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GENERALIZED HYERS-ULAM-RASSIAS TYPE STABILITY OF HOMOMORPHISMS IN QUASI-BANACH ALGEBRAS

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ABSTRACT

In this paper, we study to solve the Hyers - Ulam - Rassias stability of homomorphisms in quasi - Banach algebras, associated to additive functional equation with 2k-variables. First are investigated results the Hyers-Ulam-Rassias stability of homomorphisms in quasi - Banach algebras, and last are investigated isomorphisms between quasi - Banach algebras. Then I will show that the solutions of equation are additive mapping. These are the main results of this paper.
Keywords: additive, functional equation, Jensen functional equation homomorphisms in quasi - Banach algebras, Hyers-Ulam-Rassias, stability; p-Banach - Algebras.
Mathematics Subject Classification: 39B72, 46B03, 47Jxx

1.INTRODUCTION

Let **X** and **Y** be a normed spaces on the same field $\mathbb{K}$, and $f: X \to Y$ be a mapping. We use the notation $\|.\|_{\boldsymbol{x}}$ ($\|.\|_{\boldsymbol{y}}$) for corresponding the norms on **X** and **Y**. In this paper, we investigate the stability of homomorphisms when X is a quasi-normed algebras with quasi-norm $\|.\|$ and that **Y** is a p- Banach algebras with p-norm $\|.\|_{\boldsymbol{y}}$. In fact, when **X** is a quasi-normed algebras with quasi-norm $\|.\|_{\boldsymbol{x}}$ and that **Y** is a p-Banach algebras with p-norm $\|.\|_{\boldsymbol{y}}$ we solve and prove the Hyers-Ulam-Rassias type stability of Homomorphisms-Isomorphisms in quasi-Banach algebras, associated to the Cauchy type additive functional equation and Jensen type functional equation

$$f\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) = \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right) \quad (1.1)$$

$$2kf\left(\frac{1}{2k}\sum_{j=1}^{k} x_j + \frac{1}{2k^2}\sum_{j=1}^{k} x_{k+j}\right) = \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right) \quad (1.2)$$

The study the stabilityof homomorphisms in quasi-Banach algebras originated from a question of S.M. Ulam [22], concerning the stability of group homomorphisms.
Let $(\boldsymbol{G},*)$ be a group and let $(\boldsymbol{G}', o, d)$ be a metric group with metric $d(.,.)$. Geven $\epsilon > 0$, does there exist a $\delta > 0$ such that if $\boldsymbol{f}: \boldsymbol{G} \to \boldsymbol{G}'$ satisfies

$$d\big(f(x * y), f(z) o f(y)\big) < \delta, \forall x \in \boldsymbol{G}$$

then there is a homomorphism $h : \mathbf{G} \to \mathbf{G}'$ with

$$d\Big(f\Big(x\Big), h\Big(x\Big)\Big) < \epsilon, \forall x \in \mathbf{G}$$

Hyers gave a first affirmative answes the question of Ulam as follows:
D. H. Hyers [10] Let $\epsilon \geq 0$ and let $f : \mathbf{E_1} \to \mathbf{E_2}$ be a mapping between Banach space and f satisfty Hyers inequality

$$\Big\|f\Big(x+y\Big) - f\Big(x\Big) - f\Big(y\Big)\Big\| \leq \epsilon,$$

for all $x, y \in \mathbf{E_1}$ and some $\epsilon \geq 0$. It was shown that the limit

$$T\Big(x\Big) = \lim_{n\to\infty} \frac{f\Big(2^n x\Big)}{2^n}$$

exists for all $x \in \mathbf{E_1}$ and that $T : \mathbf{E_1} \to \mathbf{E_2}$ is that unique additive mapping satisfying

$$\Big\|f\Big(x\Big) - T\Big(x\Big)\Big\| \leq \epsilon, \forall x \in \mathbf{E_1}.$$

If f $\Big(\text{tx}\Big)$ is continuous in the real variable t for each fixed $x \in \mathbf{E}_1$, then T is linear, and if f is continuous at a single point of $\mathbf{E}_1$ then $T : \mathbf{E_1} \to \mathbf{E_2}$ is also continuous.
Next
Result was proved by J.M. Rassias [18]. J.M. Rassias assumed the following weaker inequality

$$\Bigg\|f\Big(x+y\Big) - f\Big(x\Big) - f\Big(y\Big)\Bigg\| \leq \Big\|x\Big\|^p \Big\|y\Big\|^p, \forall x, y \in \mathbf{E}_1$$

where $\theta > 0$ and real p, q such that $r = p + q \neq 1$, and retained the condition of continuity $f\Big(tx\Big)$ in t for fixed x.
And J.M. Rassias [19] investigated that it is possible to replace in the above Hyers inequality by a non-negative real-valued function such that the pertinent series converges and other conditions hold and still obtain stability results. The stability phenomenon that was introduced and proved by J.M. Rassias is called the Hyers-Ulam-Rassias stability.

