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Review Article

G- Frame Operators

NEHA AGRAWAL

Assistant Professor, Department of Mathematics, University of Delhi, Delhi

NEHAAGRAWAL

Author for Correspondence

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ABSTRACT

G-frames are generalized frames which include ordinary frames and many recent generalizations of frames. The aim of this article is to study the g- frame operators and their properties and the dual g-frames. We will also study some spectral properties of g-frame operators.

Key words and Phrases: G- frames, G-frame operator, spectrum.

INTRODUCTION

Frames are generalizations of orthonormal bases in Hilbert spaces. As for an orthonormal basis, a frame allows each element in the underlying Hilbert space to be written as an unconditionally convergent infinite linear combination of the frame elements; however in contrast to the situation for a basis, the coefficients might not be unique. G-frames were introduced by Wenchang Sun [4] in 2006.G-frames in Complex Hilbert spaceshave some properties similar to that of frames.

A sequence $\{f_i\}_{i\in I}$ of elements of a Hilbert space H is called a *frame* if there exist constants $A, B > 0$ such that for all $f \in H, A\|f\|^2 \leq \sum_{i\in I}|\langle f, f_i\rangle|^2 \leq B\|f\|^2$.

The numbers A and B are called the lower and upper frame bounds respectively. The largest number $A > 0$ and the smallest number $B > 0$ satisfying the frame inequalities for all $f \in H$ are called optimal frame bounds. Throughout this article, H and K are Hilbert spaces and $\{H_i\}_{i\in I} \subseteq K$ is a sequence of separable Hilbert spaces, where I is a subset of $\mathbb{N}$. $\mathcal{B}(H, H_i)$is the collection of all bounded linear operators from H to H_i.

2. g- FRAMES

Definition 2.1 [4]: The sequence $\{\Lambda_i \in \mathcal{B}(H, H_i) : i \in I\}$ is called a *g-frame* for H with respect to $\{H_i\}_{i\in I}$ if there exist two positive constants A and B such that for all$f \in H$, we have

$$A\|f\|^2 \le \sum_{i\in I}\|\Lambda_i f\|^2 \le B\|f\|^2 \ . \qquad (2.1)$$

The numbers A and B are called the lower and upper g- frame bounds respectively. The largest number $A > 0$ and the smallest number $B > 0$ satisfying the frame inequalities for all $f \in H$ are called optimal frame bounds.

Definition 2.2 [4]: The sequence $\{\Lambda_i\}_{i\in I}$ is called a *tight g-frame* if $A = B$ and Parseval g- frame if $A = B = 1$.

Definition 2.3 [4]: The sequence $\{\Lambda_i \in \mathcal{B}(H, H_i) : i \in I\}$ is called a *g-Bessel* sequence if there exists $B > 0$ such that $\sum_{i\in I}\|\Lambda_i f\|^2 \le B\|f\|^2$ for all $f \in H$.

Definition 2.4 [4]: Let$\{\Lambda_i \in \mathcal{B}(H, H_i) : i \in I\}$ be given. Define $(\sum_{i\in I} \oplus W_i)_{l_2}$ as

$$(\textstyle\sum_{i\in I} \oplus H_i)_{l_2} = \{\{f_i\}_{i\in I} : f_i \in H_i \ and \ \sum_{i\in I}\|f_i\|^2 < \infty\},$$

with the inner product is given by $\langle\{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle$.

Definition 2.5 [4]: Let $\{\Lambda_i \in \mathcal{B}(H, H_i) : i \in I\}$ be a g-frame for H. Then the *synthesis operator* for $\{\Lambda_i\}_{i\in I}$ is the operator $T : (\sum_{i\in I} \oplus H_i)_{l_2} \to H$ defined by $(f) = \sum_{i\in I} \Lambda_i^* f_i$,
for all $f = \{f_i\}_{i\in I} \in (\sum_{i\in I} \oplus H_i)_{l_2}$, where Λ_i^* is the adjoint of Λ_i.

Definition 2.6 [4]: The adjointT^* of the synthesis operator T is called the *analysis operator.*

The formula for the analysis operator is given by the following theorem:

Theorem 2.7 [3]: Let $\{\Lambda_i \in \mathcal{B}(H, H_i) : i \in I\}$ be a g-frame for H.Then the analysis operator for $\{\Lambda_i\}_{i\in I}$ is the operator $T^* : H \to (\sum_{i\in I} \oplus H_i)_{l_2}$ defined by $T^*f = \{\Lambda_i f\}_{i\in I}$.

