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RESEARCH ARTICLE

MODELING OF PRODUCTION PROCESS WITH INSPECTION POLICY IN TERMS OF TIME

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ABSTRACT

In this paper, we use the combined dynamic linear model of unequally spaced intervals, to find the statistical distributions for the inspection policy by terms of time, in the case of unequally spaced intervals. The researcher's work focus on the statement of theorem (1) and its proof. In addition to the statement of theorem (2), its proof, with find the prediction distribution. The results of this paper will be very useful in practical applications.

Keywords: Dynamic linear Model, Modeling, Production Process, and Inspection Policy

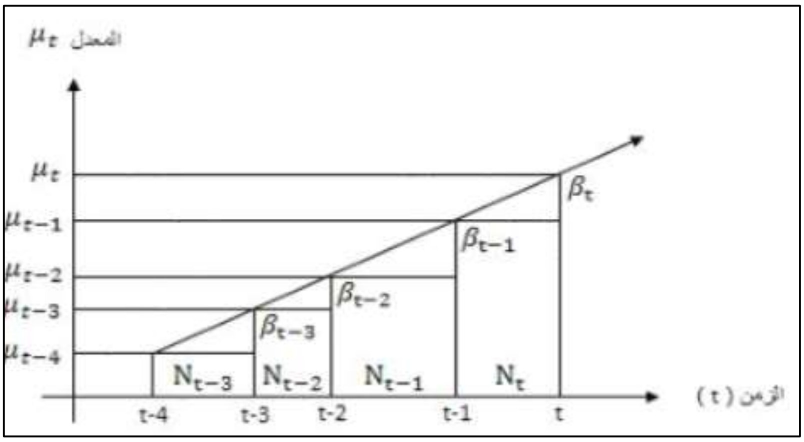
1. Introduction

The application of statistical methods as a scientific basis on the study of the production processes was first used in the first quarter of the twentieth century as a result of the development of Statistical Theory. It was proposed by the statistician Shewhart who was working in the laboratories of telephone bells in America. He tried to make use of his statistical information in finding a way to detect defects and changes in the production process. Shewhart was mainly concerned with two issues. The first was to detect any lack of control occurring in the production process as quickly as possible. The second was to maintain the production process under control (i.e. producing a high percentage of acceptable quality materials). To achieve this aim, he introduced an assisting visual means called the quality control chart in three research articles. They were published in the Journal of the American Statistical Society in 1926 and 1927. The main goal of using this chart was to discover the actual abnormal changes in the parameters of the production process.

2. Modeling of Production Process

Two types of changes often take place in the quality of the produced material in any production process. The first type is known as sudden changes that cause a change in the level of quality to another worse level suddenly. In quality control, it is called jump, and these changes are usually significant, resulting from assignable causes in the production process such as a break-in machine part, unskilled workers, poor raw material quality, etc. The second type is gradual changes, i.e. slope change. These changes are usually minor at the beginning, but over time they turn into significant changes such as corrosion of machine parts, useful life factor, gradual change in temperature and humidity, etc. In economics, these changes are called trends, and in quality control, they are called drift. This methodology can be used in conjunction with the dynamic linear model proposed by Harrison and Stevens (1976) and in particular the combined dynamic linear model of unequal time intervals where the observation is in unequally spaced intervals with the dynamic linear model proposed by Smith (1982). Dropping the restriction of equal periods in the model increases the flexibility of its use in the areas of practical life. Assuming that the slope in the level is consistent, the growth in the level is proportional with the size of the time intervals between observations, see the figure below.

(Note: $t_i - t_{i-1} \neq t_{i+1} - t_i$, i=1,2,….)



And that the level slope at time t is expressed as follows:

$$\beta_t = \frac{N_t}{N_{t-1}} \beta_{t-1}$$

For keeping the linear formula and getting acceptable and accurate results, it is preferable that the differences between (N_{t-1} and N_t) are not large. Because the slope change is possible at any moment, the random error δ_t should be added to the above slope equation, so we get:

$$\beta_t = \frac{N_t}{N_{t-1}} \beta_{t-1} + \delta_t, \qquad \delta_t \sim N(0, \sigma^2 r_\beta)$$

Consequently, the combined dynamic linear model of unequally time intervals can be written mathematically as follows:

$$x_t = \mu_t + v_t \qquad , v_t \sim N(0, \sigma^2) \tag{1}$$

$$\begin{aligned} \mu_t &= \mu_{t-1} + \beta_t + \delta_t \qquad , \ \delta_t \sim N(0, \sigma^2 r_\mu) \\ \beta_t &= \frac{N_t}{N_{t-1}} \beta_{t-1} + \Delta_t \qquad , \ \Delta_t \sim N(0, \sigma^2 r_\beta) \end{aligned} \tag{2}$$

Where

t: Time index ($t = 1,2,3, \dots$).

