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SUBSTITUTION METHOD USING THE LAPLACE TRANSFORMATION FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS INVOLVING MORE THAN TWO INDEPENDENT VARIABLES

F.Mubarak[1], M.Z.Iqbal[2], A. Moazzam[3], U.Amjed[4], M.U.Naeem[5]

[1,2,3,4] Department of Mathematics and Statistics, University of Agriculture Faisalabad, Jail Road Faisalabad, Pakistan.

[5] Department of information technology, Government College University Faisalabad, Sahiwal campus, Pakpattan chowk Sahiwal.

Email: farihamubarak97@gmai.com[1], mzts2004@uaf.edu.pk[2], alimoazzam7309723@gmail.com[3], usmanamjed33@gmail.com[4], myuz9692@gmail.com[5]

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ABSTRACT

In this work, we suggest the method of substitution of Laplace transformation to solve the partial differential equations involving more than two independent variables. In this work, using the substitution method and the Laplace transform, the inverse Laplace transform, the partial differential equations involving more than two independent variables with initial conditions, are solve less mathematical operations than the separation method of variables. By using the substitution method of Laplace solves the partial linear differential equations of partial mixed derivatives and using the substitution method of Laplace with variational iteration method solves the partial non-linear differential equation of partial mixed derivatives.

Keywords: Partial linear differential equations, partial non-linear differential equations, Adomain decomposition technique.

1. Introduction

A partial nonlinear differential equation is a partial differential equation containing nonlinear components in math and physics. However, they can be difficult to resolve numerically or conceptually at times. There are a variety of ways for obtaining approximate solution to such equations. They are utilized in science and mathematics to answer issues like the Poincaré conjecture and the Calabi conjecture, and they explain a wide range of physical phenomena, from gravity to fluid mechanics. They are complex to analyze since there are few generic approaches that work for all of these equations, and each one must typically be addressed as a distinct issue. The characteristics of the operator that defines the PDE are generally used to distinguish between a linear and a partial non-linear differential equation. A partial difference equation (PDE) in mathematics is an equation that inflicts a connection in the middle of several partial derivatives of the multi-variable function. A partial differential equation is the kinds of differential equations that carry unidentified multi-variables with their partial-derivatives. Handibag and Karande have studied the partial differential equations of nth order derivatives [7] of linear and nonlinear equation by using the substitution technique of Laplace transform. Substitution method of Laplace is applicable on many integral equations like: Integro-Differential Equations Including Partial Mixed Derivatives and Volterra integro-differential equations [6-8]. The variational iteration technique [10] and Adomain Polynomial [2,3] has been broadly used to resolve non-linear problems further and extra advantages must been identified, and various changes have been proposed to alleviate the disadvantages that arise during the solution operation. Laplace substitution method [5] involving mixed partial derivative is an easy process to discover the correct solution of the (PDEs) with fewer calculations as contrast to separation method of variables [9]. The Laplace transform in mathematics is called for its innovator Pierre-Simon Laplace that is an integral transform who established the transform during working on probability theory. The Laplace transforms *L* transfigures a function of positive variable *t* to a imaginary frequency *s* [1-4].

The major objective of this work is to present a method for solving partial linear or nonlinear differential equations with more than two independent variables with initial conditions that include more than two independent variables. In part 2, we will present this strong approach; in next section, we will spread on it to various instances; and in portion 4, we will draw particular conclusions.

Definition (1): [1]

The Laplace transformation of a function *f(t)* is given below and convergent in given interval.

$$L\{g(t)\} = \int_0^\infty e^{-st} g(t)dt = G(s); \quad t > 0,$$

Transformation for some functions: [1]

We are going to find Laplace transformation for some functions, like fix function, Polynomial's, trigonometric function and other functions.

