we know that the (i, j)th entry of P_n^{-1} is $(-1)^{i+j}\binom{i}{j}$

Making use of this vital information, let us determine the inverse of the Pascal matrix of order 5 i.e. P_5 provided in (4.1).

In fact, we get $P_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3 & -3 & 1 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 \\ 5 & -10 & 10 & -5 & 1 \end{pmatrix}$ (4.8)

5. Power Matrices

5.1 Let S_n be a $n \times n$ square matrix whose (i, j) th entry is i^j where $1 \le i, j \le n$. We call S_n to be a Power matrix of order *n*, since its entries are perfect powers.

Example of S_5 is shown in (5.1).

$$S_5 = \begin{pmatrix} 1^1 & 1^2 & 1^3 & 1^4 & 1^5 \\ 2^1 & 2^2 & 2^3 & 2^4 & 2^5 \\ 3^1 & 3^2 & 3^3 & 3^4 & 3^5 \\ 4^1 & 4^2 & 4^3 & 4^4 & 4^5 \\ 5^1 & 5^2 & 5^3 & 5^4 & 5^5 \end{pmatrix} \quad (5.1)$$

We now use the Pascal and Power matrices to determine the sum of powers of first *n* natural numbers.

5.2 Theorem 3

Let *n*, *m* and *r* be positive integers. Then the sum of *m*th power of first *n* natural numbers is given by

$1^m + 2^m + 3^m + \cdots + n^m = \sum_{r=1}^{m} \binom{n+1}{r+1} a_{r,m}$ (5.2) where $a_{r,m}$ is the (r, m) th entry of $P_n^{-1} S_n$

Proof: Let us assume that $n \ge m, r$. Let $A_n = (a_{i,j})$ be the matrix whose (r, m) th entry is $a_{r,m}$ in (5.2). Now we observe the fact that for any two positive integers *k* and *m*, k^m can be written as linear combination of the binomial coefficients $\binom{k}{r}$.

For example, $k^2 = \binom{k}{1} + 2\binom{k}{2}, k^3 = \binom{k}{1} + 6\binom{k}{2} + 6\binom{k}{3}$. In this viewpoint, and considering $a_{r,m}$ as coefficients as defined in (5.2), we have $k^m = \sum_{r=1}^{m} \binom{k}{r} a_{r,m}$ (5.3).

Now, the (k, m)th entry of $P_n A_n$ will be $\sum_{r=1}^{m} \binom{k}{r} a_{r,m}$ which from (5.3) is k^m.

Thus, the (k,m)th entry of P_nA_n is k^m which by definition of the power matrix is (k,m)th entry of S_n. From this, we see that $P_nA_n = S_n$ (5.4). Since from theorem 2, we know that P_n is invertible, from (5.4) we get $A_n = P_n^{-1}S_n$ (5.5). Thus, the coefficients $a_{r,m}$ is the (r,m) th entry of $P_n^{-1}S_n$.

Now in (5.3), if we sum for each value of *k* from 1 to *n*, and using (3.1), we obtain

$$\sum_{k=1}^{n} k^m = \sum_{k=1}^{n}\sum_{r=1}^{m}\binom{k}{r}a_{r,m} = \sum_{r=1}^{m} a_{r,m}\sum_{k=1}^{n}\binom{k}{r} = \sum_{r=1}^{m} a_{r,m}\sum_{k=r}^{n}\binom{k}{r} = \sum_{r=1}^{m} a_{r,m}\binom{n+1}{r+1}$$

Therefore, $1^m + 2^m + 3^m + \cdots + n^m = \sum_{r=1}^{m}\binom{n+1}{r+1}a_{r,m}$ where the coefficients $a_{r,m}$ is the (r,m) th entry of $P_n^{-1}S_n$.

This proves (5.2) and completes the proof.

5.3 Determining sum of powers of natural numbers

In this section, using the identity established in (5.2), we can determine the sum of first, second, third, fourth and fifth powers of natural numbers all at once.

