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ON THE CUBIC EQUATION WITH EIGHT UNKNOWNS

$$x^3 + y^3 + z^3 + w^3 = U^3 + V^3 + P^3 + Q^3$$

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ABSTRACT

We obtain infinitely many non-zero octuple (x, y, z, w, U, V, P, Q) satisfying the cubic equation with eight unknown's $x^3 + y^3 + z^3 + w^3 = U^3 + V^3 + P^3 + Q^3$. Various interesting relations between the solutions, Polygonal numbers, Pyramidal numbers, and Centered Pyramidal numbers are obtained.

KEYWORDS

Cubic equations with eight unknowns, integral solutions, special numbers, figurative numbers, centered pyramidal numbers

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NOTATIONS

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

$\Pr_n$: Pronic number of rank n

$Ct_{m,n}$: centered Polygonal number of rank n with size m

$Cf_{3,n,3}$: Centered Triangular Pyramidal number of rank n

$Cf_{3,n,6}$: Centered Hexogonal Pyramidal number of rank n

$F_{4,n,3}$: Fourth Dimensional Figurate Triangular number of rank n

$F_{4,n,4}$: Fourth Dimensional Figurate Square number of rank n

$F_{4,n,6}$: Fourth Dimensional Figurate Hexogonal number of rank n

INTRODUCTION

The cubic diophantine equations offer an unlimited field of research by reason of their variety[1,2,3]. In particular,one may refer [4-9] for cubic equations with five unknowns.

It seems that much work has not been done in finding integral solutions of cubic equations with multivariables.This has motivated as to search for cubic equations.This communication concerns with the problem of finding integer solution to the cubic equation with eight variables given by $x^3+y^3+z^3+w^3=U^3+V^3+P^3+Q^3$.A few interesting relations between the solutions, special numbers, figurative numbers and centered pyramidal numbers are obtained.

METHOD OF ANALYSIS

The diophantine equation representing a cubic equation with eight unknowns is

$$x^3+y^3+z^3+w^3=U^3+V^3+P^3+Q^3 \qquad (1)$$

It is well-known that (1) is satisfied by octuple $(2,3,10,11,1,5,8,12)$.In what follows the process of obtaining other choices of non-zero distinct integral solutions to (1) are illustrated.

PATTERN

Introducing the linear transformations

$$\left.\begin{aligned} x&=a+b+c\\ y&=a-b-c\\ z&=-a-b+c\\ w&=-a+b-c\\ U&=a+e+f\\ V&=a-e-f\\ P&=-a-e+f\\ Q&=-a+e-f \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

in (1),it simplifies to

$$bc=ef \qquad (3)$$

Again,the substitution of the linear transformations

$$b=u+v, c=u-v, e=p+q, f=p-q \qquad (4)$$

in (3) gives $\quad u^2+q^2=p^2+v^2 \qquad (5)$

The choice of $\quad v=kq, \quad k>1 \qquad (6)$

in (5) leads to $\quad u^2=(k^2-1)q^2+p^2,\ k>1 \qquad (7)$

which is satisfied by

$$\left.\begin{aligned} q&=2rs\\ p&=(k^2-1)r^2-s^2\\ u&=(k^2-1)r^2+s^2 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (8)$$

Using (8),(6) and (4) in (2),the corresponding non-zero integral solutions of (1) are represented by

$$x(k,r,s) = a + 2(k^2 - 1)r^2 + 2s^2$$
$$y(k,r,s) = a - 2(k^2 - 1)r^2 - 2s^2$$
$$z(k,r,s) = -a - 4krs$$
$$w(k,r,s) = -a + 4krs$$
$$U(k,r,s) = a + 2(k^2 - 1)r^2 - 2s^2$$
$$V(k,r,s) = a - 2(k^2 - 1)r^2 + 2s^2$$
$$P(r,s) = -a - 4rs$$
$$Q(r,s) = -a + 4rs$$

Properties

1. $x(k,r,s) - y(k,r,s) - U(k,r,s) + V(k,r,s) \equiv 0 \pmod 8$

2. $[Q(r,r) + a](P(r+1,1) + a) = -8P_r^5$

3. $[Q(r,r) - P(r,r)][w(k,r+1,1) - z(k,r+1,1)] = 128kP_r^5$

4. $z(k,r,s) + w(k,r,s) + P(r,s) + Q(r,s) \equiv 0 \pmod 4$

5. $\mathrm{U}(2s,s,s) - \mathrm{V}(2s,s,s) - 96\mathrm{F}_{4,s,6} + 48\mathrm{Cf}_{3,s,6} + 80\mathrm{t}_{3,s} \equiv 0 \pmod{40}$