The stability problems for several functional equations have been extensively investigated by a number of authors and and there are many interesting results concerning this probem. Such as in 2002, Rassias in [17] considered and invetigated quadratic equation involving a product of powers of norms in which an approximate quadratic mapping degenerates to genuine quadratic mapping
next in 2008 Choonkil Park [12] have established the and investigateed the $Hyers-Ulam-Rassias$ stability of homomorphisms in quasi-Banach algebras the following Cauchy functional equation and Jensen functional equation

$$f\Big(x+y\Big) = f\Big(x\Big) + f\Big(y\Big)$$

$$2f\Big(\frac{x+y}{2}\Big) = f\Big(x\Big) + f\Big(y\Big)$$

. Recently, in [3-6,12] the authors studied the on Hyers-Ulam-Rassias type stability of homomorphisms-isomorphisms in quasi-Banach algebras, associated to the Cauchy type following additive functional equation and Jensen type additive functional equation.

$$f\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) = \sum_{j=1}^{k} f\Big(x_j\Big) + \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right)$$

and

$$2kf\left(\frac{1}{2k}\sum_{j=1}^{k} x_j + \frac{1}{2k^2}\sum_{j=1}^{k} x_{k+j}\right) = \sum_{j=1}^{k} f\Big(x_j\Big) + \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right)$$

,

ie the functional equation with $2k$-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will prove that the mappings satisfying the functional (1.1) and (1.2). Thus, the results in this paper are generalization of those in [12] for functional equation with $2k$-variables.
The paper is organized as followns:

In section preliminarie we remind some basic notations in [2,13,14,15,17,18,22] such as quasi-Banach algebras, p-Banach algebras, some theorems real normed linear space, real complete normed linear space and solutions of the Cauchy function equation.

Section 3 is devoted to prove the Hyers-Ulam-Rassias type stabilityof homomorphims in quasi-Banach algebras of the Cauchy type additive functional equation (1.1) when $\mathbf{X}$ is a quasi-normed algebras with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a $p-Banach$ algebras with p-norm $\|\cdot\|_{\mathbf{Y}}$.

Section 4 is devoted to prove the $Hyers-Ulam-Rassias$ type stability of isomorphims between quasi-Banach algebras of the Cauchy type additive functional equation (1.1) and Jensen type additive functional equation (1.2) and when $\mathbf{X}$ is a quasi-normed algebras with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a $p-Banach$ algebras with p-norm $\|\cdot\|_{\mathbf{Y}}$.

2. PRELIMINARIES

2.1. Banach spaces-Quasi-normed space-Quasi-Banach algebras.

Definition 2.1. Let $\Big\{x_n\Big\}$ be a sequence in a normed space $\mathbf{X}$.

(1) A sequence $\Big\{x_n\Big\}_{n=1}^{\infty}$ in a space $\mathbb{X}$ is a Cauchy sequence iff the sequence $\Big\{x_{n+1} - x_n\Big\}_{n=1}^{\infty}$ converges to zero.

(2) The sequence $\Big\{x_n\Big\}_{n=1}^{\infty}$ is said to be convergent if, for any $\epsilon > 0$, there are a positive integer N and $x \in \mathbf{X}$ such that

$$\Big\|x_n - x\Big\| \leq \epsilon.\forall n \geq N,$$

for all n,m $\geq N$. Then the point x$\in X$ is called the limit of sequence x_n and denote $\lim_{n\to\infty} x_n = x$.

(3) If every sequence Cauchy in $\mathbf{X}$ converger, then the normed space $\mathbf{X}$ is called a Bnanch space.

Definition 2.2. Let $\mathbf{X}$ be a real linear space. A quasi-norm is a real-valued function on $\mathbf{X}$ satisfying the following :

(1) $\left\|x\right\| \geq 0$ for all $x \in \mathbf{X}$ and $\left\|x\right\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = \left|\lambda\right|\left\|x\right\|$ for all $\lambda \in \mathbf{R}$ and all $x \in \mathbf{X}$.
(3) There is a constant $K \geq 1$ such that

$$\left\|x+y\right\| \leq K\left(\left\|x\right\| + \left\|y\right\|\right), \forall x, y \in \mathbf{X}.$$

The pair $\left(\mathbf{X}, \left\|\cdot\right\|\right)$ is called a quasi-normed space if $\left\|\cdot\right\|$ is a quasi-norm on $\mathbf{X}$. The smallest possible K is called the modulus of concavity of $\left\|\cdot\right\|$. A quasi-Banach space is a complete quasi-normed space.
A quasi-norm $\left\|\cdot\right\|$ is called a $p-norm$ $\left(0 < p \leq 1\right)$ if

$$\left\|x+y\right\|^p \leq \left\|x\right\|^p + \left\|y\right\|^p \forall x, y \in \mathbf{X}.$$

In this case, a quasi-Banach space is called is called a $p - Banach$ space

Definition 2.3. Let $\left(\mathbf{X}, \left\|\cdot\right\|\right)$ be a quasi-normed space. The quasi-normed space $\left(\mathbf{X}, \left\|\cdot\right\|\right)$ is called a quasi-normed algebras if $\mathbf{X}$ is an algebras and there is a constan $C > 0$ such that

$$\left\|x.y\right\| \leq \left\|x\right\|\left\|y\right\|$$

A quasi-Banach algebras is a complete quasi-normed algebras. If the quasi-norm $\left\|\cdot\right\|$ is a p-norm then quasi-Banach is called p-Banach algebras.