Table 1.Transformation for some functions

Serial no.	Function g (t)	Transformed function G(s) and their convergence
1.	1	$1/s, \quad s>0$
2.	e^{bt}	$1/s-b, \quad s>0$
3	$t^n, \quad n=1,2,3,......$	$n!/s^{n+1}, \quad s\geq 0$
4.	$t^p, \quad p>-1$	$\Gamma(p+1)/s^{p+1}, \quad s\geq 0$
5.	$\sqrt{t}$	$\sqrt{\pi}/2s^{\frac{3}{2}}$
6.	$t^n e^{bt}, \ n=1,2,3,...$	$n!/(s-b)^{n+1}$
7.	$sin(bt), \quad t\geq 0$	$b/s^2+b^2, \quad s>0$
8.	$cos(bt), \quad t\geq 0$	$s/s^2+b^2, \quad s>0$
9.	$sinh(bt), \quad t\geq 0$	$b/s^2-b^2, \quad s>\lvert b\rvert$
10.	$cosh(bt), \quad t\geq 0$	$s/s^2-b^2, \quad s>0$

Theorem (1): [2]

The differential feature of the Laplace transformation is among the most significant properties in linear systems thinking. It aids in the transformation of differential equations with constant coefficients into polynomial solutions with complex coefficients. We may find the output signal in frequency response by solving these algebraic problems. We can retrieve the matching time domain output signal by obtaining the reverse of Laplace transformation. The property of time derivatives is given as,

$$L\{g'(t)\} = sG(\mathrm{s}) - g(0^-)$$

$$L\{g''(t)\} = s^2G(\mathrm{s}) - g(0^-) - g'(0^-)$$

....

$$L\{g^{(n)}(t)\} = s^nG(\mathrm{s}) - s^{n-1}g(0^-) - s^{n-2}g'(0^-) - .... - g^{(n-1)}(0^-).$$

2. Substitution method of Laplace for more than two independent variables:

The generalization of the non-homogenous partial differential equations and its initial condition is:

Consider a partial differential equation

$$Lu(v,w,z) + Ru(u,w,z) = h(v,w,z), \tag{1}$$

Its initial conditions,

$$u(v,0,z) = f(v,z), \qquad u_w(v,w,0) = g(v,w), \qquad u_z(0,w,z) = j(w,z). \tag{2}$$

Hare $L\dfrac{\partial^3}{\partial v\partial w\partial z}$, is the resting linear terms which contain partial derivatives of at most first order $Ru(v,w,z)$ regarding to v, w or z and $h(v,w,z)$ remains the origin term. Equation (1) described as

$$\frac{\partial^3}{\partial v \partial w \partial z} + Ru(v,w,z) = h(v,w,z)$$

Equation (2) can be in the form,

$$\frac{\partial}{\partial v}\left(\frac{\partial^2 u}{\partial w \partial z}\right) + Ru(v,w,z) = h(v,w,z) \quad (3)$$

putting $\frac{\partial^2 u}{\partial w \partial z} = V$ in equation (3) we conclude,

$$\frac{\partial V}{\partial v} + Ru(v,w,z) = h(v,w,z). \quad (4)$$

Preceding the Laplace transform of equation (4) w. r. t, '*v*' and applying differential property of Laplace,

$$sV(s,w,z) - V(0,w,z) = L_v[h(v,w,z) - Ru(v,w,z)]$$

$$V(s,w,z) = \frac{1}{s}V_z(0,w,z) + \frac{1}{s}L_v[h(v,w,z) - Ru(v,w,z)]$$

$$V(s,w,z) = \frac{1}{s}\, j(w,z) + \frac{1}{s}L_v[h(v,w,z) - Ru(v,w,z)] \quad (5)$$

Now proceeding the Laplace inverse transform of (5) w. r. t '*v*',

$$V(v,w,z) = j(w,z) + \frac{1}{s}L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z) - Ru(v,w,z)]\right] \quad (6)$$

Again substituting the value of $V(v,w,z)$ in (6) we obtain,

$$\frac{\partial^2 u(v,w,z)}{\partial w \partial z} = j(w,z) + \frac{1}{s}L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z) - Ru(v,w,z)]\right]$$

$$\frac{\partial}{\partial w}\left[\frac{\partial u}{\partial z}\right] = j(w,z) + \frac{1}{s}L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z) - Ru(v,w,z)]\right], \quad (7)$$

putting $\frac{\partial u}{\partial z} = T$ in equation (7),

$$\frac{\partial T}{\partial w} = j(w,z) + \frac{1}{s}L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z) - Ru(v,w,z)]\right]. \quad (8)$$

