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RESEARCH ARTICLE

GENERALIZED HYERS-ULAM STABILITY OF THE ADDITIVE FUNCTIONAL INEQUALITIES WITH 2n-VARIABLES IN NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT

In this paper we study to solve additive functional inequality with 2n-variables and their Hyers-Ulam stability in non-Archimedean Banach spaces. Then I will show that the solutions of inequality are additive mapping. Hyers - Ulam stability of this inequality are given and proven: These are the main results of this paper

Keywords: Non-Archimedean Banach space; generalized Hyers-Ulam stability: Jordan- von Newmann functional equation functional inequality

Let **X** and **Y** be a normed spaces on the same field $\mathbb{K}$, and $f: X \to Y$. We use the notation $\|\bullet\|$ for all the norm on both **X** and **Y**. In this paper, we investisgate additive functional inequality when **X** is non-Archimedean normed space and **Y** is non- Archimedean Banach spaces.

In fact, when **X** is non-Archimedean normed space and **Y** is non-Archimedean Banach spaces we solve and prove the Hyers-Ulam stability of forllowing additive functional in- equality.

$$\left\|\sum_{j=1}^{n} f(x_j)+\sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\| \le \left\|kf\left(\frac{\sum_{j=1}^{n} x_j}{k}+\frac{\sum_{j=1}^{n} x_{n+j}}{nk}\right)\right\|, |n|>|k| \qquad (1.1)$$

The notions of non- Archimedean normed space will remind in the next section. The Hyers-Ulam stability was first investigated for functional equation of Ulam in [6] concern- ing the stability of group homomorphisms.

The functional equation,

$$f(x+y)=f(x)+f(y)$$

it is called the Cauchy equation. In particular, each solution of the Cauchy equation is said to be an additive mapping. The Hyers [7] gave the first partial affirmative answer to the Ulam equation in Banach spaces. After that, Hyers' Theorem was generalized by Aoki's additive assignments [1] and by Rassias [8] for linear assignments considering a non-bounced Cauchy difference. Gaˇvruta [4] he obtained a generalization of the Rassias theorem by replacing the boundless Cauchy difference with a general control function in the spirit of the Rassias approach.

$$f\left(\frac{x+y}{2}\right)=\frac{1}{2}f(x)+\frac{1}{2}f(y)$$

is called the Jensen equation. For more information on functional equations see [2, 3, 9, 10].

The Hyers-Ulam stability for functional inequalities have been investigated such as in [14, 15]. Gilany showed that is if satisfies the functional inequality

$$\|2f(x)+2f(y)-f(xy^{-1})\|\leq\|f(xy)\|$$

Then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)-f(xy^{-1}) \tag{1.2}$$

See also [15,16]. Gila´nyi [13] and Fechner [9] proved the Hyers-Ulam stability of the func- tional inequality.

The stability problems for several functional equations have been extensively investigated by a number of authors and and there are many interesting results concerning this probem. In 2010 Yeol Je Cho Choonkil Park, Reza Saadati [21] investigated three-variable func- tional inequalities and next in 2018 Y.Aribou, S.kbbaj [22] proved the generalized Hyers- Ulam stability of n-variable functional inequlities in non-Archimedean Banach spaces. Recently, in [18, 19, 20, 22, 23,24] the authors studied the Hyers-Ulam stability for the following functional inequalities

$$\|f(x)+f(y)+f(z)\|\leq\left\|k\left(f\left(\frac{x+y+z}{k}\right)\right)\right\|,|k|\leq|3| \tag{1.3}$$

$$\|f(x_1)+f(x_2)+\cdots+f(x_n)\|\leq\left\|kf\left(\frac{x_1+x_2+\cdots x_n}{k}\right)\right\|,|k|\leq|n| \tag{1.4}$$

in Non-Archimedean Banach spaces.

In this paper, we solve and proved the Hyers-Ulam stability for functional inequalitie (1.1), ie the functional inequalities with 2n-variables [4, 13, 14, 18, 21, 22, 23,24]. Under suitable assumptions on spaces X and Y, we will prove that the mappings satisfying the functional inequatilie (1.1). Thus, the results in this paper are generalization of those in [18, 21, 22, 23,24] for functional inequatilie with 2n-variables.

The paper is organized as followns: In section preliminarier we remind some basic nota- tions in [18, 19, 20] such as Non-Archimedean field, Non-Archimedean normed space and Non-Archimedean Banach space.Section 3 is devoted to prove the Hyers-Ulam stability of the addive functional inequalities (1.1) when X Non-Archimedean normed space and Y Non-Archimedean Banach space.

