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# A SPECIAL TRANSCENDENTAL EQUATION WITH FIVE UNKNOWNS

$$2\sqrt[3]{y^2+x^2}+\sqrt[2]{2(X^2+Y^2)-3XY}=3(k^2+3)z^3$$

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**ABSTRACT**

The transcendental equation with five unknowns represented by the diophantine equation $2\sqrt[3]{y^2+x^2}+\sqrt[2]{2(X^2+Y^2)-3XY}=3(k^2+3)z^3$ is analyzed for its patterns of non-zero distinct integral solutions and different methods of integral solutions are illustrated.

**KEYWORDS:** Transcendental equations, integral solutions, Pyramidal number, Four dimensional figurative number, Polygonal number, Pyramidal number

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**NOTATIONS:**

$T_{m,n}$ -Polygonal number of rank $n$ with size $m$

$P_n^m$ - Pyramidal number of rank $n$ with size $m$

$PR_n$ - Pronic number of rank $n$

$OH_n$ - Octahedral number of rank $n$

$SO_n$ -Stella octangular number of rank $n$

$S_n$ -Star number of rank $n$

$J_n$ -Jacobsthal number of rank of $n$

$j_n$ - Jacobsthal-Lucas number of rank $n$

$KY_n$ -keynea number of rank $n$

$CP_{n,3}$ - Centered Triangular pyramidal number of rank $n$

$CP_{n,6}$ - Centered hexagonal pyramidal number of rank $n$

$F_{4,n,3}$ - Four Dimensional Figurative number of rank $n$ whose generating polygon is a triangle

$F_{4,n,5}$ - Four Dimensional Figurative number of rank $n$ whose generating polygon is a pentagon.

**INTRODUCTION**

Diophantine equations have an unlimited field of research by reason of their variety. Most of the Diophantine problems are algebraic equations [123]. It seems that much work has not been done to obtain integral solutions of transcendental equations. In this context one may refer [4-16]. This communication analyses a transcendental equation with five unknown given by $2\sqrt[3]{y^2+x^2}+\sqrt[2]{2(X^2+Y^2)-3XY}=3(k^2+3)z^3$. Infinitely many non-zero integer quintuples $(x,y,z,X,Y)$ satisfying the above equation are obtained.

**Method of Analysis:**

The Diophantine equation representing the transcendental equation is given by

$$2\sqrt[3]{y^2+x^2}+\sqrt[2]{2(X^2+Y^2)-3XY}=3(k^2+3)z^3 \qquad (1)$$

To start with, the transformations

$$\left.\begin{array}{l} x=p(p^2+q^2),\ y=q(p^2+q^2) \\ X=p^2-7q^2+2pq,\ Y=p^2-7q^2-2pq \end{array}\right\} \qquad (2)$$

in (1) leads to

$$p^2+3q^2=(k^2+3)z^3 \qquad (3)$$

The above equation (3) is solved through different approaches and thus,
one obtains different sets of solutions to (1)

**Pattern1:**

Let $z=a^2+3b^2$ (4)

Substituting (4) in (3) and using the method of factorisation, define

$$(p+i\sqrt{3}q)=(k+i\sqrt{3})(a+i\sqrt{3}b)^3 \qquad (5)$$

Equating real and imaginary parts in (5) we get

$$\left.\begin{array}{l} p=k(a^3-9ab^2)-9(a^2b-b^3) \\ q=(a^3-9ab^2)+3k(a^2b-b^3) \end{array}\right\} \qquad (6)$$

In view of (2), (4) and (6), the corresponding values of $x,y,X,Y$ and $z$ are represented by

$$\left.\begin{array}{l} x=p(p^2+q^2) \\ y=q(p^2+q^2) \\ X=p^2-7q^2+2pq \\ Y=p^2-7q^2-2pq \\ z=a^2+3b^2 \end{array}\right\} \qquad (7)$$

where the values of $p$ *and* $q$ are given by (6).

