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RESEARCH ARTICLE

RAMANUJAN SUMMATION AND MAGIC SQUARES

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ABSTRACT

Magic Squares is one of the richest and fabulous topic of research in recreational mathematics. Great Indian Mathematician Srinivasa Ramanujan produced a wonderful formula for summing divergent series which today is known in the name Ramanujan Summation. In this paper, I will prove a new result regarding determining Ramanujan Summation for positive integral powers of magic constants in an $n \times n$ magic square constructed using first n^2 natural numbers.

Keywords: Magic Square, Magic Constant, Bernuoulli Numbers, Binomial Expansion, Ramanujan Summation

1. Introduction

Srinivasa Ramanujan used to note down his findings in his notebooks which are considered as mathematical treasures today. Among several chapters that he wrote, in first chapter, he began with the concept of constructing Magic Squares. In this paper, I will consider magic squares of order $n \times n$ constructed using first n^2 natural numbers. The term magic constant refers to the same sum obtained upon adding any row, column or two leading diagonals of the given magic square. In this paper, I will derive a new result regarding determining Ramanujan summation for positive integral powers of magic constants for $n \times n$ magic squares for $n \geq 3$.

2. Theorem 1

For $n \geq 3$, the magic constant for any *n* × *n* magic square constructed with first n^2 natural numbers is

$$\frac{n(n^2+1)}{2} \quad (2.1)$$

Proof: First, we note that there is only one magic square of order 1 × 1 which consists of just the number 1. In this trivial case since, we cannot distinguish between row, column and diagonals we neglect the trivial magic square of order 1 × 1. It is well known that there is no magic square of order 2 × 2. Hence we can consider non – trivial magic squares of order *n* × *n* for $n \geq 3$ constructed with first n^2 natural numbers.

Since we use first n^2 natural numbers with no repetition in constructing the magic square, the sum of all numbers making up the whole magic square would be the sum given by $1+2+3+\cdots+n^2 = \frac{n^2(n^2+1)}{2}$. Since the magic constant is the identical sum of numbers in any row or column in the *n* × *n* magic square, it is obtained by dividing $\frac{n^2(n^2+1)}{2}$ upon *n*. Thus the magic constant for a *n* × *n* magic square is given by $\frac{n(n^2+1)}{2}$.

This completes the proof.

3. Bernoulli Numbers

The numbers which occur as coefficients of $\frac{x^n}{n!}$ in the Maclaurin's series expansion of $\frac{x}{e^x-1}$ are called Bernoulli numbers. The *n*th Bernoulli number is given by $\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$ (3.1)

The first few values of Bernoulli Numbers are given by

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, B_8 = -\frac{1}{30}, B_9 = 0$$

$$B_{10} = \frac{5}{66}, B_{11} = 0, B_{12} = -\frac{691}{2730}, B_{13} = 0, B_{14} = \frac{7}{6}, B_{15} = 0, B_{16} = -\frac{3617}{510}, \ldots \quad (3.2)$$

From the above values we observe that except for B_1, $B_n = 0$ for all odd values of *n*.

4. Ramanujan Summation

While trying to assign particular values of summing divergent series in connection with Riemann Zeta function, Srinivasa Ramanujan provided a formula known as Ramanujan Summation formula. In particular, Ramanujan mentioned that if *r* is a positive integer and if ζ is the Riemann zeta function, then the Ramanujan summation of *r*th powers of positive integers is given by $(RS)(1^r + 2^r + 3^r + \cdots) = (RS)\left(\sum_{k=1}^{\infty} k^r\right) = \zeta(-r) = -\frac{B_{r+1}}{r+1}$ (4.1) where B_{r+1} is the (*r*+1)th Bernoulli number as defined in (3.1).

5. Theorem 2

The Ramanujan Summation of *r*th powers of magic constants obtained from magic squares of order $n \times n$ constructed using first n^2 natural numbers were $n \geq 3$ is given by

$$(RS)\left(15^r + 34^r + 65^r + 111^r + \cdots\right) = -\frac{1}{2^r}\sum_{m=0}^{\frac{3r-1}{2}}\left[\sum_{s=0}^{r} 2^{2m-2s}\binom{3r-2s}{2m-2s}\binom{r}{s}\right]\frac{B_{3r-2m+1}}{3r-2m+1} - \frac{5^r}{2} \quad (5.1)$$

if *r* is odd

$$(RS)\left(15^r + 34^r + 65^r + 111^r + \cdots\right) = -\frac{1}{2^r}\sum_{m=0}^{\frac{3r-2}{2}}\left[\sum_{s=0}^{r} 2^{2m-2s+1}\binom{3r-2s}{2m-2s+1}\binom{r}{s}\right]\frac{B_{3r-2m}}{3r-2m} - \frac{5^r}{2} \quad (5.2)$$

if *r* is even

Proof: In view of (2.1), notice that the expression $\frac{(k+2)\left((k+2)^2+1\right)}{2}$ generates the magic constants 15, 34, 65, 111, 175, 260, . . . for k = 1, 2, 3, 4, 5, 6, . . . respectively.