2. PRELIMINARIES

2.1. Non-Archimedean normed and Banach spaces. In this subscetion we recall some basic notations [19, 20] such as Non-Archimedean fields, Non-Archimedean normed spaces and Non-Archimedean normed spaces.

A valuation is a function $|.|$from a field $\mathbb{K}$ into $[0, \infty)$ such that 0 is the unique element having the 0 valuation,

$$|r| = 0 \Leftrightarrow r = 0,$$

$$|rs| = |r||s|, \forall r, s \in \mathbb{K}$$

and the triangle inequality holds, ie;

$$|r + s| \leq |r| + |s|, \forall r, s \in \mathbb{K}$$

A field $\mathbb{K}$ is called a valued filed if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ are examples of valuation. Let us consider a valuation which satisfies a stronger condition than the triangle inaquality. If the tri triangle inequality is replaced by

$$|r + s| \leq \max\{|r| + |s|\} \forall r, s \in \mathbb{K}$$

then the function $|.|$ is called a norm -Archimedean valuational, and filed. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$, , $\forall r \in \mathbf{N}$. A trivial expamle of a non- Archimedean valuation is the function $|.|$ talking everything except for 0 into 1 and $|0| = 0$ this paper, we assume that the base field is a non-Archimedean filed, hence call it simply a filed.

Definition 2.1. Let be a vecter space over a filed K with a non -Archimedean $|.|$. A function $\|\bullet\| : X \to [0, \infty)$ is said a non -Archimedean norm if it satisfies the follwing conditions:

$$(1)\ \|x\| = 0 \ if\ and\ only\ if\ x = 0;$$

$$(2)\ \|rx\| = |r|\ \|x\|\ (r \in \mathbb{K}, x \in X);$$

$$(3)\ \|x + y\| \leq \max\{\|x\|, \|xy\|\} x, y, \in X\ hold$$

Then $(X, \|\bullet\|)$ is called a norm -Archimedean norm space.

Definition 2.2. Let, $\{x_n\}$, be a sequence in a norm -Archimedean normed space X.

1. A sequence , $\{x_n\}_{n=1}^{\infty}$in a non -Archimedean space is a Cauchy sequence if the, $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero.
2. The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be convergent if, for any $\in >\ 0$, there are a positive integer **N** and $x\ \in\ X$ such that

   $$\|x_n - x\| \leq \in \forall n \geq N,$$

   for all $n, m\ \geq\ N$. Then the point $x\ \in\ X$ is called the limit of sequence x_n, which is denoted by $\lim_{n\to\infty} x_n\ =\ x$.
3. If every sequence Cauchy in X converges, then the norm -Archimedean normed space X is called a norm -Archimedean Branch space.

2.2. Solutions of the inequalities. The functional equation

$$f\ (x\ +\ y) = f\ (x)\ +\ f\ (y)$$

is called the cauchuy equation. In particular, every solution of the cauchuy equation is said to be an *additive mapping.*

3. GENERALIZEDA HYERS-ULAM STABILITY Of THE FUNCTIONAL INEQUALITY

Now, we first study the solutions of (1.1). Note that for this inequalitie, X is non- Archimedean normed space and Y is non-Archimedean Banach spaces. Under this setting, we can show that the mapping satisfying (1.1) is additive. These results are given in the following. Let k be a fixed integer greater than n and let $n > |k|$.

Proposition 3.1. $f : X \to Y$ be a mapping such that,

$$\left\|\sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_Y \le \left\|kf\left(\frac{\sum_{j=1}^{n} x_j}{k} + \frac{\sum_{j=1}^{n} x_{n+j}}{nk}\right)\right\|_Y \tag{3.1}$$

for all x_j , $x_{n+j} \in X$ for all $j = 1 \to n$ then f is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x_j = x_{n+j} = 0$ in (3.1), we get,$\|2nf(0)\| \le \|kf(0)\|. So\ f(0) = 0$

Letting $x_{j+2} = x_{n+j} = 0$ for all $j = 1 \to n\ and\ x_1 = -x_2 = x$ in (3.1), we get

$$\|f(x) + f(-x)\|_Y \le \|kf(0)\|_Y$$

for all $x \in X$.

Hence $f(x) = -f(-x), \forall x \in X$

Letting $x_1 = x, x_2 = y, x_3 = -x - y\ and\ x_{j+3} = x_{n+j} = 0$ for all $j = 1 \to 1\ in$ (3.1), we get

$$\|f(x) + f(y) - f(x+y)\|_Y = \|f(x) + f(y) - f(-x-y)\|_Y \le \|kf(0)\|_Y = 0$$

for all $x, y \in$ X. It follows that $f(x+y) = f(x) + f(y)$ This completes the proof.