**Properties:**

1. $x(a,a)+y(a,a)+512(k^2+1)(k+1)[4CP_{a,6}.T_{4,a}.T^2{}_{3,a}-4T_{3,a^4}+2T_{4,a^2}-T_{4,a^2}.CP_{a,6}]=0$

2. $2x(a,b)y(a,b)[X(a,b)+Y(a,b)]-(X(a,b)-Y(a,b))[x^2(a,b)-7y^2(a,b)]=0$

3. The following expressions are nasty numbers:

(a) $6k[x(a,a).y(a,a)]$

(b) $\frac{6x(a,b).y(a,b)}{X(a,b)-Y(a,b)}$

(c) $\frac{6kx(a,a)}{y(a,a)}$

4. The following expressions are cubic integers:

(a) $(k-1)(k^2+1)^2[y(a,a)-x(a,a)]$

(b) $z(2^{2n},2^{2n})-4(KY_n-3J_{2n+1})$

(c) $\frac{4Y(1,a,a)}{X(1,a,a)}$

(d) $\frac{2(X(a,b)-Y(a,b))[x^2(a,b)+3y^2(a,b)]}{(k^2+3)x(a,b)y(a,b)}$

5. $2X(1,a,a).Y(1,a,a)$ is a sextic integer

6. $X(a,a)+Y(a,a)=128(k^2-7)(2P_a^5.CP_{a,6}-CP_{a,6}T_{4,a})$

7. $$X(a,a)-Y(a,a)=256k[720F_{6,a,3}-1800F_{5,a,3}+1560F_{4,a,3}-540P_a^3 +62T_{3,a}+2T_{5,a}-6T_{4,a}]$$

8. $$Y(1,a,1).-X(1,a,1)+4\{4P_a^5.CP_{a,3}-3T_{4,a}.SO_a-CP_{a,6}(T_{4,a}-2) -92T_{3,a^2}+127T_{4,a}+108P_a^5\}\equiv 0 \pmod{27}$$

**Pattern2:**

Now, rewrite (3) as,

$$p^2+3q^2=(k^2+3)z^3*1 \quad (8)$$

Also 1 can be written as

$$1=\frac{(1+i\sqrt{3})^n(1-i\sqrt{3})^n}{2^{2n}} \quad (9)$$

Substituting (4) and (9) in (8) and using the method of factorisation, define

$$(p+i\sqrt{3}q)=\frac{(k+i\sqrt{3})(1+i\sqrt{3})^n(a+i\sqrt{3}b)^3}{2^n} \quad (10)$$

$$=(k+i\sqrt{3})(cos\, n\theta + i\sin n\theta)(a+i\sqrt{3}b)^3$$

Where $r=2 \ \ and \ \ \theta = \frac{\pi}{3}$

Equating real and imaginary parts in (10) we get

$$\left.\begin{aligned} p &= (k\cos n\theta - \sqrt{3}\sin n\theta)(a^3-9ab^2) - 9(\cos n\theta + \frac{k}{\sqrt{3}}\sin n\theta)(a^2b-b^3) \\ q &= (\cos n\theta + \frac{k}{\sqrt{3}}\sin n\theta)(a^3-9ab^2) + 3(k\cos n\theta - \sqrt{3}\sin n\theta)(a^2b-b^3) \end{aligned}\right\} \quad (11)$$

Performing some algebra and considering (2), (4) and (11), the corresponding values of $x, y, X, Y$ and $z$ are represented by

$$\left.\begin{aligned} x &= p(p^2+q^2) \\ y &= q(p^2+q^2) \\ X &= p^2-7q^2+2pq \\ Y &= p^2-7q^2-2pq \\ z &= a^2+3b^2 \end{aligned}\right\} \quad (12)$$

where $p \ \ and \ \ q$ are given by (11)

**Note:**

In addition to this 1 may also be expressed in four ways

(a) $$1 = \frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{7^2} \quad (13)$$