Hence, we have

$$(RS)\left(15^r + 34^r + 65^r + 111^r + \cdots\right) = (RS)\left(\sum_{k=1}^{\infty}\left[\frac{(k+2)^r\left((k+2)^2+1\right)^r}{2^r}\right]\right) \quad (5.3)$$

Now using binomial expansion for positive integral powers we have

$$(k+2)^r\left((k+2)^2+1\right)^r = \left((k+2)^3+(k+2)\right)^r = (k+2)^{3r} + \binom{r}{1}(k+2)^{3r-2} + \binom{r}{2}(k+2)^{3r-4} + \cdots + \binom{r}{r}(k+2)^r$$

$$= \left[k^{3r} + \binom{3r}{1}k^{3r-1}(2) + \binom{3r}{2}k^{3r-2}(2)^2 + \cdots + \binom{3r}{3r}(2)^{3r}\right] +$$

$$\binom{r}{1}\left[k^{3r-2} + \binom{3r-2}{1}k^{3r-3}(2) + \binom{3r-2}{2}k^{3r-4}(2)^2 + \cdots + \binom{3r-2}{3r-2}(2)^{3r-2}\right] +$$

$$\binom{r}{2}\left[k^{3r-4} + \binom{3r-4}{1}k^{3r-5}(2) + \binom{3r-4}{2}k^{3r-6}(2)^2 + \cdots + \binom{3r-4}{3r-4}(2)^{3r-4}\right] + \cdots +$$

$$\binom{r}{r}\left[k^r + \binom{r}{1}k^{r-1}(2) + \binom{r}{2}k^{r-2}(2)^2 + \cdots + \binom{r}{r}(2)^r\right]$$

Collecting like powers of *k* and simplifying we get

$$\left(k+2\right)^r\left((k+2)^2+1\right)^r=k^{3r}+2^1\binom{3r}{1}k^{3r-1}+\left[2^2\binom{3r}{2}+\binom{r}{1}\right]k^{3r-2}+\left[2^3\binom{3r}{3}+2^1\binom{3r-2}{1}\binom{r}{1}\right]k^{3r-3}+$$
$$\left[2^4\binom{3r}{4}+2^2\binom{3r-2}{2}\binom{r}{1}+\binom{r}{2}\right]k^{3r-4}+\left[2^5\binom{3r}{5}+2^3\binom{3r-2}{3}\binom{r}{1}+2^1\binom{3r-4}{1}\binom{r}{2}\right]k^{3r-5}+\cdots+$$
$$\left[2^{3r}+\binom{r}{1}2^{3r-2}+\binom{r}{2}2^{3r-4}+\binom{r}{3}2^{3r-6}+\cdots+\binom{r}{r}^2 2^r\right]$$

Now substituting this expression in (5.3) and using (4.1), and the fact that $\zeta(0)=-\frac{1}{2}$ we have

$$(RS)\left(15^r+34^r+65^r+111^r+\cdots\right)=(RS)\left(\sum_{k=1}^{\infty}\left[\frac{\left(k+2\right)^r\left((k+2)^2+1\right)^r}{2^r}\right]\right)$$

$$=\frac{1}{2^r}\left\{\begin{array}{l}(RS)\left(\sum_{k=1}^{\infty}k^{3r}\right)+\left[2^1\binom{3r}{1}\right](RS)\left(\sum_{k=1}^{\infty}k^{3r-1}\right)+\left[2^2\binom{3r}{2}+\binom{r}{1}\right](RS)\left(\sum_{k=1}^{\infty}k^{3r-2}\right)\\+\left[2^3\binom{3r}{3}+2^1\binom{3r-2}{1}\binom{r}{1}\right](RS)\left(\sum_{k=1}^{\infty}k^{3r-3}\right)+\left[2^4\binom{3r}{4}+2^2\binom{3r-2}{2}\binom{r}{1}+\binom{r}{2}\right](RS)\left(\sum_{k=1}^{\infty}k^{3r-4}\right)\\+\left[2^5\binom{3r}{5}+2^3\binom{3r-2}{3}\binom{r}{1}+2^1\binom{3r-4}{1}\binom{r}{2}\right](RS)\left(\sum_{k=1}^{\infty}k^{3r-5}\right)+\cdots\\+\left[2^{3r}+\binom{r}{1}2^{3r-2}+\binom{r}{2}2^{3r-4}+\binom{r}{3}2^{3r-6}+\cdots+\binom{r}{r}^2 2^r\right](RS)\left(\sum_{k=1}^{\infty}k^0\right)\end{array}\right\}$$