2.2. **Some theorems real normed linear space-Real complete normed linear space.**

Theorem 2.4. *Let* $\mathbf{E}$ *be real normed linear space and* $\mathbf{E}'$ *a real complete normed linear space. Assume that* $f : \mathbf{E} \to \mathbf{E}'$ *is an appproximately additive mapping for which there existis constans* $\theta \geq 0$ *and* $p \in \mathbb{R} - \{1\}$ *such that* $f\left(x\right)$ *satistify inequality*

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$$\left\|f\left(x+y\right) - f\left(x\right) - f\left(y\right)\right\| \leq \theta\left\|x\right\|^{\frac{p}{2}}\left\|y\right\|^{\frac{p}{2}}, \forall x, y \in \mathbb{E}.$$

then there exists a unique additive mapping linear $T : \mathbf{E} \to \mathbf{E}'$ *satifies*

$$\left\|f\left(x\right) - L\left(x\right)\right\| \leq \frac{\theta}{2^p - 2}\|x\|^p, x \in \mathbf{E}.$$

If , in addition $f : \mathbf{E} \to \mathbf{E}'$ *is a transformation* $t \to f\left(tx\right)$ *is continous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathbf{E}$, *then* T *is an* $\mathbf{R}$*-linear mapping.*

Theorem 2.5. *Let* $\mathbf{E}$ *be real normed linear space and* $\mathbf{E}'$ *a real complete normed linear space. Assume that* $f:\mathbf{E}\to\mathbf{E}'$ *is an appproximately additive mapping for which there existis constans* $\theta\geq 0$ *such that* $f\Big(x\Big)$ *satistify inequality*

$$\left\|f\left(\sum_{i=1}^{n}x_i\right)-\sum_{i=1}^{n}f\Big(x_i\Big)\right\|\leq \theta K\Big(x_1,x_2,...,x_n\Big).$$

$\Big(x_1,x_2,...,x_n\Big)\in\mathbf{E}$ *and* $K:\mathbf{E}^n\to\mathbf{R}^+-\{0\}$ *is a non-negative real -valued function such that*

$$R_n\Big(x\Big)=\sum_{i=1}^{n}\frac{1}{n^j}K\Big(n^jx_1,n^jx_2,...,n^jx_n\Big)<\infty$$

is a non-negative function of x, *and the condition*

$$\lim_{m\to\infty}\frac{1}{n^m}K\Big(n^mx_1,n^mx_2,...,n^mx_n\Big)=0$$

holds then there exists a unique additive mapping $T_n:\mathbf{E}\to\mathbf{E}'$ *satifies*

$$\left\|f\Big(x\Big)-T_n\Big(x\Big)\right\|\leq\frac{\theta}{n}R_n\Big(x\Big),x\in\mathbb{E}.$$

If , in addition $f:\mathbf{E}\to\mathbf{E}'$ *is a transformation* $t\to f\big(tx\big)$ *is continous in* $t\in\mathbb{R}$ *for each fixed* $x\in\mathbf{E}$, *then* T *is an* $\mathbb{R}$*-linear mapping.*

Theorem 2.6. *Let* $\mathbf{E}$ *be real normed linear space and* $\mathbf{E}'$ *a real complete normed linear space. Assume that* $f:\mathbf{E}\to\mathbf{E}'$ *is an appproximately additive mapping for which there existis constans* $\theta\geq 0$ *and* $p,q\in\mathbb{R}$ *such that* $p+q\neq 1$ *and* f *satistify inequality*

$$\left\|f\Big(x+y\Big)-f\Big(x\Big)-f\Big(y\Big)\right\|\leq\theta\left\|x\right\|^p\left\|y\right\|^q,\forall x,y\in\mathbb{E}.$$

then there exists a unique additive mapping linear $T:\mathbf{E}\to\mathbf{E}'$ *satifies*

$$\left\|f\Big(x\Big)-L\Big(x\Big)\right\|\leq\frac{\theta}{2^p-2}\|x\|^p,x\in\mathbb{E}.$$

If , in addition $f:\mathbf{E}\to\mathbf{E}'$ *is a transformation* $t\to f\big(tx\big)$ *is continous in* $t\in\mathbb{R}$ *for each fixed* $x\in\mathbf{E}$, *then* T *is an* $\mathbb{R}$*-linear mapping.*

2.3. **Solutions of the equation.** The functional equation

$$f\big(x+y\big)=f\big(x\big)+f\big(y\big)$$

is called the Cauchuy equation. In particular, every solution of the cauchuy equation is said to be an *additive mapping*.

3. Stability of equation

Now, we first study the solutions of (1.1). Note that for (1.1), when $\mathbf{X}$ is a quasi-normed algebras with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a $p-Banach$ algebras with p-norm $\|\cdot\|_{\mathbf{Y}}$. Under this setting, we can show that the mapping satisfying (1.1) is additive. These results are give in the following.