6. Each of the following is a nasty number [3]

(a) $30[y(r,r,r) - x(r,r,r) - 6\,\mathrm{Pr}_a - 6Cf_{3,r,6} + F_{4,r,3}]$

(b) $Q(r,3r) - P(r,3r)$

(c) $6\left[\dfrac{w(\alpha^2,r,s) - z(\alpha^2,r,s)}{Q(r,s) - P(r,s)}\right]$

(d) $6[x(3r,r,r) - y(3r,r,r) + z(3r,r,r) - w(3r,r,r) + P(r,r) + Q(r,r)]$

7. $w(r,2r,4r) - z(r,2r,4r)$ is a cubic integer

PATTERN

Rewrite (7) as $(k^2 - 1)q^2 + p^2 = u^2 * 1$ (9)

Assume $u = (k^2 - 1)a^2 + b^2$ (10)

and write 1 as $$1=\frac{\left(1+i\sqrt{k^2-1}\right)\left(1-i\sqrt{k^2-1}\right)}{k^2} \qquad (11)$$

substituting (11),(10) in (9),and employing method of factorization,define

$$p+i\sqrt{k^2-1}q=\frac{\left(b+i\sqrt{k^2-1}a\right)^2\left(1-i\sqrt{k^2-1}\right)}{k} \qquad (12)$$

Equating real and imaginary parts in(12),we get

$$\left.\begin{aligned} p&=\frac{1}{k}\left[b^2-(k^2-1)a^2-2ab(k^2-1)\right]\\ q&=\frac{1}{k}\left[b^2-(k^2-1)a^2+2ab\right]\end{aligned}\right\}\dots\dots\dots\dots \qquad (13)$$

Choosing a by kA, b by kB in (13),(10),(6) and employing (2), the corresponding non-zero integral solutions of (1) are obtained as

$$\begin{aligned}
x(k,r,s)&=a+2k^2\left[(k^2-1)A^2-2AB+(B^2+AB)\right]\\
y(k,r,s)&=a-2k^2\left[(k^2-1)A^2-2AB+(B^2+AB)\right]\\
z(k,r,s)&=-a+2k^2\left[(k^2-1)A^2-2AB-(B^2+AB)\right]\\
w(k,r,s)&=-a-2k^2\left[(k^2-1)A^2-2AB-(B^2+AB)\right]\\
U(k,r,s)&=a+k\left[2B^2-2(k^2-1)A^2-2AB(k^2-2)-2k^2AB\right]\\
V(k,r,s)&=a-k\left[2B^2-2(k^2-1)A^2-2AB(k^2-2)-2k^2AB\right]\\
P(r,s)&=-a-k\left[2B^2-2(k^2-1)A^2-2AB(k^2-2)+2k^2AB\right]\\
Q(r,s)&=-a+k\left[2B^2-2(k^2-1)A^2-2AB(k^2-2)+2k^2AB\right]
\end{aligned}$$

Properties

1. $x(A,2A,A)-y(A,2A,A)-4Cf^2_{3,A,6}+48F_{4,A,6}-48Cf_{3,A,3}-16t_{4,A}\equiv 0 \pmod{24}$
2. $Q(A,A,2)-P(A,A,2)-4Cf_{3,A,6}-32t_{3,A}\equiv 0 \pmod 4$
3. $w(A,A,A)-z(A,A,A)+4Cf^2_{3,A,6}$ is a nasty number
4. $18\left[V(A^2,A,A)-U(A^2,A,A)+144F_{4,A,4}-48Cf_{3,A,6}-24\Pr_a-48t_{4,A}\right]$ is a cubic integer

PATTERN

Rewrite (7) as $$u^2-p^2=(k^2-1)q^2 \qquad (14)$$

(14) can be written in the form of ratio as

$$\frac{u+p}{q}=\frac{(k^2-1)q}{u-p}=\frac{A}{B}\ , \quad B>0 \qquad (15)$$

which is equivalent to the system of double equations

$$\left.\begin{aligned} Bu + Bp - Aq = 0 \\ -Au + Ap + (k^2 - 1)Bq = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots \quad (16)$$

Applying the method of cross multiplication, we get

$$\left.\begin{aligned} q &= 2AB \\ p &= A^2 - (k^2 - 1)B^2 \\ u &= A^2 + (k^2 - 1)B^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots \quad (17)$$

