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ON THE HOMOGENEOUS BI-QUADRATIC EQUATION WITH FIVE UNKNOWNS

$$[k(x^2+y^2)-(2k-1)xy](x^2-y^2)=4k(z^2-w^2)T^2$$

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ABSTRACT

The biquadratic equation with 5 unknown given by $[k(x^2+y^2)-(2k-1)xy](x^2-y^2)=4k(z^2-w^2)T^2$ is analyzed for its patterns of non – zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

KEY WORDS

Quadratic equation, Integral solutions, Special polygonal numbers, Pyramidal numbers.

INTRODUCTION

Biquadratic diophantine equations, homogeneous and non- homogeneous, have aroused the interest of numerous Mathematicians since antiquity as can be seen from [1-7]. In the context one may refer [8-24] for varieties of problems on the Diophantine equations with two, three and four variables. This communication concerns with the problems of determining non-zero integral solutions of yet another biquadratic equation in 5 unknowns represented by $[k(x^2+y^2)-(2k-1)xy](x^2-y^2)=4k(z^2-w^2)T^2$. A few interesting relations between the solutions and special polygonal numbers are presented.

NOTATIONS USED

- $t_{m,n}$ - Polygonal number of rank n with size m.
- P_n^m - Pyramidal number of rank n with size m.
- $Ct_{m,n}$ - Centered polygonal number of rank n with size m.
- gn_a - Gnomonic number of rank a
- SO_n - Stella octangular number of rank n
- pr_n - Pronic number of rank n
- $CP_{m,n}$ - Centered pyramidal number of rank n with size m

METHOD OF ANALYSIS

The diophantine equation representing the biquadratic equation with five unknowns under consideration is

$$[k(x^2+y^2)-(2k-1)xy](x^2-y^2)=4k(z^2-w^2)T^2 \quad (1)$$

The substitution of the linear transformations

$$x=u+v,\, y=u-v,\, z=uv+1,\, w=uv-1 \quad (2)$$

in (1) leads to $u^2+(4k-1)v^2=4kT^2$ (3)

Different patterns of solutions of (1) are presented below

Pattern -1

Write (3) as $u^2-v^2=4k(T^2-v^2)$ (4)

which implies $\frac{u+v}{k(T+v)}=\frac{4(T-v)}{u-v}=\frac{p}{q}, q\neq 0$ (5)

Using the method of cross ratio, we get

$$u=u(k,p,q)=kp^2+4q^2-8kpq$$

$$v=v(k,p,q)=kp^2-4q^2$$

$$T=T(k,p,q)=-kp^2-4q^2+2pq$$

Hence in view of (2) the corresponding solutions of (1) are

$$x=x(k,p,q)=2kp^2-8kpq$$

$$y=y(k,p,q)=8q^2-8kpq$$

$$z=z(k,p,q)=k^2p^4-8k^2p^3q+32kpq^3-16q^4+1$$

$$w=w(k,p,q)=k^2p^4-8k^2p^3q+32kpq^3-16q^4-1$$

$$T=T(k,p,q)=-kp^2-4q^2+2pq$$

A few Numerical examples are presented in the table below:

p	q	x	y	z	w	T
1	1	$-6k$	$8-8k$	$-7k^2+32k-15$	$-7k^2+32k-17$	$-k-2$
1	2	$-14k$	$32-16k$	$-15k^2+256k-255$	$-15k^2+256k-257$	$-k-12$
2	1	$-8k$	$8-16k$	$-48k^2+64k-15$	$-48k^2+64k-17$	$-4k$
2	2	$-24k$	$32-32k$	$-112k^2+512k-255$	$-112k^2+512k-257$	$-4k-8$
3	1	$-6k$	$8-24k$	$-135k^2+96k-15$	$-135k^2+96k-17$	$2-9k$

From the table it is observed that $x^2+w^2=y^2+z^2$

and by definition, the numbers $49k^4-448k^3+1298k^2-1088k+289$,

$$225k^4-7680k^3+73442k^2-131584k+66049\ ,$$

$$2304k^4-6144k^3+5792k^2-2176k+289\ ,$$

$$12544k^4 - 114688k^3 + 320288k^2 - 263168k + 66049$$
and $$18225k^4 - 25920k^3 + 13842k^2 - 3264k + 289$$

represent the second order Ramanujan numbers.