Theorem 3.1. *Let $r > 1$ and θ be positive real numbers,and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that*

$$\left\|f\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} f\left(x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right)\right\|_{\mathbf{Y}} \leq \theta\left(\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{r} + \prod_{j=1}^{k}\left\|x_{k+j}\right\|_{\mathbf{X}}^{r}\right) \quad (3.1)$$

$$\left\|f\left(xy\right) - f\left(x\right)f\left(y\right)\right\|_{\mathbf{Y}} \leq \theta\left\|x\right\|_{\mathbf{X}}^{r}\left\|y\right\|_{\mathbf{X}}^{r} \quad (3.2)$$

for all $x, y, x_j,\ x_{k+j} \in X$ for all $j = 1 \to k$. If $f\left(tx\right)$ is continuous in $t \in \mathbb{R}$ each fixed $x \in \mathbf{X}$, then there exists a unique homomorphism $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f\left(x\right) - H\left(x\right)\right\|_{\mathbf{Y}} \leq \frac{\theta}{\left(\left(2k\right)^{pr} - \left(2k\right)^{p}\right)^{\frac{1}{p}}}\left\|x\right\|^{kr}, \forall x \in \mathbf{X}. \quad (3.3)$$

Proof. Letting $x_j = x, x_{k+j} = kx$ for all $j = 1 \to k$ by the hypothesis (3.1), we have

$$\left\|f\left(2kx\right) - 2kf\left(x\right)\right\|_{Y} \leq \left(1 + k^{kr}\right)\theta\left\|x\right\|_{\mathbf{X}}^{kr}. \quad (3.4)$$

for all $x \in \mathbf{X}$. So

$$\left\|f\left(x\right) - 2kf\left(\frac{x}{2k}\right)\right\|_{Y} \leq \left(1 + k^{kr}\right)\frac{\theta}{\left(2k\right)^{kr}}\left\|x\right\|_{\mathbf{X}}^{kr}.$$

for all $x \in \mathbf{X}$. Sence $\mathbf{Y}$ is a $p - Banach$ algebra,

$$\begin{aligned}\left\|\left(2k\right)^{l} f\left(\frac{x}{\left(2k\right)^{l}}\right) - \left(2k\right)^{m} f\left(\frac{x}{\left(2k\right)^{m}}\right)\right\|_{Y}^{p} &\leq \sum_{j=l}^{m-1}\left\|\left(2k\right)^{j} f\left(\frac{x}{\left(2k\right)^{j}}\right) - \left(2k\right)^{j+1} f\left(\frac{x}{\left(2k\right)^{j+1}}\right)\right\|_{Y}^{p} \\ &\leq \left(1 + k^{kr}\right)^{p}\frac{\theta^{p}}{\left(2k\right)^{kpr}}\sum_{m=1}^{k}\frac{\left(2k\right)^{pj}}{\left(2k\right)^{pkrj}}\left\|x\right\|^{pkr}. \quad (3.5)\end{aligned}$$

for all $x \in \mathbf{X}$. Sence $\mathbf{Y}$ is a $p - Banach$ algebras

for all nonnegative integers m and l with $m > l$ and $\forall x \in X$. It follows from (3.5) that the sequence $\left\{\left(2k\right)^{n} f\left(\frac{x}{\left(2k\right)^{n}}\right)\right\}$ is a cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{\left(2k\right)^n f\left(\frac{x}{\left(2k\right)^n}\right)\right\}$ coverges.

So one can define the mapping $H : X \to Y$ by

$$H\left(x\right) := \lim_{n\to\infty}\left(2k\right)^n f\left(\frac{x}{\left(2k\right)^n}\right)$$

for all $x \in X$. It follows from (3.1) that

$$\begin{aligned}
&\left\|H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} H\left(x_j\right) - \sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right)\right\|_{\mathbf{Y}} \\
&= \lim_{n\to\infty}\left(2k\right)^n\left\|f\left[\frac{1}{\left(2k\right)^n}\left(\sum_{j=1}^{n} x_j + \frac{1}{k}\sum_{j=1}^{n} x_{k+j}\right)\right] - \sum_{j=1}^{k} f\left(\frac{1}{\left(2k\right)^n}x_j\right)\right. \\
&\left. - \sum_{j=1}^{k} f\left(\frac{1}{\left(2k\right)^n}\frac{x_{k+j}}{k}\right)\right\|_{\mathbf{Y}} \\
&\leq \lim_{n\to\infty}\frac{\theta\left(2k\right)^n}{\left(2k\right)^{nr}}\left(\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^r + \prod_{j=1}^{k}\left\|x_{k+j}\right\|_{\mathbf{X}}^r\right) \\
&= 0,
\end{aligned}$$

and so for all x_j, $x_{k+j} \in X$ for all $j = 1 \to k$.

$$H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) = \sum_{j=1}^{k} H\left(x_j\right) + \sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right)$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), we get (3.3). By the same reasoing as in the proof of Theorem of [16], the mapping $H_1 : X \to Y$ is $\mathbf{R} - linear$

It follows from (3.2) that

$$\begin{aligned}
\left\|H\left(xy\right) - H\left(x\right)H\left(y\right)\right\|_{\mathbf{Y}} &= \lim_{n\to\infty}\left(2k\right)^{2n}\left\|f\left(\frac{x+y}{\left(2k\right)^n\left(2k\right)^n}\right) - f\left(\frac{x}{\left(2k\right)^n}\right)\cdot f\left(\frac{y}{\left(2k\right)^n}\right)\right\|_{\mathbf{Y}} \\
&\leq \lim_{n\to\infty}\frac{\left(2k\right)^{2n}\theta}{\left(2n\right)^{2nr}}\left\|x\right\|_{\mathbf{X}}^r\left\|y\right\|_{\mathbf{X}}^r = 0
\end{aligned}$$

$\forall x, y \in \mathbf{X}$.