μ_t: Represents the level or the real rate for the quality of produced material and inspected at time t, and is usually unknown exactly.

β_t: Represents the real slope for the quality of produced material and inspected at time t.

All random errors are $v_t , \delta_t , \Delta_t$ assumed to be independent events, where:

v_t : It is the result of random error in the observations or metrics used or in both.

δ_t , Δ_t: They are the result of random errors (disturbances) in the same production process.

In equation (1), we note that the value of the quality observation of the material produced is equal to the quality rate of the material produced, as well as the random error in the observation itself. It is not possible to be equal to the real quality observation and quality rate if there is no error in observation (free–error–observation). Equation(2) expresses the correlation or linear relationships of the process parameters μ_t , β_t and their evolution over time.

Now, equations (1) and (2) can be written on the formula of the combined dynamic linear model of unequal time intervals as follows:

$$x_t = F\theta_t + v_t \quad , \quad v_t \sim N(0, \sigma^2) \tag{3}$$

$$\theta_t = G_t\theta_{t-1} + \omega_t \quad , \quad \omega_t \sim N(0, \sigma^2 r) \tag{4}$$

Where v_t , ω_t are two independent random natural vectors. And

$$r = E\left(\omega_t . \omega_t^{/}\right) = \begin{bmatrix} r_\beta + r_\mu & r_\beta \\ r_\beta & r_\beta \end{bmatrix}$$

Note that:

$$F = (1 \quad 0), G_t = \begin{bmatrix} 1 & \frac{N_t}{N_{t-1}} \\ 0 & \frac{N_t}{N_{t-1}} \end{bmatrix}, \qquad \theta_t = \binom{\mu_t}{\beta_t} \ \ and \ \ \omega_t = \binom{\Delta_t + \delta_t}{\Delta_t}$$

3. Inspection Policies

Usually, several inspection policies take place on the quality of the material produced in the practical life, including:

a. Inspection in terms of the produced units: In this case, there are several inspection policies, for example:

1. Inspecting the first or the last unit of the produced units n
2. Inspecting n units for every N of the produced units
3. Inspecting all produced units (comprehensive inspection)

b. Inspection in terms of time: In this case, the units produced are examined during equal or unequal periods, for example:

1) Inspecting one unit or a sample of n units in fixed and equal periods (fixed interval inspection).
2) Inspecting one unit or n units in unequal, short-term periods (unequally spaced inspection).

We will use the sequential inspection method when finding the posterior distribution.

In this paper, we will review a comprehensive mathematical study of the state of inspection by time in unequal time intervals. In this case, the quality is inspected during intervals of time. For example, a single unit is inspected in short and unequal time intervals. This policy is represented by equations (1), (2). When modeling the production processes (according to the quality of the material produced) for the unit j (where $j = 1,2,\dots,n,$ at inspection time t), these two equations become as follows:

$$x_{j,t} = \mu_{j,t} + v_{j,t} \qquad , \ v_{j,t} \sim N(0,\sigma^2) \tag{5}$$

$$\mu_{j,t} = \mu_{j-1,t} + \beta_{j,t} + \delta_{j,t} \qquad , \ \delta_{j,t} \sim N\left(0,\sigma^2 r_\mu\right) \tag{6}$$

$$\beta_{j,t} = \frac{N_t}{N_{t-1}}\beta_{j-1,t} + \Delta_{j,t} \qquad , \Delta_{j,t} \sim N\left(0,\sigma^2 r_\beta\right) \tag{7}$$

Where $v_{j,t}$,$\delta_{j,t}$ and $\Delta_{j,t}$ are independent random errors.

Note that :

t: Represents the time of the inspected unit.

n: Represents the number of produced units.

j: Represents the directory of the produced unit.

$x_{j,t}$: Represents the value of the produced material quality j after the time t.

$\mu_{j,t}$: Represents the real slope of the produced material quality j after the time t.

$\beta_{j,t}$: Represents the slope factor of the produced material quality j after the time t.