Following the same procedure as in pattern2 we get the values of $p$ , $q$ $and$ $z$ as

$$\left.\begin{aligned} p &= 7^2[(k-12)(a^3-9ab^2)-9(4k+1)(a^2b-b^3)] \\ q &= 7^2[(4k+1)(a^3-9ab^2)+3(k-12)(a^2b-b^3)] \\ z &= 7^2(A^2+3B^2) \end{aligned}\right\} \quad (14)$$

(b) $$1 = \frac{(13+i3\sqrt{3})(13-i3\sqrt{3})}{14^2} \quad (15)$$

Following the same procedure as in pattern2 we get the values of $p$ , $q$ $and$ $z$ as

$$\left.\begin{aligned} p &= 14^2[(13k-9)(a^3-9ab^2)-9(3k+13)(a^2b-b^3)] \\ q &= 14^2[(3k+13)(a^3-9ab^2)+3(13k-9)(a^2b-b^3)] \\ z &= 14^2(a^2+3b^2) \end{aligned}\right\} \quad (16)$$

(c) $$1 = \frac{(1+i15\sqrt{3})(1-i15\sqrt{3})}{26^2} \quad (17)$$

Following the same procedure as in pattern2 we get the values of $p$ , $q$ $and$ $z$ as

$$\left.\begin{aligned} p &= 26^2[(k-45)(a^3-9ab^2)-9(15k+1)(a^2b-b^3)] \\ q &= 26^2[(15k+1)(a^3-9ab^2)+3(k-45)(a^2b-b^3)] \\ z &= 26^2(a^2+3b^2) \end{aligned}\right\} \quad (18)$$

(d) $$1=\frac{((3-\alpha^2)+i2\alpha\sqrt{3})((3-\alpha^2)-i2\alpha\sqrt{3})}{(3+\alpha^2)^2} \quad (19)$$

Following the same procedure as in pattern2 we get the values of $p$ , $q$ $and$ $z$ as

$$\left.\begin{aligned} p &= (3+\alpha^2)^2[k(3-\alpha^2)-6\alpha)(a^3-9ab^2)-9(3-\alpha^2+2\alpha k)(a^2b-b^3)] \\ q &= (3+\alpha^2)^2[(3-\alpha^2+2\alpha k)(a^3-9ab^2)+3(k(3-\alpha^2)-6\alpha)(a^2b-b^3)] \\ z &= (3+\alpha^2)^2(a^2+3b^2) \end{aligned}\right\} \quad (20)$$

Substituting the above values of $p, q$ $and$ $z$ in (2) the corresponding solutions can be obtained.

**Pattern3**

The assumption

$$z=(k^2+3)Z,\ p=(k^2+3)^2PZ, q=(k^2+3)^2QZ \quad (21)$$

in (3) yields to

$$P^2+3Q^2=Z \quad (22)$$

(i) Taking $Z=T^2$ (23)

in (22) and performing some algebra, we get

$$P=(\alpha^2-3\beta^2), T=(\alpha^2+3\beta^2), Q=2\alpha\beta \quad (24)$$

From (24), (23) and (21) we get

$$\left.\begin{aligned} p &= (\alpha^2-3\beta^2)(k^2+3)^2(\alpha^2+3\beta^2) \\ q &= 2\alpha\beta(k^2+3)^2(\alpha^2+3\beta^2)^2 \\ z &= (k^2+3)(\alpha^2+3\beta^2)^2 \end{aligned}\right\} \quad (25)$$

In view of (25) and (2), we get the corresponding integral solution of (1).

(ii) Taking $Z=T^3$ (26)

in (22),we get

$$P^2+3Q^2=T^3 \quad (27)$$

The solution of (27) as

$$P=a(a^2+3b^2), T=a^2+3b^2, Q=b(a^2+3b^2) \quad (28)$$

In view of (21), (28), we have

$$\left.\begin{aligned} p &= a(k^2+3)^2(a^2+3b^2)^4 \\ q &= b(k^2+3)^2(a^2+3b^2)^4 \\ z &= (k^2+3)(a^2+3b^2)^3 \end{aligned}\right\} \quad (29)$$

Using (29) and (2), we get the corresponding integral solution of (1).