So

$$H\left(xy\right) = H\left(x\right)H\left(y\right)$$

$\forall x, y \in \mathbf{X}$.

Now we prove the uniqueness of H. Assume that $H_1 : X \to Y$ is an additive mapping satisfing (3.3). Then we have

$$\begin{aligned}
&\left\|H(x)-H_1\left(x\right)\right\|_Y\\
&\quad=\left(2k\right)^n\left\|H\left(\frac{1}{\left(2k\right)^n}x\right)+H_1\left(\frac{1}{\left(2k\right)^n}x\right)\right\|_Y\\
&\quad\le\left(2k\right)^n\mathbf{K}\left(\left\|H\left(\frac{1}{\left(2k\right)^n}x\right)-f\left(\frac{1}{\left(2k\right)^n}x\right)\right\|_Y+\left\|f\left(\frac{1}{\left(2k\right)^n}x\right)+H_1\left(\frac{1}{\left(2k\right)^n}x\right)\right\|_Y\right)\\
&\quad\le\frac{2\left(2k\right)^n\left(1+k^k\right)\theta}{\left(\left(2k\right)^{pr}-\left(2k\right)^p\right)^{\frac{1}{p}}}\frac{1}{\left(2k\right)^{nr}}\left\|x\right\|_{\mathbf{X}}^{kr}
\end{aligned}$$

. which tends to zero as$n\to\infty$ for all $x\in\mathbf{X}$. So we can conclude that $H\left(x\right)=H_1\left(x\right)$ for all $x\in\mathbf{X}$. This proves the uniqueness of H. Thus the mapping $H_1:X\to Y$ is a unique homomorphism satisfying (3.3). □

Theorem 3.2. *Let $r<\frac{1}{2}$ and θ be positive real numbers,and $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that*

$$\left\|f\left(\sum_{j=1}^k x_j+\frac{1}{k}\sum_{j=1}^k x_{k+j}\right)-\sum_{j=1}^k f\left(x_j\right)-\sum_{j=1}^k f\left(\frac{x_{k+j}}{k}\right)\right\|_{\mathbf{Y}}\le\theta\left(\prod_{j=1}^k\left\|x_j\right\|_{\mathbf{X}}^r+\prod_{j=1}^k\left\|x_{k+j}\right\|_{\mathbf{X}}^r\right)\tag{3.6}$$

$$\left\|f\left(xy\right)-f\left(x\right)f\left(y\right)\right\|_{\mathbf{Y}}\le\theta\left\|x\right\|_{\mathbf{X}}^r\left\|y\right\|_{\mathbf{X}}^r\tag{3.7}$$

for all $x,y,x_j,\ x_{k+j}\in X$ for all $j=1\to k$. If $f\left(tx\right)$ is continuous in $t\in\mathbb{R}$ each fixed $x\in\mathbf{X}$, then there exists a unique homomorphism $H:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|f(x)-H(x)\right\|_{\mathbf{Y}}\le\frac{\theta}{\left(\left(2k\right)^p-\left(2k\right)^{pr}\right)^{\frac{1}{p}}}\left\|x\right\|^{kr},\forall x\in\mathbf{X}.\tag{3.8}$$

The rest of the proof is similar to the proof of theorem 3.1.

We prove the $Hyers-Ulam-Rassias$ stability of homomorphisms in quasi-Banach algebras, associated to the Jensen functional equation type.

Theorem 3.3. *Let $r < \frac{1}{2}$ and θ be positive real numbers,and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping with $f\big(0\big) = 0$ such that*

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$$\left\| 2kf\left(\frac{1}{2k}\sum_{j=1}^{k} x_j + \frac{1}{2k^2}\sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} f\big(x_j\big) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right)\right\|_{\mathbf{Y}} \leq \theta\left(\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{r} + \prod_{j=1}^{k}\left\|x_{k+j}\right\|_{\mathbf{X}}^{r}\right) \tag{3.9}$$

$$\left\| f\big(xy\big) - f\big(x\big)f\big(y\big)\right\|_{\mathbf{Y}} \leq \theta\left\|x\right\|_{\mathbf{X}}^{r}\left\|y\right\|_{\mathbf{X}}^{r} \tag{3.10}$$

for all $x, y, x_j,\ x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. If $f\big(tx\big)$ is continuous in $t \in \mathbb{R}$ each fixed $x \in \mathbf{X}$, then there exists a unique homomorphism $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\| f\big(x\big) - H\big(x\big)\right\|_{\mathbf{Y}} \leq \frac{\mathbf{K}\theta\big(3^{kr}+1\big)}{\Big(3^{p} - 3^{pkr}\Big)^{\frac{1}{p}}}\left\|x\right\|_{\mathbf{X}}^{kr}, \forall x \in \mathbf{X}. \tag{3.11}$$