$$=-\frac{1}{2^r}\left\{\begin{array}{l}\frac{B_{3r+1}}{3r+1}+\left[2^1\binom{3r}{1}\right]\frac{B_{3r}}{3r}+\left[2^2\binom{3r}{2}+\binom{r}{1}\right]\frac{B_{3r-1}}{3r-1}+\left[2^3\binom{3r}{3}+2^1\binom{3r-2}{1}\binom{r}{1}\right]\frac{B_{3r-2}}{3r-2}\\+\left[2^4\binom{3r}{4}+2^2\binom{3r-2}{2}\binom{r}{1}+\binom{r}{2}\right]\frac{B_{3r-3}}{3r-3}+\left[2^5\binom{3r}{5}+2^3\binom{3r-2}{3}\binom{r}{1}+2^1\binom{3r-4}{1}\binom{r}{2}\right]\frac{B_{3r-4}}{3r-4}\\+\cdots+\left[2^{3r}+\binom{r}{1}2^{3r-2}+\binom{r}{2}2^{3r-4}+\binom{r}{3}2^{3r-6}+\cdots+\binom{r}{r}^2 2^r\right]\times\frac{1}{2}\end{array}\right\}$$

Now the last term in the above expression can be simplified as

$$2^{3r}+\binom{r}{1}2^{3r-2}+\binom{r}{2}2^{3r-4}+\binom{r}{3}2^{3r-6}+\cdots+\binom{r}{r}^2 2^r=2^r\left[2^{2r}+\binom{r}{1}2^{2r-2}+\binom{r}{2}2^{2r-4}+\binom{r}{3}2^{2r-6}+\cdots+\binom{r}{r}\right]$$
$$=2^r\sum_{u=0}^{r}\binom{r}{u}\left(2^2\right)^{r-u}(1)^u=2^r\left(2^2+1\right)^r=2^r\times5^r$$

Substituting this in the last expression we get

$$(RS)\left(15^r+34^r+65^r+111^r+\cdots\right)$$
$$=-\frac{1}{2^r}\left\{\begin{matrix}\frac{B_{3r+1}}{3r+1}+\left[2^1\binom{3r}{1}\right]\frac{B_{3r}}{3r}+\left[2^2\binom{3r}{2}+\binom{r}{1}\right]\frac{B_{3r-1}}{3r-1}+\left[2^3\binom{3r}{3}+2^1\binom{3r-2}{1}\binom{r}{1}\right]\frac{B_{3r-2}}{3r-2} \\ +\left[2^4\binom{3r}{4}+2^2\binom{3r-2}{2}\binom{r}{1}+\binom{r}{2}\right]\frac{B_{3r-3}}{3r-3}+\left[2^5\binom{3r}{5}+2^3\binom{3r-2}{3}\binom{r}{1}+2^1\binom{3r-4}{1}\binom{r}{2}\right]\frac{B_{3r-4}}{3r-4}+\cdots\end{matrix}\right\}-\frac{5^r}{2} \quad (5.4)$$

If *r* is odd then 3*r*, 3*r* – 2, 3*r* – 4, . . . are all odd. Since except B_1 all Bernoulli numbers of odd subscripts are zero, from (5.4) we get

$$(RS)\left(15^r+34^r+65^r+111^r+\cdots\right)$$
$$=-\frac{1}{2^r}\left\{\frac{B_{3r+1}}{3r+1}+\left[2^2\binom{3r}{2}+\binom{r}{1}\right]\frac{B_{3r-1}}{3r-1}+\left[2^4\binom{3r}{4}+2^2\binom{3r-2}{2}\binom{r}{1}+\binom{r}{2}\right]\frac{B_{3r-3}}{3r-3}+\cdots\right\}-\frac{5^r}{2}$$
$$=-\frac{1}{2^r}\sum_{m=0}^{\frac{3r-1}{2}}\left[\sum_{s=0}^{r}2^{2m-2s}\binom{3r-2s}{2m-2s}\binom{r}{s}\right]\frac{B_{3r-2m+1}}{3r-2m+1}-\frac{5^r}{2}$$

This proves (5.1)