σ^2: Represents the observed variance.

Suppose that y be a comprehensive code, we will use the following convention:

$$y_{n,t-1} = y_{0,t} = y_t \tag{8}$$

For more information on dynamic models and Bayesian inference and their field of application, see the recent literature (Migon et al, 2005), (Mubwandarikwa et al, 2005), (Aguilar and West, 2000), (Gerlach and Kohn, 2000), (Awe et al, 2015), (Aktekin et al, 2018), (Debeko and Goshu, 2018), (Petris et al, 2009), (Roberto, 2016), (Laine, 2019), (Berry and West, 2019), (Chen et al, 2019), (Nobuhiko et al, 2014), (Gonçalves et al, 2020) and (Petridis et al, 2001).

From the above discussion, we can use the combined dynamic linear model of unequal time intervals at the real inspection time in practical applications through the use of the theorems below. The formulation of theorems and the confirmation of their validity is the researcher's contribution to Jalil (1988).

Theorem (1): If $\theta_t = \binom{\mu_t}{\beta_t}$ and if the two equations are

$$\mu_{j,t} = \mu_{j-1,t} + \beta_{j,t} + \delta_{j,t} \qquad , \ \delta_{j,t} \sim N\left(0,\sigma^2 r_\mu\right) \tag{9}$$

$$\beta_{j,t} = \frac{N_t}{N_{t-1}}\beta_{j-1,t} + \Delta_{j,t} \qquad , \Delta_{j,t} \sim N\left(0,\sigma^2 r_\beta\right) \tag{10}$$

Then, we get:

$$\theta_t = H_t\theta_{t-1} + \gamma_t$$

Where

$$H_t = G_t^n = \begin{pmatrix} 1 & \sum_{d=1}^{n} \left(\frac{h}{z}\right)^d \\ 0 & \left(\frac{h}{z}\right)^n \end{pmatrix}$$

By putting $h = N_t$, $z = N_{t-1}$, and

$$\gamma_t = G_t^{n-1}\omega_{1,t-1} + G_t^{n-2}\omega_{2,t-1} + \cdots + G_t\omega_{n-1,t-1} + \omega_{n,t-1}$$

Where

$$\omega_{j,t-1} = \begin{pmatrix} \Delta_{j,t-1} + \delta_{j,t-1} \\ \Delta_{j,t-1} \end{pmatrix}, \quad j = 1,2, \dots, n,$$

Proof: For the produced units n during unequal and short-term time intervals, by using the relationship (7), we get:

$$\beta_{1,t-1} = \frac{h}{z}\beta_{0,t-1} + \Delta_{1,t-1}$$

$$\beta_{2,t-1} = \frac{h}{z}\beta_{1,t-1} + \Delta_{2,t-1}$$

$$= \left(\frac{h}{z}\right)^2 \beta_{0,t-1} + \frac{h}{z}\Delta_{1,t-1} + \Delta_{2,t-1}$$

$$\beta_{3,t-1} = \frac{h}{z}\beta_{2,t-1} + \Delta_{3,t-1}$$

$$= \left(\frac{h}{z}\right)^3 \beta_{0,t-1} + \left(\frac{h}{z}\right)^2 \Delta_{1,t-1} + \frac{h}{z}\Delta_{2,t-1} + \Delta_{3,t-1}$$

$$\vdots$$

$$\beta_{j,t-1} = \left(\frac{h}{z}\right)^j \beta_{0,t-1} + \left(\frac{h}{z}\right)^{j-1} \Delta_{1,t-1} + \left(\frac{h}{z}\right)^{j-2} \Delta_{2,t-1} + \cdots + \frac{h}{z}\Delta_{j-1,t-1} + \Delta_{j,t-1}$$

$$\vdots$$

$$\beta_{n,t-1} = \left(\frac{h}{z}\right)^n \beta_{0,t-1} + \left(\frac{h}{z}\right)^{n-1} \Delta_{1,t-1} + \left(\frac{h}{z}\right)^{n-2} \Delta_{2,t-1} + \cdots + \frac{h}{z}\Delta_{n-1,t-1} + \Delta_{n,t-1}$$

$$= \left(\frac{h}{z}\right)^n \beta_{0,t-1} + \sum_{j=1}^{n} \left(\frac{h}{z}\right)^{n-j} \Delta_{j,t-1}$$