Theorem 3.2. Let $r < 1, \theta$ be non-negative real and $f : X \to Y$ be an odd mapping such that

$$\left\|\sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_Y \le \left\|kf\left(\frac{\sum_{j=1}^{n} x_j}{k} + \frac{\sum_{j=1}^{n} x_{n+j}}{nk}\right)\right\|_Y +, \theta(\sum_{j=1}^{n} \|x_j\|_X^r + \sum_{j=1}^{n} \|x_{n+j}\|_X^r) \tag{3.2}$$

for all x_j , $x_{n+j} \in X$ for all $j = 1 \to n$ then Then there exists a unique additive mapping $H : X \to Y$ such that,

$$\|f(x) + H(-x)\|_Y \le (n+1)\theta\|x\|_X^r$$

Proof. Letting $x_1 = x, x_{j+1} = 0, x_{n+j} = -x$ for all $j = 1 \to 1\ in$ (3.2), we get

$$\left\|f(x) + nf\left(\frac{x}{y}\right)\right\|_Y = \left\|f(x) + nf\left(-\frac{x}{y}\right)\right\|_Y \le \|kf(0)\|_Y + (1+n)\theta\|x\|_X^r$$

And so

$$\left\|f(x) + nf\left(\frac{x}{y}\right)\right\|_Y \le (1+n)\theta\|x\|_X^r$$

Hence we have

$$\left\|n^{m+1} f\left(\frac{x}{n^{m+1}}\right) - n^m f\left(\frac{x}{n^m}\right)\right\|_Y \le \frac{1+n}{|n|^{(r-1)m}}\theta\|x\|_X^r \tag{3.3}$$

$\forall m, n > 1, \forall x \in X$. It follows from (3.3) that the sequence $\left\{n^p f\left(\frac{x}{n^p}\right)\right\}$ is a cauchy sequence for all $x \in X$. Since Y is a Non-Archimedean Banach space, the sequence $\left\{n^p f\left(\frac{x}{n^p}\right)\right\}$ coverges.

So one can de ne the mapping $H : X \to Y$ by

$$H(x) := \lim_{p\to\infty} n^p f\left(\frac{x}{n^p}\right)$$

for all $x \in X$. Now, let $T : X \to Y$ be another additive mapping satistify (3.3) then we have

$$\|H(x) - T(x)\| = \left\|n^p h\left(\frac{x}{n^p}\right) - n^p T\left(\frac{x}{n^p}\right)\right\| \le \max\left\{\left\|n^p h\left(\frac{x}{n^p}\right) - n^p h\left(\frac{x}{n^p}\right)\right\|, \left\|n^q T\left(\frac{x}{n^p}\right) - n^p T\left(\frac{x}{n^p}\right)\right\|\right\} \le \frac{1+n}{|n|^{(r-1)p}}\theta\|x\|_X^r \qquad (3.4)$$

which tends to zero as $q \to \infty$for all $x \in X$. So we can conclude that $H(x)= T(x)$for all $x \in X$. This proves the uniqueness of H. It follows from (3.2) that,

$$\left\|H(x_1) + H(x_2) + \cdots + H(x_n) + H\left(\frac{x_{n+1}}{n}\right) + H\left(\frac{x_{n+2}}{n}\right) + \cdots + H\left(\frac{x_{n2}}{n}\right)\right\|_Y \lim_{p\to\infty}\left\|n^m f\left(\frac{x_1}{n^m}\right) + n^m f\left(\frac{x_2}{n^m}\right) + \cdots + n^m f\left(\frac{x_n}{n^m}\right) + n^m f\left(\frac{x_{n+1}}{n^{m+1}}\right) + n^m f\left(\frac{x_{n+2}}{n^{m+1}}\right) + \cdots + n^m f\left(\frac{x_{2n}}{n^{m+1}}\right)\right\|_Y \le \lim_{p\to\infty}\left\|n^m kf\left(\frac{x_1+x_2+\cdots+x_n}{n^m k}\right) + \left(\frac{x_{n+1}+x_{n+2}+\cdots+x_{2n}}{n^m.k}\right)\right\|_Y + \lim_{p\to\infty}\frac{|n|^m}{|n|^{mr}}\theta\left(\sum_{j=1}^n\|x_j\|_X^r + \sum_{j=1}^n\|x_{n+j}\|_X^r\right) = \left\|kf\left(\frac{\sum_{j=1}^n x_j}{k} + \frac{\sum_{j=1}^n x_{n+j}}{n.k}\right)\right\|_Y \qquad (3.5)$$

for all x_j , $x_{n+j} \in X$, for all $j = 1 \to n$ and so

$$\left\|\sum_{j=1}^n H(x_j) + \sum_{j=1}^n H\left(\frac{x_{n+j}}{n}\right)\right\|_Y \le \left\|kH\left(\frac{\sum_{j=1}^n x_j}{k} + \frac{\sum_{j=1}^n x_{n+j}}{n.k}\right)\right\|_Y \qquad (3.6)$$

for all x_j , $x_{n+j} \in X$ for all $x , y \in X$ By lemma 3.1, the mapping $H : X \to Y$ is additive