For this, from (5.5), first we need to compute $A_5 = P_5^{-1}S_5$. Using (4.8) and (5.1), we get

$$A_5 = P_5^{-1}S_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3 & -3 & 1 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 \\ 5 & -10 & 10 & -5 & 1 \end{pmatrix} \times \begin{pmatrix} 1^1 & 1^2 & 1^3 & 1^4 & 1^5 \\ 2^1 & 2^2 & 2^3 & 2^4 & 2^5 \\ 3^1 & 3^2 & 3^3 & 3^4 & 3^5 \\ 4^1 & 4^2 & 4^3 & 4^4 & 4^5 \\ 5^1 & 5^2 & 5^3 & 5^4 & 5^5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 & 30 \\ 0 & 0 & 6 & 36 & 150 \\ 0 & 0 & 0 & 24 & 240 \\ 0 & 0 & 0 & 0 & 120 \end{pmatrix} \quad (5.6)$$

We can derive the sum of first five powers of natural numbers by selecting respective column entries of A_5 obtained in (5.6).

From (5.2), with *m* = 1, we get

$$1+2+3+\cdots+n = \binom{n+1}{2}a_{1,1} = \binom{n+1}{2}\times 1 = \binom{n+1}{2} = \frac{n(n+1)}{2} \quad (5.7)$$

where $a_{1,1}$ = 1 is the entry in the first column of A_5 in (5.6)

Similarly, from (5.2), with *m* = 2, and from second column of A_5 we get

$$1^2+2^2+3^2+\cdots+n^2 = \sum_{r=1}^{2}\binom{n+1}{r+1}a_{r,2} = \binom{n+1}{2}\times 1 + \binom{n+1}{3}\times 2 = \frac{n(n+1)(2n+1)}{6} \quad (5.8)$$

For *m* = 3, using the third column of A_5 we get

$$1^3+2^3+3^3+\cdots+n^3=\sum_{r=1}^{3}\binom{n+1}{r+1}a_{r,3}=\binom{n+1}{2}\times 1+\binom{n+1}{3}\times 6+\binom{n+1}{4}\times 6=\left[\frac{n(n+1)}{2}\right]^2 \quad (5.9)$$

For *m* = 4, using the fourth column of A_5 we get

$$1^4+2^4+3^4+\cdots+n^4=\sum_{r=1}^{4}\binom{n+1}{r+1}a_{r,4}=\binom{n+1}{2}\times 1+\binom{n+1}{3}\times 14+\binom{n+1}{4}\times 36+\binom{n+1}{5}\times 24$$

$$=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \quad (5.10)$$

For *m* = 5, using the fifth column of A_5 we get

$$1^5+2^5+3^5+\cdots+n^5=\sum_{r=1}^{5}\binom{n+1}{r+1}a_{r,5}=\binom{n+1}{2}\times 1+\binom{n+1}{3}\times 30+\binom{n+1}{4}\times 150$$

$$+\binom{n+1}{5}\times 240+\binom{n+1}{6}\times 120=\frac{n^2(n+1)^2(2n^2+2n-1)}{12} \quad (5.11)$$

Equations (5.7) to (5.11) provide the sum of first five powers of natural numbers.

6. Conclusion

Though there are several possible ways to sum the powers of natural numbers, this paper introduce a new formula for doing so through (5.2) of theorem 3. This task is accomplished by using the Pascal's matrix and Power matrix respectively. Suitable illustrations are provided wherever necessary for better understanding of the concepts involved in this paper.

We notice that the entries of P_5^{-1} are obtained from P_5 just by sprinkling some minus signs according to the positions of the entries. In particular if the sum of the indices of row and column add up to odd number, then we get the negative entry whereas, if the sum of indices is even, then we get the same entry as it is. This property is applicable for computing P_n^{-1} from given P_n almost effortlessly.

Similarly in section 5.3, upon computing $A_5 = P_5^{-1}S_5$ we have not only determined the sum of fifth powers of natural numbers, but the matrix A_5 enabled us to compute the sum of first, second, third and fourth powers of first *n* natural numbers simultaneously through its columns respectively, which is an added advantage for us. Thus by determining $A_k = P_k^{-1}S_k$ and using its first *k* columns, we can determine sum of first, second, third, and so on up to *k*th powers of first *n* natural numbers simultaneously. This is the main advantage behind the formula derived in (5.2) of theorem 3. In fact, in (5.2), we can also show that $a_{r,m} = r!\times S(m,r)$ where $S(m,r)$ are Stirling's numbers of second kind. Surprisingly enough, $a_{r,m} = r!\times S(m,r)$ is the number of onto functions (surjectives) from a set with *m* elements to a set with *r* elements. These kind of inter-relationships between completely different concepts is one of the key aspects of mathematical research. This paper has thus produced a new way of arriving results for well known and age old problem.

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