Preceding the Laplace transform w. r. t '*w*' of (8) and applying differential property of Laplace,

$$sT(v,s,z) - T(v,0,z) = L_w\left[\, j(\mathrm{w,z}) + L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z) - Ru(v,w,z)]\right]\right]$$

$$T(v,s,z) = \frac{1}{s}T(v,0,z) + \frac{1}{s}L_w\left[\, j(\mathrm{w,z}) + L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z) - Ru(v,w,z)]\right]\right]$$

$$T(v,s,z)=\frac{1}{s}f(v,z)+\frac{1}{s}L_w\left[j(\mathrm{w},\mathrm{z})+L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z)\text{-}Ru(v,w,z)]\right]\right] \quad (9)$$

Proceeding the Laplace inverse transform of (9) w. r. t *'w'*,

$$T(v,s,z)=f(v,z)+\frac{1}{s}L^{-1}{}_w\left[\frac{1}{s}L_w\left[j(\mathrm{w},\mathrm{z})+L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z)\text{-}Ru(v,w,z)]\right]\right]\right] \quad (10)$$

Again substitute the value of *T (v, w, z)* in (10),

$$\frac{\partial u(v,w,z)}{\partial z}=f(v,z)+\frac{1}{s}L^{-1}{}_w\left[\frac{1}{s}L_w\left[j(w,z)+L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z)\text{-}Ru(v,w,z)]\right]\right]\right] \quad (11)$$

Now we have in variables *v, w* and *z* first order partial differential equation. Preceding the Laplace transformation of (11) w. r. t *'z'* we obtain,

$$su(v,w,s)-u(v,w,0)=L_z\left[f(v,z)+L^{-1}{}_w\left[\frac{1}{s}L_w\left[j(w,z)+L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z)\text{-}Ru(v,w,z)]\right]\right]\right]\right]$$

$$u(v,w,s)=\frac{1}{s}u(v,w,0)+\frac{1}{s}L_z\left[f(v,z)+L^{-1}{}_w\left[\frac{1}{s}L_w\left[j(w,z)+L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z)\text{-}Ru(v,w,z)]\right]\right]\right]\right]$$

$$u(v,w,s)=\frac{1}{s}g(v,w)+\frac{1}{s}L_z\left[f(v,z)+L^{-1}{}_w\left[\frac{1}{s}L_w\left[\begin{matrix}j(w,z)+\\ L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z)\text{-}Ru(v,w,z)]\right]\end{matrix}\right]\right]\right] \quad (12)$$

Proceeding the inverse Laplace transform of equation (12) w. r. t, 'z',

$$u(v,w,z)=g(v,w)+L_z^{-1}\left[\frac{1}{s}L_z\left[f(v,z)+L^{-1}{}_w\left[\frac{1}{s}L_w\left[\begin{matrix}j(w,z)+\\ L_v^{-1}\left[\frac{1}{s}L_v[h(v,w,z)\text{-}Ru(v,w,z)]\right]\end{matrix}\right]\right]\right]\right].$$

The exact solution of given problem is provided by above equation.

3. Applications:

Example 1: Find the solution of PDE,

$$\frac{\partial^3 u}{\partial v\partial w\partial z}=\sin v\ \sin w\ \sin z, \quad (13)$$

with initial conditions,

$$u(v,0,z)=1+\cos v, \qquad u_w(v,w,0)=-2\sin w, \qquad u_z(0,w,z)=\cos z.$$

Solution: In this initial value problem $Lu(v,w,z)=\frac{\partial^3 u}{\partial v\partial w\partial z}$, $h(v,w,z)=\sin v\sin w\sin z$ and general non-linear term $Ru(v,w,z)=0$,

Equation (13) can be written as,

$$\frac{\partial}{\partial v}\left(\frac{\partial^2 u}{\partial w\partial z}\right)=\sin v\sin w\sin z \quad (14)$$

Let $\frac{\partial^2 u}{\partial w\partial z}=U$ then the equation (14) becomes,

$$\frac{\partial U(v,w,z)}{\partial v}=\sin v\ \sin w\ \sin z \quad (15)$$

Taking the Laplace transform of (15) w. r. t, *'v'*

$$sU(s,w,z)-U_z(0,w,z)=\left(\frac{1}{1+s^2}\right)\sin w\ \sin z.$$

$$sU(s,w,z)-\cos z=\left(\frac{1}{1+s^2}\right)\sin w\ \sin z$$

$$U(s,w,z)=\frac{1}{\mathrm{s}(1+s^2)}\sin w\ \sin z+\frac{\cos z}{s}, \quad (16)$$