By using the convention $y_{n,t-1} = y_{0,t} = y_t$, we get:

$$\beta_t = \left(\frac{h}{z}\right)^n \beta_{t-1} + \sum_{j=1}^{n} \left(\frac{h}{z}\right)^{n-j} \Delta_{j,t-1} \tag{11}$$

Now by using equation (9) and equation (11) that we obtained from the above results, we get:

$$\mu_{1,t-1} = \mu_{0,t-1} + \beta_{1,t-1} + \delta_{1,t-1}$$

$$= \mu_{0,t-1} + \frac{h}{z}\beta_{0,t-1} + \Delta_{1,t-1} + \delta_{1,t-1}$$

$$\mu_{2,t-1} = \mu_{1,t-1} + \beta_{2,t-1} + \delta_{2,t-1}$$

$$= \mu_{0,t-1} + \left[\frac{h}{z} + \left(\frac{h}{z}\right)^2\right] \beta_{0,t-1} + \left[1 + \frac{h}{z}\right] \Delta_{1,t-1} + \Delta_{2,t-1} + \delta_{1,t-1} + \delta_{2,t-1}$$

$$\mu_{3,t-1} = \mu_{2,t-1} + \beta_{3,t-1} + \delta_{3,t-1}$$

$$= \mu_{0,t-1} + \left[\frac{h}{z} + \left(\frac{h}{z}\right)^2 + \left(\frac{h}{z}\right)^3\right]\beta_{0,t-1} + \left[1 + \frac{h}{z} + \left(\frac{h}{z}\right)^2\right]\Delta_{1,t-1} +$$

$$+\left(1 + \frac{h}{z}\right)\Delta_{2,t-1} + \Delta_{3,t-1} + \delta_{1,t-1} + \delta_{2,t-1} + \delta_{3,t-1}$$

$$\vdots$$

$$\mu_{j,t-1} = \mu_{0,t-1} + \left[\sum_{d=1}^{n}\left(\frac{h}{z}\right)^d\right]\beta_{0,t-1} + \left[\sum_{d=1}^{n}\left(\frac{h}{z}\right)^{d-1}\right]\Delta_{1,t-1} +$$

$$+\left[\sum_{d=2}^{n}\left(\frac{h}{z}\right)^{d-2}\right]\Delta_{2,t-1} + \cdots + \Delta_{j,t-1} + \delta_{1,t-1} + \delta_{2,t-2} + \cdots + \delta_{j,t-1}$$

By using the geometric sequence laws, we get:

$$\sum_{d=1}^{n}\left(\frac{h}{z}\right)^d = \frac{h^{n+1} - z^n}{hz^n - z^{n+1}}$$

$$\sum_{d=1}^{n}\left(\frac{h}{z}\right)^{d-1} = \frac{h^n - z^n}{hz^{n-1} - z^n}$$

$$\sum_{d=2}^{n}\left(\frac{h}{z}\right)^{d-2} = \frac{h^{n-1} - z^{n-1}}{hz^{n-2} - z^{n-1}}$$

Therefore,

$$\mu_{j,t-1} = \mu_{0,t-1} + \left[\frac{h^{n+1} - hz^n}{hz^n - z^{n+1}}\right]\beta_{0,t-1} + \left[\frac{h^n - z^n}{hz^{n-1} - z^n}\right]\Delta_{1,t-1} +$$

$$+\left[\frac{h^{n-1} - z^{n-1}}{hz^{n-2} - z^{n-1}}\right]\Delta_{2,t-1} + \cdots + \Delta_{j,t-1} + \delta_{1,t-1} + \delta_{2,t-1} + \cdots + \delta_{j,t-1}$$

$$\vdots$$

$$\mu_{n,t-1} = \mu_{0,t-1} + \left[\frac{h^{n+1} - hz^n}{hz^n - z^{n+1}}\right]\beta_{0,t-1} + \sum_{j=1}^{n}\frac{h^{n-j+1} - z^{n-j+1}}{hz^{n-j} - z^{n-j+1}}\Delta_{j,t-1} + \sum_{j=1}^{n}\delta_{j,t-1}$$

By using the convention $y_{n,t-1} = y_{0,t} = y_t$, we get:

$$\mu_t = \mu_{t-1} + \left[\frac{h^{n+1}-hz^n}{hz^n-z^{n+1}}\right]\beta_{t-1} + \sum_{j=1}^{n}\frac{h^{n-j+1}-z^{n-j+1}}{hz^{n-j}-z^{n-j+1}}.\Delta_{j,t-1} + \sum_{j=1}^{n}\delta_{j,t-1} \quad (12)$$