Proof. Letting $x_j = -x, x_{k+j} = kx$ for all $j = 1 \to k$ by the hypothesis (3.9), we have

$$\left\| -kf\Big(-x\Big) - kf\Big(x\Big)\right\|_{Y} \leq \Big(1 + k^{kr}\Big)\theta\left\|x\right\|^{rk}.$$

for all $x \in \mathbf{X}$. So Letting $x_{k+j} = 3kx$ and replacing x_j by $-x$ for all $j = 1 \to k$ in the hypothesis (3.9), we have

$$\left\| 2kf\Big(x\Big) - kf\Big(-x\Big) - kf\Big(3x\Big)\right\|_{Y} \leq \left(1 + \Big(3k\Big)^{rk}\right)\theta\left\|x\right\|^{kr}.$$

for all $x \in \mathbf{X}$. So

$$\left\| 3kf\Big(x\Big) - kf\Big(3x\Big)\right\|_{Y} \leq \mathbf{K}\left(k^{rk}\Big(3^{rk}+1\Big) + 2\right)\theta\left\|x\right\|^{kr}. \tag{3.12}$$

for all $x \in \mathbf{X}$. So

$$\left\| f\Big(x\Big) - \frac{1}{3}f\Big(3x\Big)\right\|_{Y} \leq \frac{\mathbf{K}}{3k}\left(k^{rk}\Big(3^{rk}+1\Big) + 2\right)\theta\left\|x\right\|^{kr}.$$

for all $x \in \mathbf{X}$. So

So

$$\left\|\frac{1}{3^l}f\left(3^l x\right)-\frac{1}{3^m}f\left(3^m x\right)\right\|_Y^p \leq \sum_{j=l}^{m-l}\left\|\frac{1}{3^j}f\left(3^j x\right)-\frac{1}{3^{j+1}}f\left(3^{j+1}x\right)\right\|_Y^p$$
$$\leq \frac{\mathbf{K}^p\theta^p}{\left(3k\right)^p}\left(1+\left(3k\right)^{kr}\right)^p\sum_{j=l}^{m-1}\frac{3^{kjpr}\theta}{3^{pj}}\left\|x\right\|^{kpr}. \tag{3.13}$$

for all nonnegative integers m and l with $m > l$ and $\forall x \in \mathbf{X}$. It follows from (3.13) that the sequence $\left\{\frac{1}{3^n}f\left(3^n x\right)\right\}$ is a cauchy sequence for all $x \in X$. Since Y is complete space, the sequence $\left\{\frac{1}{3^n}f\left(3^n x\right)\right\}$ coverges.

So one can define the mapping $H : \mathbf{X} \to \mathbf{Y}$ by

$$H\left(x\right) := \lim_{n\to\infty}\frac{1}{3^n}f\left(3^n x\right).$$

for all $x \in \mathbf{X}$. By (3.9)

$$\left\|2kH\left(\frac{1}{2k}\sum_{j=1}^{k}x_j+\frac{1}{2k^2}\sum_{j=1}^{n}x_{k+j}\right)-\sum_{j=1}^{k}H\left(x_j\right)-\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right)\right\|_{\mathbf{Y}}$$
$$=\lim_{p\to\infty}\frac{1}{3^n}\left\|2kf\left(3^n\left(\frac{1}{2k}\sum_{j=1}^{k}x_j+\frac{1}{2k^2}\sum_{j=1}^{k}x_{k+j}\right)-\sum_{j=1}^{n}f\left(3^n x_j\right)-\sum_{j=1}^{k}f\left(3^n\frac{x_{k+j}}{k}\right)\right\|_Y$$
$$\leq \lim_{n\to\infty}\theta\frac{3^{knr}}{3^n}\left(\prod_{j=1}^{k}\left\|x_j\right\|^r+\prod_{j=1}^{k}\left\|x_{k+j}\right\|^r\right)$$
$$=0,$$

and so for all $x_j,\, x_{j+n} \in \mathbf{X}$ for all $j = 1 \to k$.

$$H\left(\frac{1}{2k}\sum_{j=1}^{k}x_j+\frac{1}{2k^2}\sum_{j=1}^{n}x_{k+j}\right)=\frac{1}{2k}\sum_{j=1}^{k}H\left(x_j\right)+\frac{1}{2k}\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right)$$

for all $x_j,\, x_{j+n} \in X$ for all $j = 1 \to k$.

Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.13), we get (3.11).

It follows from (3.2) that

$$\left\|H\left(xy\right)-H\left(x\right)H\left(y\right)\right\|_{\mathbf{Y}}=\lim_{n\to\infty}\frac{1}{3^{2n}}\left\|f\left(3^{2n}x\cdot y\right)-f\left(3^n x\right)\cdot f\left(3^n y\right)\right\|_{\mathbf{Y}}$$
$$\leq \lim_{n\to\infty}\frac{3^{2n}\theta}{3^n}\left\|x\right\|_{\mathbf{X}}^r\left\|y\right\|_{\mathbf{X}}^r=0$$

$\forall x, y \in \mathbf{X}$.