Thus, one may obtain infinitely many second order Ramanujan numbers.

A few interesting properties observed are as follows:

1. $z(q,1,q) - 4T(q,1,q^2) + 4x(q,1,q^3) - y(q,q,q) + 27t_{4,q} = Ct_{24,q}$
2. $x(p,p,-p) - y(p,p,-p) - 2T(p,p,-p) = 8p_p^5$
3. $x(k,1,k^2) - 2y(k,1,k^2) + z(k,1,k) = t_{4,k} + gn_k$
4. $x(k,p,k) - y(k,p,k) \equiv 0 \pmod k$
5. $x(12p^2, p, p^2) - y(12p^2, p, p^2)$ is a Bi quadratic integer.
6. Each of the following represents a nasty number:
   - $6\{x(k,1,k^2) - 2y(k,1,k^2) + z(k,1,k)\}$
   - $-2\{x(k,q,q) - y(k,q,q) + 2T(k,q,q)\}$
   - $3\{y(1,p,-p) + 2T(1,p,-p)\}$

Pattern-2:

Instead of (5) write (4) as

$$\frac{u+v}{T+v} = \frac{4k(T-v)}{u-v} = \frac{p}{q}, q \neq 0$$

Following a similar procedure as in pattern-1, the solutions of (1) are

$$x = x(k,p,q) = 2p^2 - 8kpq$$
$$y = y(k,p,q) = 8kq^2 - 8kpq$$
$$z = z(k,p,q) = p^4 - 8kp^3q + 32k^2pq^3 - 16k^2q^4 + 1$$
$$w = w(k,p,q) = p^4 - 8kp^3q + 32k^2pq^3 - 16k^2q^4 - 1$$
$$T = T(k,p,q) = -p^2 - 4kq^2 + 2pq$$

Properties:

1. $x(1,-p,p+1) + y(1,-p,p+1) + 2T(1,-p,p+1) + z(k,p,q) - w(k,p,q) - 1 = Ct_{24,p}$
2. $2\{x(q^2,q,q) + y(q^2,q,q)\} - w(1,1,q) = 4\{t_{4,q} - 12Cp_{8,q}\}$
3. $-4\{T(k^2(2k^2-1)^2, k, 2k^2-1) + y(k^2(2k^2-1), 1, 2k^2-1) - x(k^2(2k^2-1), 1, 2k^2-1)\}$
   $+ w(k,1,2k^2-1) + 2 = 2gn_{k^2+2} - 16SO_k$
4. $y(q+1,2q,q) + 2T(q+1,2q,q) = t_{4,2q} - 32P_q^5$
5. $x(k,p,-1) - y(k,p,-1) - 2T(k,p,-1) = 8t_{3,p}$
6. $x(2,4p^2,p^2) - y(2,4p^2,p^2)$ is a biquadratic integer.
7. Each of the following represents a nasty number:
   - $3\{x(k,p,-p) - y(k,p,-p) - 2T(k,p,-p)\}$
   - $-6\{x(k,-q,q) + 2T(k,-q,q)\}$

Pattern-3:

Assume $T = a^2 + (4k-1)b^2$ (6)

where a and b are distinct integers.