Taking the inverse Laplace transform of equation (16) w. r. t, *'v',*

$$U(v,w,z)=\left(1-\cos v\right)\sin w\ \sin z+\cos z \quad (17)$$

Substituting the value of U in (17),

$$\frac{\partial^2 u(v,w,z)}{\partial w\partial z}=\left(1-\cos v\right)\sin w\ \sin z+\cos z \quad (18)$$

Now let $\frac{\partial u}{\partial z}=T$ and equation (18) becomes,

$$\frac{\partial T(v,w,z)}{\partial w}=\left(1-\cos v\right)\sin w\ \sin z+\cos z \quad (19)$$

Taking the Laplace transform of equation (19) w. r. t*, 'w',*

$$sT(s,w,z)-T(v,0,z)=\left(1-\cos v\right)\ \frac{1}{(1+s^2)}\sin z+\frac{\cos z}{s}$$

$$T(s,w,z)=\frac{(1+\cos v)}{s}+\left(1-\cos v\right)\ \frac{1}{\mathrm{s}(1+s^2)}\sin z+\frac{\cos z}{s^2} \quad (20)$$

Taking the inverse of transformation of equation (20) w. r. t, *'w'*, we get,

$$T(v,w,z)=(1+\cos v)+\left(1-\cos v\right)\ (1\text{-}\cos w)\sin z+w\cos z$$

Now replace the value of *T,*

$$\frac{\partial u}{\partial z}T(v,w,z)=(1+\cos v)+\left(1-\cos v\right)\ (1\text{-}\cos w)\sin z+w\cos z. \quad (21)$$

Equation (21) contains only one partial derivative that is *'z'.* Proceeding the Laplace transform w. r. t, *'z'.,*

$$su(v,w,s)-u(v,w,0)=\frac{(1+\cos v)}{s}+\left(1-\cos v\right)\ (1\text{-}\cos w)\left(\frac{1}{(1+s^2)}\right)+w\frac{s}{(1+s^2)}$$

$$u(v,w,s)=\frac{-2\,\text{sinw}}{s}+\frac{(1+\cos v)}{s^2}+\left(1-\cos v\right)\ (1\text{-}\cos w)\left(\frac{1}{s}-\frac{1}{(1+s^2)}\right)+\frac{w}{(1+s^2)}. \qquad (22)$$

Taking the inverse of transformation of equation (22) w. r. t, *'z'*,

$$u(v,w,z)=-2\sin w+(1+\cos v)z+\left(1-\cos v\right)\ (1\text{-}\cos w)\left(1-\cos z\right)+w\sin z\,.$$

The exact solution of problem is provided by above equation.

Example 2: Consider the partial differential equation

$$\frac{\partial^3 f}{\partial v\partial w\partial z}-e^v-e^{-z}\cos w=0, \qquad (23)$$

With initial conditions,

$$f(v,0,z)=0, \qquad f_w(v,w,0)=1, \qquad f_z(0,w,z)=0.$$

Solution: The equation (23) can be expressed as,

$$\frac{\partial}{\partial v}\left(\frac{\partial^2 f}{\partial w\partial z}\right)=e^v+e^{-z}\cos w \qquad (24)$$

Substituting $\frac{\partial^2 f}{\partial w\partial z}=M$ in equation (24),

$$\frac{\partial M}{\partial v}=e^v+e^{-z}\cos w. \qquad (25)$$

Proceeding the Laplace transform of (25) w. r. t, *'v'*,

$$sM(s,w,z)-M(0,w,z)=\frac{1}{s-1}+\frac{e^{-z}\cos w}{s}$$

$$sM(s,w,z)-0=\frac{1}{s-1}+\frac{e^{-z}\cos w}{s}$$

$$M(s,w,z)=\frac{1}{s(s-1)}+\frac{e^{-z}\cos w}{s^2} \qquad (26)$$

Preceding the inverse Laplace transform of (26) w. r. t. *'v'*, we get,

$$M(v,w,z)=(-1+e^v)+ve^{-z}\cos w$$

Replacing the value of $\frac{\partial^2 f}{\partial w\partial z}=M$ in above equation,

$$\frac{\partial^2 f(v,w,z)}{\partial w\partial z}=(-1+e^v)+ve^{-z}\cos w \qquad (27)$$