Thus, from equations (11) and (12), the required final formula can be reached as follows:

$$\theta_t = \binom{\mu_t}{\beta_t}$$

$$= \begin{bmatrix} \mu_{t-1} + \left[\frac{h^{n+1} - hz^n}{hz^n - z^{n+1}}\right]\beta_{t-1} + \sum_{j=1}^{n}\frac{h^{n-j+1} - z^{n-j+1}}{hz^{n-j} - z^{n-j+1}}\Delta_{j,t-1} + \sum_{j=1}^{n}\delta_{j,t-1} \\ \left(\frac{h}{z}\right)^n \beta_{t-1} + \sum_{j=1}^{n}\left(\frac{h}{z}\right)^{n-j}\Delta_{j,t-1} \end{bmatrix}$$

$$= \begin{bmatrix} \mu_{t-1} + \left[\frac{h^{n+1}-hz^{n}}{hz^{n}-z^{n+1}}\right]\beta_{t-1} \\ \left(\frac{h}{z}\right)^{n}\beta_{t-1} \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{n}\frac{h^{n-j+1}-z^{n-j+1}}{hz^{n-j}-z^{n-j+1}}\Delta_{j,t-1} + \sum_{j=1}^{n}\delta_{j,t-1} \\ \sum_{j=1}^{n}\left(\frac{h}{z}\right)^{n-j}\Delta_{j,t-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{h^{n+1}-hz^{n}}{hz^{n}-z^{n+1}} \\ 0 & \left(\frac{h}{z}\right)^{n} \end{bmatrix}\begin{bmatrix} \mu_{t-1} \\ \beta_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & \frac{h^{n}-hz^{n-1}}{hz^{n-1}-z^{n}} \\ 0 & \left(\frac{h}{z}\right)^{n-1} \end{bmatrix}\begin{bmatrix} \Delta_{1,t-1}+\delta_{1,t-1} \\ \Delta_{1,t-1} \end{bmatrix} + \begin{bmatrix} 1 & \frac{h^{n-1}-hz^{n-2}}{hz^{n-2}-z^{n-1}} \\ 0 & \left(\frac{h}{z}\right)^{n-2} \end{bmatrix}.$$

$$.\begin{bmatrix} \Delta_{2,t-1}+\delta_{2,t-1} \\ \Delta_{2,t-1} \end{bmatrix} + \cdots + + \begin{bmatrix} 1 & \frac{h}{z} \\ 0 & \frac{h}{z} \end{bmatrix}\begin{bmatrix} \Delta_{n-1,t-1}+\delta_{n-1,t-1} \\ \Delta_{n-1,t-1} \end{bmatrix} + \begin{bmatrix} \Delta_{n,t-1}+\delta_{n,t-1} \\ \Delta_{n,t-1} \end{bmatrix}$$

$$= {G_t}^{n}\theta_{t-1} + {G_t}^{n-1}\omega_{1,t-1} + G_t^{n-2}\omega_{2,t-1} + \cdots + G_t\omega_{n-1,t-1} + \omega_{n,t-1}$$

By putting both $H_t = G_t^n$, and

$$\gamma_t = G_t^{n-1}\omega_{1,t-1} + G_t^{n-2}\omega_{2,t-1} + \cdots + G_t\omega_{n-1,t-1} + \omega_{n,t-1}$$

We get:

$$\theta_t = H_t\theta_{t-1} + \gamma_t$$

Thus, the proof was achieved.

From Theorem (1) above, we can get the variance-covariance matrix of $\omega_{j,t-1}$ as follows:

Since $\omega_{j,t-1} = \begin{pmatrix} \Delta_{j,t-1}+\delta_{j,t-1} \\ \Delta_{j,t-1} \end{pmatrix}$, Then

$$Cov(\omega_{j,t-1}) = E\begin{pmatrix} \Delta_{j,t-1}+\delta_{j,t-1} \\ \Delta_{j,t-1} \end{pmatrix}\left(\Delta_{j,t-1}+\delta_{j,t-1} \;\; \Delta_{j,t-1}\right)$$

$$= E\begin{bmatrix} (\Delta_{j,t-1}+\delta_{j,t-1})^2 & (\Delta_{j,t-1}+\delta_{j,t-1})\Delta_{j,t-1} \\ \Delta_{j,t-1}(\Delta_{j,t-1}+\delta_{j,t-1}) & (\Delta_{j,t-1})^2 \end{bmatrix}$$

$$= \sigma^2\begin{bmatrix} r_\beta + r_\mu & r_\beta \\ r_\beta & r_\beta \end{bmatrix}$$

$$= \sigma^2 r$$

Therefore, put $r = \begin{bmatrix} r_\beta + r_\mu & r_\beta \\ r_\beta & r_\beta \end{bmatrix}$