Write $4k$ as $4k = (1+i\sqrt{4k-1})(1-i\sqrt{4k-1})$ (7)

Using (6) & (7) in (3) and employing the method of factorization, define

$$(u + i\sqrt{4k-1}v) = (1+i\sqrt{4k-1})(a+i\sqrt{4k-1}b)^2$$

Equating the real and imaginary parts, we have

$$u = u(k,a,b) = a^2 - (4k-1)b^2 - 2(4k-1)ab$$

$$v = v(k,a,b) = a^2 - (4k-1)b^2 + 2ab$$

Hence in view of (2), the corresponding solutions of (1) are

$$x = x(k,a,b) = 2a^2 - 2(4k-1)b^2 + 2(2-4k)ab$$

$$y = y(k,a,b) = -8kab$$

$$z = z(k,a,b) = a^4 + 2(2-4k)a^3b - 6(4k-1)a^2b^2 + 2(4k-1)(4k-2)ab^3 + (4k-1)^2b^4 + 1$$

$$w = w(k,a,b) = a^4 + 2(2-4k)a^3b - 6(4k-1)a^2b^2 + 2(4k-1)(4k-2)ab^3 + (4k-1)^2b^4 - 1$$

$$T = T(k,a,b) = a^2 + (4k-1)b^2$$

Properties:

1. $4w(k,1,b) + y((4k-1)(4k-2),1,b^3) - 4T(k,(4k-1)b^2,b)$
$= 16(1-2k)\,Pr_b + 2(3-20k)(gn_{b^2} - 1)$
2. $x(1,a,-1) + 2T(1,a,-1) = 4Pr_a$
3. $x(1,a,b) - 2T(1,a,b) \equiv 0 \pmod 4$
4. $4x(k,a^2,b) + y((2-4k),a^2,b) + 8T(k,a^2,b)$ is a Bi quadratic integer.
5. Each of the following represents a nasty number:
   - $3\{x(a(3a-2),a,1) - y(a(3a-2),a,1) + 8t_{8,a}\}$
   - $6\{4\{x(k,a,b) + 2T(k,a,b)\} + y(2-4k,a,b)\}$
   - $-3\{2x(1,a,a) - y(1,a,a)\}$

Pattern-4:

From (3), $u^2 = 4kT^2 - (4k-1)v^2$ (8)

Introducing the linear transformations

$$T = P \pm (4k-1)Q$$
$$v = P \pm 4kQ \quad (9)$$

in (8), it simplifies to

$$P^2 = 4k(4k-1)Q^2 + u^2$$

whose solution is given by

$$u = u(k,r,s) = 4k(4k-1)r^2 - s^2$$

$$P = P(k,r,s) = 4k(4k-1)r^2 + s^2$$

$$Q = Q(k,r,s) = 2rs$$

Using the values of P and Q in (9) and taking (2), the corresponding solutions of (1) are given by

$$x = x(k,r,s) = 8k(4k-1)r^2 + 8krs$$

$$y = y(k,r,s) = -2s^2 - 8krs$$

$$z = z(k,r,s) = 16k^2(4k-1)^2 r^4 + 32k^2(4k-1)r^3 s - 8krs^3 - s^4 + 1$$

$$w = w(k,r,s) = 16k^2(4k-1)^2 r^4 + 32k^2(4k-1)r^3 s - 8krs^3 - s^4 - 1$$

$$T = T(k,r,s) = 4k(4k-1)r^2 + s^2 + 2(4k-1)rs$$

CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

REFERENCE

[1]. Dickson.L.E., History of Theory of numbers, vol.2:Diophantine Analysis, New York, Dover, 2005.

[2]. Mordell L.J., Diophantine Equations, Academic press, London (1969).

[3]. Carmichael.R.D.,The theory of numbers and Diophantine Analysis, NewYork, Dover,1959.

[4]. Lang, S. Algebraic N.T., Second ed. Newyork: Chelsea, 1999.

[5]. Weyl,H. Algebraic theory of numbers, Princeton, NJ: Princeton University press, 1998.

[6]. Oistein Ore, Number theory and its History, NewYork , Dover, 1988.