Theorem 3.3. Let $r < 1, \theta$ be non-negative real and $f : X \to Y$ be an odd mapping such that

$$\left\|\sum_{j=1}^n f(x_j) + \sum_{j=1}^n f\left(\frac{x_{n+j}}{n}\right)\right\|_Y \le \left\|kf\left(\frac{\sum_{j=1}^n x_j}{k} + \frac{\sum_{j=1}^n x_{n+j}}{n.k}\right)\right\|_Y + \theta\left(\sum_{j=1}^n\|x_j\|_X^r + \sum_{j=1}^n\|x_{n+j}\|_X^r\right) \qquad (3.7)$$

for all x_j , $x_{n+j} \in X$, for all $j = 1 \to n$. Then there exists a unique additive mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\|_Y \le \frac{(n+1)}{|n|^m}\theta\|x\|_X^r$$

Proof. Letting $x_1 = x, x_{j+1} = 0, and\ x_{n+j} = -x$ for all $j = 1 \to 1$ in (3.7), we get

$$\left\|f(x) - \frac{1}{n}f(nx)\right\|_Y \le \frac{(n+1)|n|^r}{|n|^m}\theta\|x\|_X^r$$

The rest of the proof is similar to the proof of theorem 3.2

Theorem 3.4. Let $r < \frac{1}{2n}$, θ be non-negative real and $f : X \to Y$ be an odd mapping such that

$$\left\|\sum_{j=1}^n f(x_j) + \sum_{j=1}^n f\left(\frac{x_{n+j}}{n}\right)\right\|_Y \le \left\|kf\left(\frac{\sum_{j=1}^n x_j}{k} + \frac{\sum_{j=1}^n x_{n+j}}{n.k}\right)\right\|_Y + \theta\left(\prod_{j=1}^n\|x_j\|_X^r + \prod_{j=1}^n\|x_{n+j}\|_X^r\right) \qquad (3.8)$$

for all x_j , $x_{n+j} \in X$, for all $j = 1 \to n$. Then there exists a unique additive mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\|_Y \le \frac{1}{|n|^{nr}}\theta\|x\|_X^r$$

Proof. Letting $x_1 = x, x_{j+1} = 0, and\ x_{n+j} = -x$ for all $j = 1 \to 1$ in (3.8), we get

$$\left\|f(x)-nf\left(\frac{x}{n}\right)\right\|_Y \leq \frac{1}{|n|^{nr}}\theta\|x\|_X^{nr}$$

The rest of the prooft is similar to the prooft of theorem 3.2

Theorem 3.5. Let $r > \frac{1}{2n}$, be non-negative real and $f : X \to Y$ be an odd mappingsuch that

$$\left\|\sum_{j=1}^{n} f(x_j) + \sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_Y \leq \left\|kf\left(\frac{\sum_{j=1}^{n} x_j}{k} + \frac{\sum_{j=1}^{n} x_{n+j}}{n.k}\right)\right\|_Y + \theta\left(\prod_{j=1}^{n}\|x_j\|_X^r + \prod_{j=1}^{n}\|x_{n+j}\|_X^r\right) \quad (3.9)$$

for all x_j , $x_{n+j} \in X$, for all $j = 1 \to n$. Then there exists a unique additive mapping $H : X \to Y$ such that

$$\|f(x)-H(x)\|_Y \leq \frac{|n|^{nr}}{|n|}\theta\|x\|_X^{nr}$$

Proof. Letting $x_1 = x, x_{j+1} = 0, and\ x_{n+j} = -x$ for all $j = 1 \to 1$ in (3.9), we get

$$\left\|f(x)-\frac{1}{n}f(nx)\right\|_Y \leq \frac{|n|^{nr}}{|n|}\theta\|x\|_X^{nr}$$

The rest of the prooft is similar to the prooft of theorem 3.2

4. Conclusion

In this paper, I have shown that the solutions of the functional inequalities are additive mapping. The Hyers-Ulam stability for these given from theorems. These are the main results of the paper, which are the general of the results [17, 18, 19, 21, 22, 23,24] .

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