Proof: Let $f \in H$ and let $g = \{g_i\}_{i\in I} \in (\sum_{i\in I} \oplus H_i)_{l_2}$. Then

$$\langle T^*f, g\rangle = \langle f, Tg\rangle = \langle f, \sum_{i\in I} \Lambda_i^* g_i\rangle = \sum_{i\in I}\langle \Lambda_i f, g_i\rangle = \langle\{\Lambda_i f\}_{i\in I}, \{g\}_{i\in I}\rangle = \langle\{\Lambda_i f\}_{i\in I}, g\rangle$$

Therefore $T^*f = \{\Lambda_i f\}_{i\in I}$. ∎

3. g- FRAME OPERATOR

Definition 3.1 [4]: Let $\{\Lambda_i \in \mathcal{B}(H, H_i) : i \in I\}$ be a g-frame for H with respect to $\{H_i\}_{i\in I}$. The *g- frame operator* $S: H \to H$is defined as $Sf = \sum_{i\in I} \Lambda_i^* \Lambda_i f$ for all $f \in H$.

Theorem 3.2 [4]: Let $\{\Lambda_i \in \mathcal{B}(H, H_i) : i \in I\}$ be a g-frame for H with respect to $\{H_i\}_{i\in I}$ with frame bounds C and D. Then the frame operator S for $\{\Lambda_i\}_{i\in I}$ is a positive, self-adjoint, invertible operator on H with $CI_H \le S \le DI_H$ where I_H is the identity operator on H. We further have the formula $f = \sum_{i\in I} \Lambda_i^* \Lambda_i S^{-1} f = \sum_{i\in I} S^{-1}\Lambda_i^* \Lambda_i f$ *for all* $f \in H$.

Proof: Let $f \in H$. Since$\{\Lambda_i\}_{i\in I}$ is a g- frame with respect to $\{H_i\}_{i\in I}$ with frame bounds C and D, we have $C\|f\|^2 \le \sum_{i\in I}\|\Lambda_i f\|^2 \le D\|f\|^2$. Now,

$$\|S\| = \sup\{\langle Sf, f\rangle: \|f\| = 1\} = \sup\{\textstyle\sum_{i\in I}\langle \Lambda_i^* \Lambda_i f, f\rangle : \|f\| = 1\}$$
$$= \sup\{\textstyle\sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle : \|f\| = 1\} = \sup\{\sum_{i\in I}\|\Lambda_i f\|^2 : \|f\| = 1\} \le D$$

and$\langle Sf, f\rangle = \sum_{i\in I}\|\Lambda_i f\|^2 \ge 0$. Therefore S is a bounded, positive operator and hence is self-adjoint. Also $\langle Cf, f\rangle = C\|f\|^2 \le \sum_{i\in I}\|\Lambda_i f\|^2 = \langle Sf, f\rangle \le D\|f\|^2 = \langle Df, f\rangle$, which implies $CI_H \le S \le DI_H$. Again, $C\|f\|^2 \le \langle Sf, f\rangle \le \|Sf\|\|f\|$which implies $\|Sf\| \ge C\|f\|$ and hence S is one- one. Now let $g \in H$ be such that $\langle Sf, g\rangle = 0$ for all $f \in H$ then $\langle f, Sg\rangle = 0$ which gives $Sg = 0$ and hence $g = 0$.Therefore $SH = H$. Hence S is invertible and $\|S^{-1}\| \le \frac{1}{C}$.

For any $f \in H, f = SS^{-1}f = S^{-1}Sf = \sum_{i\in I} \Lambda_i^* \Lambda_i S^{-1} f = \sum_{i\in I} S^{-1}\Lambda_i^* \Lambda_i f$ (3.1)

■

4. DUAL g- FRAME

If in equation (3.1), we let $\tilde{\Lambda_i} = \Lambda_i S^{-1}$ then $\tilde{\Lambda_i}^* = S^{-1}\Lambda_i^*$ the equation reduces to

$$f = \sum_{i \in I} \Lambda_i^* \tilde{\Lambda_i} f = \sum_{i \in I} \tilde{\Lambda_i}^* \Lambda_i f$$

Theorem 4.1 [4]: The sequence $\{\tilde{\Lambda_i} \in \mathcal{B}(H, H_i) : i \in I\}$ is a g-frame for H with respect to $\{H_i\}_{i \in I}$ with frame bounds $\frac{1}{D}$ and $\frac{1}{C}$.

Proof: For $f \in H$,

$$\sum_{i \in I} \|\tilde{\Lambda_i} f\|^2 = \sum_{i \in I} \|\Lambda_i S^{-1} f\|^2 = \sum_{i \in I} \langle \Lambda_i S^{-1} f, \Lambda_i S^{-1} f\rangle = \sum_{i \in I} \langle \Lambda_i^* \Lambda_i S^{-1} f, S^{-1} f\rangle$$

$$= \langle SS^{-1} f, S^{-1} f\rangle = \langle f, S^{-1} f\rangle \leq \|S^{-1} f\|\|f\| \leq \frac{1}{C}\|f\|^2$$

and $\|f\|^2 = \sum_{i \in I} \langle \tilde{\Lambda_i}^* \Lambda_i f, f\rangle = \sum_{i \in I} \langle \Lambda_i f, \tilde{\Lambda_i} f\rangle \leq (\sum_{i \in I} \|\Lambda_i f\|^2)^{1/2} (\sum_{i \in I} \|\tilde{\Lambda_i} f\|^2)^{1/2}$

$$\leq D^{1/2} \|f\| \left(\sum_{i \in I} \|\tilde{\Lambda_i} f\|^2 \right)^{1/2}$$

which implies $\sum_{i \in I} \|\tilde{\Lambda_i} f\|^2 \geq \frac{1}{D}\|f\|^2$. Therefore $\{\tilde{\Lambda_i} \in \mathcal{B}(H, H_i) : i \in I\}$ is a g-frame for H with respect to $\{H_i\}_{i \in I}$ with frame bounds $\frac{1}{D}$ and $\frac{1}{C}$. ■