And also, from Theorem (1) above, we can get the variance-covariance matrix of γ_t as follows:

$$Cov(\gamma_t) = \left[G_t^{n-1}r(G_t^{n-1})^{/} + G_t^{n-2}r(G_t^{n-2})^{/} + \cdots + G_t r {G_t}^{/} + r\right]\sigma^2$$

By using the properties of multiplying and adding matrices, we can come up with the following:

$$Cov(\gamma_t) = \sigma^2 W$$

Where

$$W = \begin{bmatrix} nr_\beta\left[1+\sum_{j=1}^{n}\left(\frac{h}{z}\right)^{j}\right]^2 + nr_\mu & r_\beta\left[1+\sum_{j=1}^{n}\left(\frac{h}{z}\right)^{j}\right]\sum_{j=1}^{n}\left(\frac{h}{z}\right)^{n-j} \\ r_\beta\left[1+\sum_{j=1}^{n}\left(\frac{h}{z}\right)^{j}\right]\sum_{j=1}^{n}\left(\frac{h}{z}\right)^{n-j} & r_\beta\sum_{j=1}^{n}\left(\frac{h}{z}\right)^{2(n-j)} \end{bmatrix}$$

For simplification, we put

$$w_1 = 1 + \sum_{j=1}^{n} \left(\frac{h}{z}\right)^j = \frac{h^{n+1} - z^{n+1}}{hz^n - z^{n+1}}$$

$$w_2 = \sum_{j=1}^{n} \left(\frac{h}{z}\right)^{n-j} = \frac{h^n - z^n}{hz^{n-1} - z^n}$$

$$w_3 = \sum_{j=1}^{n} \left(\frac{h}{z}\right)^{2(n-j)} = \frac{h^{2n} - z^{2n}}{h^2 z^{2n-2} - z^{2n}}$$

Thus, the matrix W becomes as follows:

$$W = \begin{bmatrix} nr_\beta w_1^2 + nr_\mu & r_\beta w_1 w_2 \\ r_\beta w_1 w_2 & r_\beta w_3 \end{bmatrix} \quad (13)$$

According to this, we can reformulate the combined dynamic linear model of unequal time intervals, at the inspection time t as follows:

Observation equation $x_t = F\theta_t + v_t \quad , \quad v_t \sim N(0, \sigma^2)$ (14)

System equation $\theta_t = H_t\theta_{t-1} + \gamma_t \quad , \quad \gamma_t \sim N(0, \sigma^2 W)$ (15)

Thus, after obtaining equations (14) and (15), using the Bayesian theorem, taking advantage of the Markov process, using sequential analysis, and referring to West and Harrison (1997), we can get the posterior probability distributions of process parameters from the repeated procedures. Considering that both σ^2,r_μ and r_β are known, we can formulate the following theorem:

Theorem (2): According to the combined dynamic linear model of unequal time intervals, at the inspection time t, in equations(14) and (15) If the prior distribution of $\theta_0$before any observation is normal distribution,

$$i.e \qquad (\theta_0 \backslash D_0) \sim N(m_0 \,, \sigma^2 C_0)$$

Then the posterior distribution of θ_t at time t, is also normal distribution

$$i.e \qquad (\theta_t \backslash D_t) \sim N(m_t, \sigma^2 C_t)$$

Where the available information D_t is $D_t = (x_t, x_{t-1}, \dots, x_1, r_\mu, r_\beta, \sigma^2)$. It is about previous information and other known constants.