Now again substituting $\frac{\partial f}{\partial z}=N$ in (27),

$$\frac{\partial N(v,w,z)}{\partial w}=(e^{v}-1)+ve^{-z}\cos w \tag{28}$$

Taking the Laplace transform of equation (28) w. r. t. *'w',*

$$sN(v,s,z)-N(v,0,z)=\frac{(e^{v}-1)}{s}+ve^{-z}\frac{s}{1+s^{2}} \tag{29}$$

After applying the initial condition and simplification (29) becomes,

$$N(v,s,z)=\frac{(e^{v}-1)}{s^{2}}+ve^{-z}\frac{1}{(1+s^{2})}. \tag{30}$$

Taking the inverse of transformation of (30) w. r. t, *'w'*,

$$N(v,w,z)=w(e^{v}-1)+ve^{-z}\sin w.$$

Again substituting the value of *N,*

$$\frac{\partial f(v,w,z)}{\partial z}=w(e^{v}-1)+ve^{-z}\sin w. \tag{31}$$

Proceeding the Laplace transform of (31) w. r .t, *'z'* and applying the initial condition,

$$f(v,w,s)=\frac{1}{s}+\frac{w(e^{v}-1)}{s^{2}}+v\sin w\left(\frac{1}{s(1+s)}\right) \tag{32}$$

Proceeding the inverse of transformation of (32) w. r. t. *'z',*

$$f(v,w,\mathrm{z})=1+zw(e^{v}-1)+v\sin w\left(1-e^{-z}\right). \tag{33}$$

The exact solution of problem is given by (33) equation.

Example 3: Consider the partial differential equation

$$\frac{\partial^{4} f}{\partial v\partial w\partial z\partial j}=\cos j-e^{v}+6w^{3}z^{2}$$
(34)

with initial conditions,

$$f_{v}(v,w,0,j)=0, \qquad f_{w}(v,w,0,j)=e^{-w}, \qquad f_{z}(v,w,z,0)=1, \qquad f_{t}(0,w,z,j)=j.$$

where v, w, z, and j are variables.

Solution:- Equation (34) can be written as,

$$\frac{\partial}{\partial v}\left(\frac{\partial^{3} f}{\partial w\partial z\partial j}\right)=\cos j-e^{v}+6w^{3}z^{2} \tag{35}$$

Substituting $\frac{\partial^{3} f}{\partial w\partial z\partial j}=U$ in equation (35),

$$\frac{\partial U}{\partial v}=\cos j-e^{v}+6w^{3}z^{2} \tag{36}$$

Proceeding the Laplace transform of equation (36) w. r. t, *'v'* and applying the differential property of Laplace,

$$sU(s,w,z,j)-U(0,w,z,j)=\frac{\cos j}{s}-\frac{1}{s-1}+\frac{6w^3z^2}{s}$$

after simplification,

$$U(s,w,z,j)=\frac{j}{s}+\frac{\cos j}{s^2}-\left(-\frac{1}{s}+\frac{1}{(s-1)}\right)+\frac{6w^3z^2}{s^2} \quad (37)$$

Proceeding the inverse Laplace transform of equation (37) w. r. t, *'v'*,

$$U(v,w,z,j)=j+v\cos j-\left(-1+e^v\right)+6vw^3z^2$$

again substitute the value of $\frac{\partial^3 f}{\partial w\partial z\partial j}=U$,

$$\frac{\partial^3 f(v,w,z,j)}{\partial w\partial z\partial j}=j+v\cos j-\left(-1+e^v\right)+6vw^3z^2, \quad (38)$$

now putting $\frac{\partial^2 f}{\partial z\partial j}=N$ in equation (38),

$$\frac{\partial N}{\partial w}=j+v\cos j-\left(-1+e^v\right)+6vw^3z^2 \quad (39)$$