Definition 4.2 [4]: The sequence $\{\tilde{\Lambda_i} : i \in I\}$ is called the *dual g- frame* of $\{\Lambda_i : i \in I\}$.

Theorem 4.3: The g- frames $\{\Lambda_i : i \in I\}$ and $\{\tilde{\Lambda_i} : i \in I\}$ are dual with respect to each other.

Proof: Let $\tilde{S}$ be the g – frame operator associated with $\{\tilde{\Lambda_i} : i \in I\}$. Then for all $f \in H$,

$$S\tilde{S} f = \sum_{i \in I} S \tilde{\Lambda_i}^* \tilde{\Lambda_i} f = \sum_{i \in I} S(\Lambda_i S^{-1})^* (\Lambda_i S^{-1}) f = \sum_{i \in I} SS^{-1} \Lambda_i^* \Lambda_i S^{-1} f = SS^{-1} f = f$$

which implies $\tilde{S} = S^{-1}$ and $\tilde{\Lambda_i} \tilde{S}^{-1} = \Lambda_i S^{-1} S = \Lambda_i$. Therefore $\{\Lambda_i : i \in I\}$ and $\{\tilde{\Lambda_i} : i \in I\}$ are dual g- frames with respect to each other. ■

Theorem 4.4: The sequence $\{\Lambda_i \in \mathcal{B}(H, H_i) : i \in I\}$ is a Parseval g- frame if and only if $S = I_H$.

Proof: Let $\{\Lambda_i \in \mathcal{B}(H, H_i) : i \in I\}$ be a Parseval g- frame of subspaces then $A = B = 1$ in equation (2.1). Also by theorem 3.2, $AI_H \leq S \leq BI_H$, therefore $S = I_H$. Otherway let $S = I_H$. Then for all $f \in H$,

$$\|f\|^2 = \langle f, f\rangle = \langle Sf, f\rangle = \sum_{i \in I} \langle \Lambda_i^* \Lambda_i f, f\rangle = \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f\rangle = \sum_{i \in I} \|\Lambda_i f\|^2$$

Therefore $\{\Lambda_i \in \mathcal{B}(H, H_i) : i \in I\}$ is a Parseval g- frame. ■

5. SPECTRUM OF g- FRAME OPERATOR

Definition 5.1 [2]: A complex number λ is said to be in the spectrum of a bounded linear operator T on a Hilbert space H, if $(T - \lambda I)$ is not invertible. The spectrum of bounded linear operator T is denoted by $\sigma(T)$ and its complement, the resolvent set is $(T) = \mathbb{C} \setminus \sigma(T)$.

Theorem 5.2 [2]: Let U be a bounded self- adjoint operator on a Hilbert space . Let $\rho(U)$ denotes the resolvent spectrum of U. A scalar $\lambda \in \rho(U)$ if and only if $(U - \lambda I)$ is bounded below.

Theorem 5.3: Let S be the g- frame operator on a Hilbert space H.Then $\sigma(S)$, the spectrum of S is a subset of the set of real numbers.

Proof: Let $\lambda \in \sigma(S)$. Let $\lambda = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$.Let if possible $\beta \neq 0$. Then for each $x \in H$, $\langle (S - \lambda)x, x\rangle = \langle Sx, x\rangle - \lambda\langle x, x\rangle$.

Therefore $\overline{\langle (S - \lambda)x, x\rangle} = \langle Sx, x\rangle - \bar{\lambda}\langle x, x\rangle$.

This implies $\overline{\langle (S - \lambda)x, x\rangle} - \langle (S - \lambda)x, x\rangle = (\lambda - \bar{\lambda})\langle x, x\rangle = 2i\beta\langle x, x\rangle = 2i\beta\|x\|^2$

Therefore $|2i\beta\|x\|^2| = \left|\overline{\langle (S - \lambda)x, x\rangle} - \langle (S - \lambda)x, x\rangle\right|$

which implies $2|\beta|\|x\|^2 \leq 2|\langle (S-\lambda)x\ ,x\rangle| \leq 2\|(S-\lambda I)x\|\|x\|$
If $x=0$ then $|\beta|\|x\| = \|(S-\lambda I)x\|$ and if $x \neq 0$ then $\|(S-\lambda I)x\| \geq |\beta|\|x\|$which implies that$\lambda \in \rho(S)$, a contradiction. Hence $\beta = 0$ and therefore $\sigma(S) \subset \mathbb{R}$. ■

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