Proof:

By using mathematical induction, we conclude that at time $t-1$ is

$$(\theta_{t-1} \backslash D_{t-1}) \sim N(m_{t-1}, \sigma^2 C_{t-1})$$

Where the available information is

$$D_{t-1} = (x_{t-1}, x_{t-2}, \dots, x_1, r_\mu, r_\beta, \sigma^2).$$

From this we can get the following distributions:

1) The prior distribution of parameter θ_t at time t: From equation (15), we get

$$E(\theta_t \backslash D_{t-1}) = E[(H_t\theta_{t-1} + \gamma_t) \backslash D_{t-1}]$$
$$= H_t m_{t-1}$$

$$var(\theta_t \backslash D_{t-1}) = var[(H_t\theta_{t-1} + \gamma_t)\backslash D_{t-1}]$$

$$= H_t var(\theta_{t-1}\backslash D_{t-1})H_t^{/} + var(\gamma_t \backslash D_{t-1})$$

$$= H_t \sigma^2 C_{t-1} H_t^{/} + \sigma^2 W$$

$$= \sigma^2 \left(H_t C_{t-1} H_t^{/} + W\right)$$

$$= \sigma^2 R_t$$

Thus, we put, $R_t = H_t C_{t-1} H_t^{/} + W$

Where R_t is a positive, definite, and symmetric matrix. We can formulate this probability distribution mathematically as follows:

$$(\theta_t \backslash D_{t-1}) \sim N(H_t m_{t-1}, \sigma^2 R_t)$$

2) Posterior distribution: We can obtain the posterior distribution by using the Bayes' theorem which has the following formula:

$$P(\theta_t \backslash D_t) \propto P(\theta_t \backslash D_{t-1}) \,.\, P(x_t \backslash D_t)$$

Where $P(x_t \backslash D_t)$is the likelihood function. We get the expectation and variance of this function from equation (14) as follows:

$$E(x_t \backslash \theta_t) = E[(F\theta_t + v_t)\backslash D_t] = F\theta_t$$

$$var(x_t \backslash \theta_t) = var[(F\theta_t + v_t)\backslash D_t] = \sigma^2$$

We can formulate this probability distribution, mathematically as follows

$$(x_t \backslash \theta_t) \sim N(F\theta_t, \sigma^2)$$

Thus, the posterior probability distribution is given as follows:

$$P(\theta_t \backslash D_t) \propto Exp\left[\frac{-1}{2\sigma^2}(\theta_t - H_t m_{t-1})^{/} R_t^{-1}(\theta_t - H_t m_{t-1})\right].$$

$$.\, Exp\left[\frac{-1}{2\sigma^2}(x_t - F\theta_t)^{/}(x_t - F\theta_t)\right]$$

By using multiplying, merging the similar quantities together, and merging the constant quantities with a constant of proportionality, we can conclude that:

$$P(\theta_t \backslash D_t) \propto Exp\left[\frac{-1}{2\sigma^2}(\theta_t - m_t)^{/} C_t^{-1}(\theta_t - m_t)\right]$$

By putting:

$$m_t = H_t m_{t-1} + A_t(x_t - f_t) \tag{16}$$

Where m_t represents the expectation of the posterior probability distribution at time t , and it is called Kalman Filter, and $(x_t - f_t)$ is called prediction error, where $f_t = FH_t m_{t-1}$ and

$$C_t^{/} = R_t - A_t F R_t \tag{17}$$

Where $C_t^{/}$ represents the variance-covariance matrix of the posterior probability distribution at time t, and that

$$A_t = R_t F^{/} \left(1 + F R_t F^{/}\right)^{-1} \tag{18}$$

Where A_t is called Kalman Factor. Thus, we can formulate the posterior probability distribution mathematically as follows:

$$(\theta_t \backslash D_t) \sim N(m_t , \sigma^2 C_t)$$

4. Prediction Distribution

Finding prediction distribution is very important because it provides us with future information. We can at least know what will be the rate of the quality of the material produced. If the variance of this distribution is large, this increases the uncertainty about the production process. Hence, we prepare to make the right decision. To get the prediction distribution one step ahead, we should formulate the combined dynamic linear model for unequal time intervals, defined by equations (14) and (15) in the future format as follows:

$$x_{t+1} = F\theta_{t+1} + v_{t+1} \quad , \quad v_{t+1} \sim N(0, \sigma^2)(\tag{19}$$

$$\theta_{t+1} = H_t \theta_t + \gamma_{t+1} \quad , \quad \gamma_{t+1} \sim N(0, \sigma^2 W) \tag{20}$$

Where v_{t+1} and γ_{t+1} are disjoint and independent events. Accordingly, we can reach the following results:

