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RESEARCH ARTICLE

# ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION $X^2 + Y^2 = 29Z^2$

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**ABSTRACT**

The ternary quadratic homogeneous equation representing homogeneous cone given by $x^2 + y^2 = 29z^2$ is analyzed for its non-zero distinct integer points on it. Four different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relations between the solutions and special number patterns namely Polygonal number and Nasty number are presented. Also knowing an integer solution satisfying the given cone, three triples of integers generated from the given solution are exhibited.

**Keywords**: Ternary homogeneous quadratic, integral solutions

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## INTRODUCTION

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1,19]. For an extensive review of various problems, one may refer [2-18]. This communication concerns with yet another interesting ternary quadratic equation $x^2 + y^2 = 29z^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

**NOTATION:**

$T_{m,n}$ – Polygonal number of rank n with size m.

## 2.METHOD OF ANALYSIS

The ternary quadratic equation studied for its non-zero distinct integer solutions is given by

$$x^2 + y^2 = 29z^2 \tag{1}$$

To start with, note that (1) is satisfied by $(\pm 5, \pm 2, \pm 1)$. However, we have other patterns of solutions which are illustrated as follows:

### 2.1 Pattern 1:

Assume $z = z(a,b) = a^2 + b^2,\ where\ a,b > 0$ (2)

and Write 29 as $29 = (2+5i)(2-5i)$ (3)

Substituting (2) & (3) in (1) and employing the method of factorization, define

$x + iy = (2+5i)(a+ib)^2$

Equating the real and imaginary parts in the above equation, we get

$x = x(a,b) = 2a^2 - 2b^2 - 10ab\ ,\ y = y(a,b) = 5a^2 - 5b^2 + 4ab$ (4)

Thus, (2) and (4) represent the distinct integer points on the cone (1).

**Properties:**

- $x(A+1, A+1) + 2T_{6,A} \equiv 0\ (mod\ 2)$
- $y(2,2B) + 2T_{10,B} \equiv 0\ (mod\ 2)$
- $x(2A,1) - y(2A,1) - T_{18,A} - T_{10,A} \equiv 0\ (mod\ 3)$
- $x(2B, 2B+1) + T_{22,B} + T_{14,B} \equiv -26 (mod\ 42)$
- $6\{y(A,A)\} = 6 \times 4A^2$ is a Nasty number.

**2.2 Pattern 2:**

Equation (1) can be written as

$\frac{x+5z}{2z+y} = \frac{2z-y}{x-5z} = \frac{A}{B},\ where\ B \neq 0$ (5)

This equation is equivalent to the following two equations:

$Bx - Ay + (5B-2A)z = 0, -Ax - By + (2B+5A)z = 0$ (6)

By the method of cross multiplication, we get the integral solutions of (1) to be

$$\left.\begin{aligned} x = x(A,B) &= -5A^2 + 5B^2 - 4AB \\ y = y(A,B) &= 2A^2 - 2B^2 - 10AB \\ z = z(A,B) &= -A^2 - B^2 \end{aligned}\right\} \qquad (7)$$

**Properties:**

- $x(A,1) + T_{8,A} + T_{6,A} \equiv 5\ (mod\ 7)$
- $x(A+1, A+1) + 2T_{6,A} \equiv 0\ (mod\ 2)$
- $y(B+4, B+4) + T_{16,B} + T_{8,B} \equiv 0\ (mod\ 2)$
- $x(A,1) + y(A,1) + z(A,1) + 2T_{6,A} \equiv 0\ (mod\ 2)$
- $x(1,B) - y(1,B) - z(1,B) - T_{12,B} - T_{8,B} \equiv 0\ (mod\ 6)$

**Note** :

Equation (5) can also be written in the following way:

1. $\frac{x+5z}{2z-y} = \frac{2z+y}{x-5z} = \frac{A}{B} \quad where\ B \neq 0$

Proceeding as above in pattern 2, different choice of integer solutions to (1) is obtained.

**2.3 Pattern 3:**

Equation (1) can be written as $x^2 = 29z^2 - y^2$ (8)

Assume $x = x(a,b) = 29a^2 - b^2\ ,\ where\ a,b > 0$ (9)

Using (9) in (8) and employing the method of factorization, define

$(\sqrt{29}a + b)^2 = (\sqrt{29}z + y)$

Equating the rational and irrational factors, we get

$$\left.\begin{aligned} y = y(a.b) &= 29a^2 + b^2 \\ z = z(a,b) &= 2ab \end{aligned}\right\} \qquad (10)$$

Thus (9) &(10) represents the distinct integer points on the cone (1).

**Properties:**

- $x(A, A+1) - 2T_{30,A} \equiv -1 \pmod{24}$
- $y(B, B+2) - T_{34,B} - T_{30,B} \equiv 0 \pmod{4}$
- $x(A+1, A+2) - T_{32,B} - T_{28,B} \equiv 0 \pmod{5}$
- $x(A, A-1) + 2y(A, A-1) + 2z(A, A-1) - T_{104,A} - T_{84,A} \equiv 1 \pmod{3}$
- $x(A, 1) + 29z(A, 1) - T_{40,A} - T_{22,A} \equiv -1 \pmod{85}$
- $6\{x(A, 2A)\} = 6 \times 25A^2$ is a Nasty number.