Proceeding the Laplace transform of equation (39) w .r. t, *'w'*,

$$sN(v,s,z,j)-N(v,0,z,j)=\frac{j}{s}+\frac{v\cos j}{s}-\frac{\left(-1+e^v\right)}{s}+\frac{6v.6z^2}{s^4}$$

applying the initial condition,

$$sN(v,s,z,j)-0=\frac{j}{s}+\frac{v\cos j}{s}-\frac{\left(-1+e^v\right)}{s}+\frac{6v.6z^2}{s^4},$$

$$N(v,s,z,j)=\frac{j}{s^2}+\frac{v\cos j}{s^2}-\frac{\left(-1+e^v\right)}{s^2}+\frac{36vz^2}{s^5}. \quad (40)$$

Proceeding the inverse of transformation of (40) w. r. t, *'w'*,

$$N(v,w,z,j)=jw+vw\cos j-w\left(-1+e^v\right)+\frac{3}{2}vw^4z^2.$$

Again substituting the value of $\frac{\partial^2 f}{\partial z\partial j}=N$,

$$\frac{\partial^2 f(v,w,z,j)}{\partial z\partial j}=jw+vw\cos j-w\left(-1+e^v\right)+\frac{3}{2}vw^4z^2, \quad (41)$$

now putting $\frac{\partial f}{\partial j} = M$ in equation (41),

$$\frac{\partial M}{\partial z} = jw + vw\cos j - w\left(-1+e^{v}\right) + \frac{3}{2}vw^{4}z^{2}. \tag{42}$$

Proceeding the Laplace transform of (42) w. r. t, *'z'* and applying the differential property of Laplace,

$$sM(v,w,s,j) - M(v,w,0,j) = \frac{jw}{s} + \frac{vw\cos j}{s} - \frac{w\left(-1+e^{v}\right)}{s} + \frac{3}{2}.vw^{4}.\frac{2}{s^{3}},$$

$$M(v,w,s,j) = \frac{e^{-w}}{s} + \frac{jw}{s^{2}} + \frac{vw\cos j}{s^{2}} - \frac{w\left(-1+e^{v}\right)}{s^{2}} + 3vw^{4}.\frac{1}{s^{4}}. \tag{43}$$

Proceeding the inverse Laplace transform of equation (43) w. r. t, 'z',

$$M(v,w,z,j) = ze^{-w} + jwz + vwz\cos j - wz\left(-1+e^{v}\right) + \frac{vw^{4}z^{3}}{2},$$

Again substitute the value of $\frac{\partial f}{\partial j} = M$ in above equation,

$$\frac{\partial f(v,w,z,j)}{\partial j} = ze^{-w} + jwz + vwz\cos j - wz\left(-1+e^{v}\right) + \frac{vw^{4}z^{3}}{2}. \tag{44}$$

Proceeding the Laplace transform of equation (44) w. r. t, *'j'* and apply the differential property of Laplace,

$$sf(v,w,z,s) - f(v,w,z,0) = \frac{ze^{-w}}{s} + \frac{wz}{s^{2}} + vwz\left(\frac{s}{s^{2}-1}\right) - \frac{wz\left(-1+e^{v}\right)}{s} + \frac{vw^{4}z^{3}}{2s},$$

$$f(v,w,z,s) = \frac{1}{s} + \frac{ze^{-w}}{s^{2}} + \frac{wz}{s^{3}} + vwz\left(\frac{1}{s^{2}-1}\right) - \frac{wz\left(-1+e^{v}\right)}{s^{2}} + \frac{vw^{4}z^{3}}{2s^{2}}. \tag{45}$$

Proceeding the inverse of transform of (45) w. r. t, *'j'*,

$$f(v,w,z,\mathrm{j}) = j + zje^{-w} + \frac{wzj^{2}}{2} + vwz\sin j - wzj\left(-1+e^{v}\right) + \frac{vw^{4}z^{3}j}{2}.$$

Example 4: Consider a non-homogenous partial differential equation

$$\frac{\partial^{3} f}{\partial v\partial w\partial z} + \frac{\partial f}{\partial v} = we^{v+z}, \tag{46}$$

with initial conditions,

$$f(v,0,z) = 0, \qquad f_{w}(v,w,0) = 1, \qquad f_{z}(0,w,z) = e^{w}, \qquad f_{z}(0,w,z) = z.$$