From the equation (20), we get:

$$(\theta_{t+1} \backslash D_t) \sim N(H_t m_t , \ \sigma^2 R_{t+1}) \tag{21}$$

Where

$$R_{t+1} = H_t C_t H^{/} + W(\tag{22}$$

and from the equations (19), (20), and (21), we can get:

$$(x_{t+1} \backslash D_t) \sim N(\hat{x}_{t+1} \ , \sigma^2 \ \widehat{\times}_{t+1})$$

Where

$$\hat{x}_{t+1} = F H_t m_t$$

$$\hat{X}_{t+1} = F R_{t+1} F^{/} + 1$$

In the case of practical applications using this model, we can put:

$$C_t = \begin{bmatrix} c_{1,t} & c_{2,t} \\ c_{2,t} & c_{3,t} \end{bmatrix} \tag{23}$$

$$A_t = \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} \tag{24}$$

$$m_t = E(\theta_t \backslash D_t) = E\begin{pmatrix} \mu_t \\ \beta_t \end{pmatrix} \backslash \quad D_t \Big) = \begin{pmatrix} u_t \\ b_t \end{pmatrix}$$

$$R_{\mathrm{t}} = \begin{bmatrix} \mathrm{r}_{1,\mathrm{t}} & \mathrm{r}_{2,\mathrm{t}} \\ \mathrm{r}_{2,\mathrm{t}} & \mathrm{r}_{3,\mathrm{t}} \end{bmatrix} \tag{25}$$

Under the laws of the geometric series:

$$H_t = \begin{bmatrix} 1 & \dfrac{h^{n+1} - hz^n}{hz^n - z^{n+1}} \\ 0 & \left(\dfrac{h}{z}\right)^n \end{bmatrix}$$

Now from the equations (16), (17), (18), and (25), we can access the recursive equations as follows:

$$a_{1,t} = \frac{r_{1,t}}{1+r_{1,t}} \quad \text{and} \quad a_{2,t} = \frac{r_{2,t}}{1+r_{1,t}}$$

For ease, we put $Z = \frac{h^{n+1}-hz^n}{hz^n-z^{n+1}}$. Therefore,

$$f_t = u_{t-1}Zb_{t-1}$$

$$u_t = u_{t-1} + Zb_{t-1} + a_{1,t}(x_t - f_t)$$

$$b_t = \left(\frac{h}{z}\right)^n b_{t-1} + a_{2,t}(x_t - f_t)$$

$$r_{1,t} = c_{1,t-1} + 2Zc_{2,t-1} + Z^2c_{3,t-1} + nr_\beta w_1^2 + nr_\mu$$

$$r_{2,t} = \left(\frac{h}{z}\right)^n c_{2,t-1} + \left(\frac{h}{z}\right)^n Zc_{3,t-1} + r_\beta w_1 w_2$$

$$r_{3,t} = \left(\frac{h}{z}\right)^{2n} c_{3,t-1} + r_\beta w_3$$

$$c_{1,t} = r_{1,t} - \frac{(r_{1,t})^2}{1+r_{1,t}}, \quad c_{2,t} = r_{2,t} - \frac{(r_{1,t})(r_{2,t})}{1+r_{1,t}} \quad \text{and} \quad c_{3,t} = r_{3,t} - \frac{(r_{2,t})^2}{1+r_{1,t}}$$

Also $\hat{x}_{t+1} = u_t + Zb_t$ and $\hat{X}_{t+1} = r_{1,t+1} + 1$

Where $r_{1,t+1}$, we get it from equation (22) and will be in the following formula:

$$r_{1,t+1} = c_{1,t} + 2\,Zc_{2,t} + Z^2c_{3,t} + nr_\beta w_1^2 + nr_\mu$$

We point out that we need all the above-mentioned recurring relationships to find the value of u_t. It is possible to program these relationships by using a software program in the case of practical applications.

5. Conclusion

In this study, we have explained how to find the recurring relationships of the combined dynamic linear model of unequal time intervals to study the inspection policies by time in the case that r_μ ,r_β and σ^2 are known. From the above discussion, we can use the combined dynamic linear model of unequal time intervals at the real inspection time in practical applications through the use of the above theorems. the researcher's work focus on the statement of the theory of theorem (1) and its prove. In addition to the statement of theorem (2) and its prove, with find the prediction distribution, The results of this paper will be very useful in practical applications. In the end, we recommend finding the above statistical distributions in case r_μ ,r_β and σ^2 are unknown.

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