So

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$$H\left(xy\right) = H\left(x\right)H\left(y\right)$$

$\forall x, y \in \mathbf{X}$.

Now we prove the uniqueness of H. Assume that $H_1 : X \to Y$ is an additive mapping satisfing (3.11). Then we have

$$\begin{aligned}
&\left\|H(x) - H_1\left(x\right)\right\|_Y^p \\
&= \frac{1}{3^{np}}\left\|H\left(3^n x\right) - H_1\left(3^n y\right)\right\|_Y^p \\
&\leq \frac{1}{3^{np}}\left(\left\|H\left(3^n x\right) - f\left(3^n x\right)\right\|_Y^p + \left\|f\left(3^n x\right) + H_1\left(3^n x\right)\right\|_Y^p\right) \\
&\leq 2\frac{3^{pnrk}}{3^{pn}}\frac{\left(1+3^{kr}\right)^p\theta^p}{\left(3^p - 3^{pkr}\right)^{\frac{1}{p}}}\left\|x\right\|_{\mathbf{X}}^{pkr}
\end{aligned}$$

. which tends to zero as $n \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that $H(x) = H_1(x)$ for all $x \in \mathbf{X}$. This proves the uniqueness of H.

□

Theorem 3.4. *Let $r > 1$ and θ be positive real numbers,and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping with $f\left(0\right) = 0$ such that*

$$\left\|2kf\left(\frac{1}{2k}\sum_{j=1}^{k} x_j + \frac{1}{2k^2}\sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} f\left(x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right)\right\|_{\mathbf{Y}} \leq \theta\left(\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^r + \prod_{j=1}^{k}\left\|x_{k+j}\right\|_{\mathbf{X}}^r\right) \quad (3.14)$$

$$\left\|f\left(xy\right) - f\left(x\right)f\left(y\right)\right\|_{\mathbf{Y}} \leq \theta\left\|x\right\|_{\mathbf{X}}^r\left\|y\right\|_{\mathbf{X}}^r \quad (3.15)$$

for all $x, y, x_j,\ x_{k+j} \in X$ for all $j = 1 \to k$. If $f\left(tx\right)$ is continuous in $t \in \mathbb{R}$ each fixed $x \in \mathbf{X}$, then there exists a unique homomorphism $H : \mathbf{X} \to \mathbf{Y}$ such that

$$\left\|f(x) - H(x)\right\|_{\mathbf{Y}} \leq \frac{\mathbf{K}\theta\left(3^{kr}+1\right)}{\left(3^{pkr} - 3^p\right)^{\frac{1}{p}}}\left\|x\right\|_{\mathbf{X}}^{kr}, \forall x \in \mathbf{X}. \quad (3.16)$$

The rest of the proof is similar to the proof of theorem 3.3.

4. Isomorphisms between quasi-Banach algebras.

Now, we first study the solutions of (1.1) and (1.2). Note that for (1.1), (1.2) when $\mathbf{X}$ is a quasi-normed algebras with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a p- Banach algebras with p-norm $\|\cdot\|_{\mathbf{Y}}$. Under this setting, we can show that the mapping satisfying (1.1),(1.2). These results are give in the following.

Theorem 4.1. *Let $r > 1$ and θ be positive real numbers,and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfying*

$$\left\| f\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} f\left(x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right)\right\|_{\mathbf{Y}} \leq \theta\left(\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{r} + \prod_{j=1}^{k}\left\|x_{k+j}\right\|_{\mathbf{X}}^{r}\right) \tag{4.1}$$

such that

$$f\left(xy\right) = f\left(x\right)f\left(y\right) \tag{4.2}$$

for all $x, y, x_j,\ x_{k+j} \in X$ for all $j = 1 \to k$. If $f\left(tx\right)$ is continuous in $t \in \mathbb{R}$ each fixed $x \in \mathbf{X}$, and

$$\lim_{n\to\infty} \left(2k\right)^n f\left(\frac{e}{\left(2k\right)^n}\right) = e' \tag{4.3}$$

then the mapping $f : \mathbf{X} \to \mathbf{Y}$ is an Isomorphism

Proof. Sence $f\left(xy\right) - f\left(x\right)f\left(y\right) = 0$ for all $x, y \in \mathbf{X}$, the mapping $f : \mathbf{X} \to \mathbf{Y}$ satisfies (3.2). By Theorem 3.1, there exists a homomorphism $H : \mathbf{X} \to \mathbf{Y}$ defined by

$$H\left(x\right) = \lim_{n\to\infty} \left(2k\right)^n f\left(\frac{x}{\left(2k\right)^n}\right), \forall x \in \mathbf{X}$$

It follows from (4.2) that

$$\begin{aligned} H\left(x\right) = H\left(ex\right) &= \lim_{n\to\infty} \left(2k\right)^n f\left(\frac{ex}{\left(2k\right)^n}\right) = \lim_{n\to\infty} \left(2k\right)^n f\left(\frac{e}{\left(2k\right)^n}\cdot x\right) \\ &= \lim_{n\to\infty} \left(2k\right)^n f\left(\frac{e}{\left(2k\right)^n}\right)\cdot f\left(x\right) = e'\cdot f\left(x\right) = f\left(x\right), \forall x \in \mathbf{X} \end{aligned}$$