Solution: Equation (47) can be written in the form,

$$\frac{\partial}{\partial v}\left(\frac{\partial^{2} f}{\partial w\partial z}\right) + \frac{\partial f}{\partial v} = we^{v+z}, \tag{48}$$

putting $\frac{\partial^2 f}{\partial w \partial z} = F$ in equation (48),

$$\frac{\partial F}{\partial v} + \frac{\partial f}{\partial v} = we^{v+z} \tag{49}$$

Proceeding the Laplace transform of equation (49) w. r. t, 'v',

$$sF(s,w,z) - F(0,w,z) + sf(s,w,z) - f(0,w,z) = \frac{we^z}{s-1}$$

Applying the initial condition,

$$sF(s,w,z) - e^w + sf(s,w,z) - z = \frac{we^z}{s-1}$$

$$F(s,w,z) = -f(s,w,z) + \frac{e^w}{s} + \frac{z}{s} + \frac{we^z}{s(s-1)} \tag{50}$$

Proceeding the inverse Laplace transform of equation (50) w. r. t, 'v',

$$F(v,w,z) = -f(v,w,z) + e^w + z + we^z\left(-1+e^v\right).$$

Again substitution the value of F,

$$\frac{\partial^2 f(v,w,z)}{\partial w \partial z} = -f(v,w,z) + e^w + z + we^z\left(-1+e^v\right), \tag{51}$$

now putting $\frac{\partial f}{\partial z} = M$ in equation (51),

$$\frac{\partial M}{\partial w} = -f(v,w,z) + e^w + z + we^z(-1+e^v) \tag{52}$$

Proceeding the Laplace transform of equation (52) w. r. t, 'w' and apply the differential property of Laplace,

$$sM(v,s,z) - M(v,0,z) = -L_w[f(v,w,z)] + \frac{1}{s-1} + \frac{z}{s} + \frac{1}{s^2}e^z(e^v - 1)$$

Applying the initial condition,

$$sM(v,s,z) - 0 = -L_w[f(v,w,z)] + \frac{1}{s-1} + \frac{z}{s} + \frac{1}{s^2}e^z(e^v - 1)$$

$$M(v,s,z) = -\frac{1}{s}L_w[f(v,w,z)] + \frac{1}{s(s-1)} + \frac{z}{s^2} + \frac{1}{s^3}e^z(e^v - 1) \tag{53}$$

Proceeding the inverse Laplace transform of equation (53) w. r. t, 'w',

$$M(\mathrm{v,w,z}) = -L_w^{-1}\left[\frac{1}{s}L_w[f(v,w,z)]\right] + (-1+e^w) + zw + \frac{w^2}{2}e^z(e^v - 1)$$

$$\frac{\partial f(v,w,z)}{\partial z} = -L_w^{-1}\left[\frac{1}{s}L_w[f(v,w,z)]\right] + (-1+e^w) + zw + \frac{w^2}{2}e^z(e^v-1). \tag{54}$$

Proceeding the Laplace transform of equation (54) w. r. t, 'z',

$$sf(v,w,s) - f(v,w,0) = -L_z\left[L_w^{-1}\left[\frac{1}{s}L_w[f(v,w,z)]\right]\right] + \frac{(-1+e^w)}{s} + \frac{w}{s^2} + \frac{w^2(e^v-1)}{2(s-1)}$$

Applying the initial condition and simplification,

$$f(v,w,s) = \frac{1}{s} - \frac{1}{s}L_z\left[L_w^{-1}\left[\frac{1}{s}L_w[f(v,w,z)]\right]\right] + \frac{(-1+e^w)}{s^2} + \frac{w}{s^3} + \frac{w^2(e^v-1)}{2s(s-1)} \tag{55}$$

Proceeding the inverse Laplace transform of (55) w. r. t, 'z',

$$f(v,w,z) = 1 - L_z^{-1}\left[\frac{1}{s}L_z\left[L_w^{-1}\left[\frac{1}{s}L_w[f(v,w,z)]\right]\right]\right] + z(-1+e^w) + \frac{z^2w}{2} + \frac{w^2(e^z-1)(e^v-1)}{2}.$$