[7]. T.Nagell, Introduction to Number theory, Chelsea(Newyork) 1981.

[8]. Cohn J.H.E., The Diophantine equation $y(y+1)(y+2)(y+3) = 2x(x+1)(x+2)(x+3)$ Pacific J.Math. 37, 331-335, 1971.

[9]. Leabey.W.J and Hsu. D.F, "The Diophantine equation $y^4 = x^3 + x^2 + 1$", Rocky Mountain J.Math. Vol.6, 141-153 , 1976.

[10]. Mihailov, "On the equation $x(x+1) = y(y+1)z^2$", Gaz. Mat. Sec.A 78, 28-30, 1973.

[11]. Gopalan M.A., and Anbuselvi, R., "Integral Solutions of ternary quadratic equation $x^2 + y^2 = z^4$", Acta Ciencia Indica, Vol XXXIV M, No. 1, 297-300, 2008.

[12]. Gopalan M.A., Manju Somanath and Vanitha.N., "Parametric integral solutions of $x^2 + y^3 = z^4$"., Acta Ciencia Indica, Vol XXXIII M, No. 4, 1261-1265 , 2007.

[13]. Cross J.T., "In the Gaussian Integers $\alpha^4 + \beta^4 \neq \gamma^4$", Math, Magazine,66, 105-108,1993.

[14]. Sandorszobo some fourth degree Diophantine equation in Gaussian integers: Electronic Journal of combinatorial Number theory, Vol .4, 1-17, 2004.

[15]. Gopalan M.A., Vijayasankar.A and Manju Somanath, "Integral solutions of Note on the Diophantine equation $x^2 - y^2 = z^4$"., Impact J.Sci Tech; Vol 2(4), 149-157,2008.

[16]. Gopalan M.A and V.Pandichelvi, "On the solutions of the Biquadratic equation $(x^2 - y^2)^2 = (z^2 - 1)^2 + W^4$" ,International Conference on Mathematical Methods and Computations, Trichirapalli, 24-25, July 2009.

[17]. Gopalan M.A and Janaki.G., "Integral Solutions of Ternary quadratic equations $x^2 - y^2 + xy = z^4$" Impact Journal Vol 2(2), 71-76,2008.

[18]. Gopalan M.A and V.Pandichelvi, "On ternary quadratic Diophantine equation $x^2 + ky^3 = z^4$", Pacific Asian Journal of Mathematics, Vol 2, No 1-2, 57-62,Jan-Dec, 2008.

[19]. Gopalan M.A and J.Kaligarani, "On quadratic equation in 5 unknowns $x^4 - y^4 = 2(z^2 - W^2)p^2$", Bulletion of pure and applied sciences , Vol 28E, No 2,305-311, 2009.

[20]. Gopalan M.A and S.Vidhyalakshmi, S.Devibala, "Ternary Quartic Diophantine Equation $2^{4n+3}(x^3 - y^3) = z^4$, Impact Journal of Science and Technology, **4**(1),57-60, (2010).

[21]. Gopalan M.A and A.Vijayasankar and Manju Somanath, "Integral Solutions of $x^2 - y^2 = z^4$" Impact Journal of Science and Technology,**2**(4), 149-157, 2010.

[22]. Gopalan M.A and J.Kaligarani, On Quadratic Equation In Five Unknowns $x^4 - y^4 = (z + W)p^3$,Bessal J Math , 1(1),pg 49-57, 2011.

[23]. Gopalan M.A and P.Shanmuganandham., "On the biquadratic equation $x^4 + y^4 + (x + y)z^3 = 2(k^2 + 3)^{2n} w^4$"Bessal J Math ,2(2), pg 87-91, 2012.

[24]. Gopalan M.A and B.Sivakami., integral solutions of quartic equation with four unknowns $x^3 + y^3 + z^3 = 3xyz + 2(x + y)w^3$, Antarctica J.Math.,10(2).,151-159.,2013.