**2.4 Pattern 4:**

Equation (1) can be written as

$$29z^2 - y^2 = x^2 * 1 \tag{11}$$

Write 1 as

$$1 = \frac{(\sqrt{29}+5)(\sqrt{29}-5)}{4} \tag{12}$$

Using (9) & (12) in (11) and applying the method of factorization, define

$$\sqrt{29}z + y = \left(\sqrt{29}a + b\right)^2 \frac{(\sqrt{29}+5)}{2}$$

Equating the rational and irrational factors, we get

$$\left.\begin{aligned} y = y(a,b) &= \tfrac{1}{2}[145a^2 + 5b^2 + 58ab] \\ z = z(a,b) &= \tfrac{1}{2}[29a^2 + b^2 + 10ab] \end{aligned}\right\} \tag{13}$$

Replacing a by 2A and b by 2B in the above equation, the corresponding integer solutions to (1) are given by

$$\left.\begin{aligned} x = x(A,B) &= 116A^2 - 4B^2 \\ y = y(A,B) &= 290A^2 + 10B^2 + 116AB \\ z = z(A,B) &= 58A^2 + 2B^2 + 20AB \end{aligned}\right\} \tag{14}$$

**Properties**:

- $y(A, 1) - T_{282,A} - T_{302,A} \equiv 0 \pmod{2}$
- $z(A, 2) - T_{102,A} - T_{18,A} \equiv 0 \pmod{8}$
- $x(1, B) - y(1, B) + z(1, B) + T_{18,B} + T_{10,B} \equiv 0 \pmod{2}$
- $x(A, -2) - z(A, -2) - T_{82,A} - T_{38,A} \equiv 0 \pmod{3}$
- $y(1, B) - 2z(1, B) - T_{10,B} - T_{6,B} \equiv 14 \pmod{80}$
- $z(3, 5A) - 2T_{52,A} \equiv 0 \pmod{3}$

**Note**:

Instead of (12) we can also write 1 as

$$1 = \frac{(\sqrt{29}+2)(\sqrt{29}-2)}{25}$$

Proceeding as above in pattern 4, we can get different choices of integer solutions to (1).

**3.Generation of integer solutions:**

Let $(x_0, y_0, z_0)$ be any given integer solution of (1), Then, each of the following triples of non-zero distinct integers based on $x_0, y_0, z_0$ also satisfies (1).

**Triple 1:** $(2^n x_0, y_n, z_n)$

Here $y_n = \frac{1}{10\sqrt{29}}\left[5\sqrt{29}A_n y_0 + 145B_n z_0\right]$

$z_n = \frac{1}{10\sqrt{29}}\left[5B_n y_0 + 5\sqrt{29}A_n z_0\right]$

where $A_n = \left(27 + 5\sqrt{29}\right)^n + \left(27 - 5\sqrt{29}\right)^n$

$B_n = \left(27 + 5\sqrt{29}\right)^n - \left(27 - 5\sqrt{29}\right)^n.$

**Triple 2:** $(x_n, 5^n y_0, z_n)$

Here $x_n = \frac{2^n}{4}\left((29 - (-1)^n 25)x_0 + 145(-1 + (-1)^n)z_0\right)$

$z_n = \frac{2^n}{4}\left(5(1 - (-1)^n)x_0 + (-29 + (-1)^n 25)z_0\right)$

**Triple 3:** $(x_n, y_n, 5^n z_0)$

Here $x_n = 5^{n-1}\left((4 + (-1)^n)x_0 + 2(1 + (-1)^n)y_0\right)$

$z_n = 5^{n-1}\left(2(1 - (-1)^n)x_0 + (1 + 4(-1)^n)y_0\right)$

**4.Remarkable Observation:**

Let $x, y$ be represent the positive distinct integer solution of (1).

Let $p, q$ be two non-zero distinct positive integers such that $p = x + \frac{y}{2}, q = \frac{y}{2}$

Note that $p > q > 0$. Treat $p, q$ as the generators of the Pythogorean triangle $S(\alpha, \beta, \gamma)$

Where $\alpha = 2pq,\ \beta = p^2 - q^2,\ \gamma = p^2 + q^2$. Let A & P represent the area and perimeter of S.

This Pythogorean triangle $S$ is such that

(i) $3\gamma - \alpha - 2\beta \equiv 0 \ (mod\ 29)$

(ii) $\gamma + \alpha - \frac{8A}{P} \equiv 0 \ (mod\ 29)$

(iii) $5\gamma - 3\alpha - 4\beta + \frac{8A}{P} \equiv 0 \ (mod\ 29)$

**5.CONCLUSION**

In this paper, we have presented four different patterns of non-zero distinct integer solutions of the homogeneous cone given by $x^2 + y^2 = 29z^2$. Since the ternary quadratic Diophantine equations are rich in variety, one may search for other choices of Diophantine equations to find their corresponding integer solutions.

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