In view of (17),(6) and (2), the corresponding non-zero integral solutions of (1) are given by

$$x(k, A, B) = a + 2A^2 + 2(k^2 - 1)B^2$$
$$y(k, A, B) = a - 2A^2 - 2(k^2 - 1)B^2$$
$$z(k, A, B) = -a - 4kAB$$
$$w(k, A, B) = -a + 4kAB$$
$$U(k, A, B) = a + 2A^2 - 2(k^2 - 1)B^2$$
$$V(k, A, B) = a - 2A^2 + 2(k^2 - 1)B^2$$
$$P(A, B) = -a - 4AB$$
$$Q(A, B) = -a + 4AB$$

PATTERN

Introduction of the linear substitution

$$\left.\begin{aligned} x = a + b, y = a - b, U = a + c, V = a - c \\ P = A + B, Q = A - B, z = A + C, w = A - C \end{aligned}\right\} \ldots\ldots\ldots \quad (18)$$

in(2) leads to $a(b^2 - c^2) = A(B^2 - C^2)$ (19)

Taking $a = kA \quad , \ k > 1$ (20)

in(19) it gives $B^2 + kc^2 = kb^2 + C^2$ (21)

(21) can be written as

$$B^2 + kc^2 = (kb^2 + C^2) * 1 \quad (22)$$

Write as 1 as

$$1 = \frac{\left[(k-1) + i2\sqrt{k}\right]\left[(k-1) - i2\sqrt{k}\right]}{(k+1)^2} \quad (23)$$

Substituting (23) in (22) ,and employing the method of factorization, define

$$B + ic\sqrt{k} = \frac{\left[C(k-1) - 2kb\right] + i\sqrt{k}\left[b(k-1) + 2C\right]}{k+1}$$

Equating real and imaginary parts,we get

$$\left.\begin{aligned} B &= \frac{1}{k+1}\left(C(k-1) - 2kb\right) \\ c &= \frac{1}{k+1}\left(b(k-1) + 2C\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots \quad (24)$$

Replacing b by $(k+1)b$, C by $(k+1)C$ (25)

Using (25),(24),(20) in (8) and (2), the corresponding non-zero integral solutions of (1) are followed by

$$\mathrm{x = kA + (k+1)b}$$
$$\mathrm{y = kA - (k+1)b}$$
$$\mathrm{z = A + (k+1)C}$$
$$\mathrm{w = A - (k+1)C}$$
$$\mathrm{U = kA + b(k-1) + 2C}$$
$$\mathrm{V = kA - b(k-1) - 2C}$$
$$\mathrm{P = A + C(k-1) - 2kb}$$
$$\mathrm{Q = A - C(k-1) + 2kb}$$

PATTERN

Consider the identity

$$(a+b)^3 + (a-b)^3 - (a+c)^3 - (a-c)^3 = 6a(b^2 - c^2) \quad (26)$$

Let A,B,C be three non-zero distinct integers such that

$$a = B - C, b = \frac{B+C+A}{2}, c = \frac{B+C-A}{2} \quad (27)$$

Using (27) in (26),we have

$$\text{L.H.S of (26)} = 6A(B^2 - C^2)$$
$$= (A+B)^3 + (A-B)^3 - (A-C)^3 - (A-C)^3 \quad (28)$$

Substituting (27) in L.H.S of (28) and simplifying, the corresponding values of octuple (x, y, z, w, U, V, P, Q) are given by

$$x = \frac{3B - C + A}{2}$$
$$y = \frac{B - 3C - A}{2}$$
$$z = (A+B)$$
$$w = (A-B)$$
$$U = \frac{3B - C - A}{2}$$
$$V = \frac{B - 3C + A}{2}$$
$$P = (A+C)$$
$$Q = (A-C)$$

PATTERN

Taking $u = \alpha + \beta, p = \alpha - \beta$ (29)

in (14) gives $4\alpha\beta = (k^2 - 1)q^2$, $\mathrm{k > 1}$ (30)

Assumption $\alpha = (k^2 - 1)\beta$ (31)

in (30)leads to $q = 2\beta$

In view of (31) ,(29),(60,(4) and (2),the corresponding integral solutions of (1) are represented by

$$x = a + 2k^2\beta$$
$$y = a - 2k^2\beta$$
$$z = -a - 4k\beta$$
$$w = -a + 4k\beta$$
$$U = a + 2k^2\beta - 4\beta$$
$$V = a - 2k^2\beta + 4\beta$$
$$P = -a - 4\beta$$
$$Q = -a + 4\beta$$

CONCLUSION

One may search for other pattern of solutions and their corresponding properties.

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