It is to be noted that, the solution of (27) may also be written as

$$P=(a^3-9ab^2), Q=3(a^2b-b^3), T=a^2+3b^2 \quad (30)$$

Using (30) and (2), we get the corresponding integral solution of (1).

(iii) Taking $Z=T^n$ (31)

in (22) and performing some algebra, we get

$$\left.\begin{aligned} p &= \frac{1}{2}[(a+i\sqrt{3}b)^n+(a-i\sqrt{3}b)^n](k^2+3)^2(a^2+3b^2)^n \\ q &= \frac{1}{2i\sqrt{3}}[(a+i\sqrt{3}b)^n-(a-i\sqrt{3}b)^n](k^2+3)^2(a^2+3b^2)^n \\ z &= (k^2+3)(a^2+3b^2)^n \end{aligned}\right\} \quad (32)$$

Using (32) and (2), we get the corresponding integral solution of (1).

**Properties:**

1. $4x(a,a)-(k^2+3)^3z^3(a,a).CP_{a^n,3}.GL_n(2,-4)[GL_n{}^2(2,-4)+2GF_n{}^2(2,-4)]=0$

2. $2y(a,a)-(k^2+3)^3GF_n(2,-4)z^3(a,a)(2P^5_{a^n}-T_{4,a^n})[GL_n{}^2(2,-4)+2GF_n{}^2(2,-4)]=0$

3. $X(a,a)-Y(a,a)=4^{2n}(k^2+3)^4GL(2,-4).GF_n(2,-4)[6CP_{a^n,3}.(OH_{a^n})-3T_{3,a^{2n}}+2T_{4,a^n}]$

4. $X(a,a).Y(a,a)-4^{4n-2}(k^2+3)^8(CP_{a^{4n},6})[GL^4(2,-4)+28^2GF_n{}^4(2,-4)-72GL_n{}^2(2,-4)GF_n{}^2(2,-4)]=0$

5. $6z(a,a)-4^n(k^2+3)[S_{a^n}+6(2CP_{a^n,6}-SO_{a^n})-1]=0$

(iv) Replacing

$$p=Pz, q=Qz \quad (33)$$

in (3) yields to

$$p^2+3q^2=(k^2+3)z \quad (34)$$

Taking $z=t^2$ (35)

in (34) and writing it as a system of double equations, we have

$$\left.\begin{aligned} p-kt &= t-q \\ p+kt &= 3(t+q) \end{aligned}\right\} \quad (36)$$

and Solving we get

$$p=4(k+3)\alpha^3, q=4(k-1)\alpha^3, z=4\alpha^2 \quad (37)$$

Using (2) and (37), we get the corresponding integral solution of (1)

Writing the system of double equations differently as

$$\left.\begin{aligned} p-kt &= 3(t-q) \\ p+kt &= t+q \end{aligned}\right\} \quad (38)$$

Solving we get

$$p=4(3-k)\alpha^3, q=4(k+1)\alpha^3, z=4\alpha^2 \quad (39)$$

Using (2) and (39), we get the corresponding integral solution of (1)

**Properties:**

1. $X(a,a,1)+Y(a,a,1)=48[4P_a^5.CP_{a,6}-SO_a.T_{4,a}-CP_{a,6}]$
2. $6(2k-k^2+3)[X(p,q)-Y(p,q)]$ is a nasty number.
3. $(1-k)^4 z(p,q)[X(p,q)-Y(p,q)]$ is a quintic integer.
4. $z(a,a,1)[x(a,a,1)+y(a,a,1)]$ is an integer of power seven.
5. $X(a,a,1)-Y(a,a,1)-128[4P_a^5.CP_{a,3}-2P_a^5.T_{4,a}-CP_{a,6}]=0$

**CONCLUSION**

In conclusion, one may search for different patterns of solutions to (1) and their corresponding properties.

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