If *r* is even, then 3*r* + 1, 3*r* – 1, 3*r* – 3, . . . are all odd. Since except B_1 all Bernoulli numbers of odd subscripts are zero, from (5.4) we get

$$(RS)\left(15^r+34^r+65^r+111^r+\cdots\right)$$
$$=-\frac{1}{2^r}\left\{\begin{matrix}\left[2^1\binom{3r}{1}\right]\frac{B_{3r}}{3r}+\left[2^3\binom{3r}{3}+2^1\binom{3r-2}{1}\binom{r}{1}\right]\frac{B_{3r-2}}{3r-2}+ \\ \left[2^5\binom{3r}{5}+2^3\binom{3r-2}{3}\binom{r}{1}+2^1\binom{3r-4}{1}\binom{r}{2}\right]\frac{B_{3r-4}}{3r-4}+\cdots\end{matrix}\right\}-\frac{5^r}{2}$$
$$=-\frac{1}{2^r}\sum_{m=0}^{\frac{3r-2}{2}}\left[\sum_{s=0}^{r}2^{2m-2s+1}\binom{3r-2s}{2m-2s+1}\binom{r}{s}\right]\frac{B_{3r-2m}}{3r-2m}-\frac{5^r}{2}$$

This proves (5.2) and completes the proof.

5.1 Corollary

$$(RS)\left(15+34+65+111+175+\cdots\right)=-\frac{243}{80} \quad (5.5)$$

$$(RS)\left(15^2+34^2+65^2+111^2+175^2+\cdots\right)=-\frac{1844}{105} \quad (5.6)$$

Proof: Taking r = 1 in (5.1), we have

$$(RS)(15+34+65+111+175+\cdots)=-\frac{1}{2}\sum_{m=0}^{1}\left[\sum_{s=0}^{1}2^{2m-2s}\binom{3-2s}{2m-2s}\binom{1}{s}\right]\frac{B_{4-2m}}{4-2m}-\frac{5}{2}$$

$$=-\frac{1}{2}\sum_{m=0}^{1}\left[2^{2m}\binom{3}{2m}\binom{1}{0}+2^{2m-2}\binom{1}{2m-2}\binom{1}{1}\right]\frac{B_{4-2m}}{4-2m}-\frac{5}{2}$$

$$=-\frac{1}{2}\left\{\left[2^{0}\binom{3}{0}\right]\frac{B_4}{4}+\left[2^{2}\binom{3}{2}+2^{0}\binom{1}{0}\right]\frac{B_2}{2}\right\}-\frac{5}{2}=\frac{1}{240}-\frac{13}{24}-\frac{5}{2}=-\frac{243}{80}$$

Hence $(RS)(15+34+65+111+175+\cdots)=-\frac{243}{80}$

Taking r = 2 in (5.2), we have

$$(RS)(15^2+34^2+65^2+111^2+175^2+\cdots)=-\frac{1}{2^2}\sum_{m=0}^{2}\left[\sum_{s=0}^{2}2^{2m-2s+1}\binom{6-2s}{2m-2s+1}\binom{2}{s}\right]\frac{B_{6-2m}}{6-2m}-\frac{5^2}{2}$$

$$=-\frac{1}{4}\sum_{m=0}^{2}\left[2^{2m+1}\binom{6}{2m+1}\binom{2}{0}+2^{2m-1}\binom{4}{2m-1}\binom{2}{1}+2^{2m-3}\binom{2}{2m-3}\binom{2}{2}\right]\frac{B_{6-2m}}{6-2m}-\frac{25}{2}$$

$$=-\frac{1}{4}\left\{\left[2^{1}\binom{6}{1}\right]\frac{B_6}{6}+\left[2^{3}\binom{6}{3}+2^{1}\binom{4}{1}\binom{2}{1}\right]\frac{B_4}{4}+\left[2^{5}\binom{6}{5}+2^{3}\binom{4}{3}\binom{2}{1}+2^{1}\binom{2}{1}\right]\frac{B_2}{2}\right\}-\frac{25}{2}$$

$$=-\frac{1}{84}+\frac{11}{30}-\frac{65}{12}-\frac{25}{2}=-\frac{1844}{105}$$

This completes the proof.

6. Conclusion

By considering the magic constants obtained through magic squares of order $n \times n$ for $n \geq 3$, I have determined the Ramanujan summation for rth powers of such constants. This is achieved by proving two new formulas derived in theorem 2. In particular if r is odd, then the Ramanujan summation value is given by (5.1) and if r is even then it is given by (5.2). The computations when r = 1 and 2 are provided in corollary through (5.5) and (5.6). In this sense, the result obtained in this paper connects the concept of magic square constants with that of Ramanujan summation values. By considering various values of r, we can compute Ramanujan summation for several divergent series representing powers of magic constants using equations (5.1) and (5.2). The results obtained in this paper can be considered as possible extensions to the original formulas provided by Srinivasa Ramanujan in his famous notebooks.

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