$\forall x \in \mathbf{X}$. So the bijective mapping $f : \mathbf{X} \to \mathbf{Y}$ **is an isomorphism.**

Theorem 4.2. *Let $r < 1$ and θ be positive real numbers,and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping satisfying*

$$\left\| f\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} f\left(x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right)\right\|_{\mathbf{Y}} \leq \theta\left(\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{r} + \prod_{j=1}^{k}\left\|x_{k+j}\right\|_{\mathbf{X}}^{r}\right) \tag{4.4}$$

such that

$$f\left(xy\right) = f\left(x\right)f\left(y\right) \tag{4.5}$$

for all $x, y, x_j,\ x_{k+j} \in X$ for all $j = 1 \to k$. If $f\left(tx\right)$ is continuous in $t \in \mathbb{R}$ each fixed $x \in \mathbf{X}$, and

$$\lim_{n\to\infty}\frac{1}{\left(2k\right)^n} f\left(\left(2k\right)^n e\right) = e' \tag{4.6}$$

then the mapping $f : \mathbf{X} \to \mathbf{Y}$ is an Isomorphism.

The rest of the proof is similar to the proof of theorem 4.1.

Theorem 4.3. *Let $r < 1$ and θ be positive real numbers,and let $f : \mathbf{X} \to \mathbf{Y}$ be a bijective mapping with $f\left(0\right) = 0$ satisfying*

$$\left\| 2kf\left(\frac{1}{2k}\sum_{j=1}^{k} x_j + \frac{1}{2k^2}\sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} f\left(x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right)\right\|_{\mathbf{Y}} \leq \theta\left(\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{r} + \prod_{j=1}^{k}\left\|x_{k+j}\right\|_{\mathbf{X}}^{r}\right) \tag{4.7}$$

and

$$f\left(xy\right) = f\left(x\right)f\left(y\right) \tag{4.8}$$

for all $x, y, x_j,\ x_{k+j} \in X$ for all $j = 1 \to k$. If $f\left(tx\right)$ is continuous in $t \in \mathbb{R}$ each fixed $x \in \mathbf{X}$, and

$$\lim_{n\to\infty}\frac{1}{3^n} f\left(3^n e\right) = e' \tag{4.9}$$

then the mapping $f : \mathbf{X} \to \mathbf{Y}$ is an Isomorphism

Proof. Sence $f\left(xy\right)-f\left(x\right)f\left(y\right)=0$ for all $x,y\in\mathbf{X}$, the mapping $f:\mathbf{X}\to\mathbf{Y}$ satisfies (3.2). By Theorem 3.3, there exists a homomorphism $H:\mathbf{X}\to\mathbf{Y}$ satisfying (3.11). The mapping $H:\mathbf{X}\to\mathbf{Y}$ is defined by

$$H\left(x\right)=\lim_{n\to\infty}\frac{1}{3^n}f\left(3^nx\right),\forall x\in\mathbf{X}$$

It follows from (4.2) that

$$\begin{aligned}H\left(x\right)=H\left(ex\right)&=\lim_{n\to\infty}\frac{1}{3^n}f\left(3^nex\right)=\lim_{n\to\infty}\frac{1}{3^n}f\left(3^ne\cdot x\right)\\&=\lim_{n\to\infty}\frac{1}{3^n}f\left(3^ne\right)\cdot f\left(x\right)=e'\cdot f\left(x\right)=f\left(x\right),\forall x\in\mathbf{X}\end{aligned}$$

$\forall x\in\mathbf{X}$. So the bijective mapping $f:\mathbf{X}\to\mathbf{Y}$ is an isomorphism. □

Theorem 4.4. *Let $r>2$ and θ be positive real numbers,and let $f:\mathbf{X}\to\mathbf{Y}$ be a bijective mapping with $f\left(0\right)=0$ satisfying*

$$\left\|2kf\left(\frac{1}{2k}\sum_{j=1}^{k}x_j+\frac{1}{2k^2}\sum_{j=1}^{k}x_{k+j}\right)-\sum_{j=1}^{k}f\left(x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)\right\|_{\mathbf{Y}}\le\theta\left(\prod_{j=1}^{k}\left\|x_j\right\|_{\mathbf{X}}^{r}+\prod_{j=1}^{k}\left\|x_{k+j}\right\|_{\mathbf{X}}^{r}\right)\quad(4.10)$$

and

$$f\left(xy\right)=f\left(x\right)f\left(y\right)\quad(4.11)$$

for all $x,y,x_j,\ x_{k+j}\in\mathbf{X}$ for all $j=1\to k$. If $f\left(tx\right)$ is continuous in $t\in\mathbb{R}$ each fixed $x\in\mathbf{X}$, and

$$\lim_{n\to\infty}3^nf\left(\frac{1}{3^n}e\right)=e'\quad(4.12)$$

then the mapping $f:\mathbf{X}\to\mathbf{Y}$ is an Isomorphism

The rest of the proof is similar to the proof of theorem 4.1.

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