To find the solution of nonlinear term use the variational iteration method,

$$\frac{\partial^3 f(v,w,z)}{\partial v\partial w\partial z} = \frac{\partial^3 f(1)}{\partial v\partial w\partial z} - \frac{\partial^3 f}{\partial v\partial w\partial z}\left[L_z^{-1}\left[\frac{1}{s}L_z\left[L_w^{-1}\left[\frac{1}{s}L_w[f(v,w,z)]\right]\right]\right]\right]$$
$$+\frac{\partial^3 f}{\partial v\partial w\partial z}\left(z(-1+e^w)\right) + \frac{\partial^3 f}{\partial v\partial w\partial z}\left(\frac{z^2w}{2}\right) + \frac{\partial^3 f}{\partial v\partial w\partial z}\left(\frac{w^2(e^z-1)(e^v-1)}{2}\right)$$

$$\frac{\partial^3 f(v,w,z)}{\partial v\partial w\partial z} = 0 - \frac{\partial^3 f}{\partial v\partial w\partial z}\left[L_z^{-1}\left[\frac{1}{s}L_z\left[L_w^{-1}\left[\frac{1}{s}L_w[f(v,w,z)]\right]\right]\right]\right] + 0 + 0 + 0 + e^v w e^z.$$

Now using the recursive relation and then taking *n=0,*

$$u_{n+1}(v,w,z) = u_n(v,w,z)$$
$$-\int_0^v\int_0^w\int_0^z (u_n)_{vwz}(v,w,z) + \frac{\partial^3}{\partial v\partial w\partial z}\left[L_z^{-1}\left[\frac{1}{s}L_z\left[L_w^{-1}\left[\frac{1}{s}L_w[u_n(v,w,z)]\right]\right]\right]\right] - we^{v+z}dvdwdz$$

$$u_1(v,w,z) = u_0(v,w,z).$$
$$-\int_0^v\int_0^w\int_0^z (u_0)_{vwz}(v,w,z) + \frac{\partial^3}{\partial v\partial w\partial z}\left[L_z^{-1}\left[\frac{1}{s}L_z\left[L_w^{-1}\left[\frac{1}{s}L_w[u_0(v,w,z)]\right]\right]\right]\right] - we^{v+z}dvdwdz$$

Choosing $u_0 = 1+z(-1+e^w) + \frac{z^2w}{2} + \frac{w^2(e^z-1)(e^v-1)}{2}$ by using the Adomain polynomial comparison,

$$u_1(v,w,z) = 1+z(-1+e^w) + \frac{z^2w}{2} + \frac{w^2(e^z-1)(e^v-1)}{2}$$

$$-\int_0^v\int_0^w\int_0^z 0 + \frac{\partial^3}{\partial v\partial w\partial z}\left[L_z^{-1}\left[\frac{1}{s}L_z\left[L_w^{-1}\left[\frac{1}{s}L_w[1+z(-1+e^w) + \frac{z^2w}{2} + \frac{w^2(e^z-1)(e^v-1)}{2}]\right]\right]\right]\right] - we^{v+z}dvdwdz$$

$$u_1(v,w,z)=1+z(-1+e^{w})\ +\frac{z^2w}{2}+\frac{w^2(e^z-1)(e^v-1)}{2}-\int_0^v\int_0^w\int_0^z we^{z+v}\ +e^v w^2(e^z-1)-we^{v+z}dvdwdz$$

$$u_1(v,w,z)=1+z(-1+e^{w})\ +\frac{z^2w}{2}+\frac{w^2(e^z-1)(e^v-1)}{2}-\int_0^v\int_0^w\int_0^z e^v w^2(e^z-1)dvdwdz$$

$$u_1(v,w,z)=1+z(-1+e^{w})\ +\frac{z^2w}{2}+\frac{w^2(e^z-1)(e^v-1)}{2}-\frac{w^3e^{v+z}}{3}+\frac{e^v w^3 z}{3}.$$

4. Conclusion

In this research work, we conclude that method of substitution for Laplace is appropriate and suitable for solving the non-homogenous linear partial differential equation of more than two independent variables with less computational as compared to the separation method of variable. This suggested method give the accurate solution if the non-linear term $Ru(v,w,z)=0$. The equations that involve the non-linear term cannot solve by using the substitution method of Laplace. If non-linear term is not equal to zero than we solve the equation by using the substitution method of Laplace as well as variational iteration method. As a result the method of substitution for Laplace can be functional for other equations which are used in different aspect of science.

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